TOPSIS method

- TOPSIS stands for *Technique for Order Preference by Similarity to Ideal Solution*.
- The method consists of two main phases:
  - defining a (*positive*) *ideal solution* (PIS) and an *ideal* solution *in a negative sense* (*negative ideal solution* – NIS)
  - select the preferred alternative, which should be close to the PIS and far from the NIS.

- The Euclidean distance can be used to evaluate distances.
- A ranking based on the distances to the previously defined PIS and NIS is then considered.
- The method considers “benefit” criteria separately from “cost” criteria.
- Starting with the performance matrix, six steps are typically followed.

Example [COAT]

- The performance matrix is:

	TRF	PRD	THK	CAP	MNT	FLX
	(%)	(m^2/h)	(mm)	(k€)	(min)	(-)
A1	95	600	0.23	160	10	8
A2	98	550	0.23	180	15	8
B	92	800	0.21	120	30	6
C1	96	700	0.22	140	20	7
C2	94	800	0.24	180	20	9

- CAP and MNT are “cost” criteria.

- *First step:*
  construction of the normalised matrix
- Let there be *m* alternatives and *n* evaluation criteria (in the example [COAT] *m=5*, *n=6*).
- Since the performance of the alternative a_i in criterion g_j takes a numerical value in TOPSIS, we use the notation:

$$g_j(a_i) = x_{ij} \quad i = 1,\ldots,m \quad j = 1,\ldots,n$$

- The normalised value of the performance (xij) is calculated as:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{h=1}^{m} x_{hj}^2}}$$

- This is the traditional method (*vector normalisation*).
- Other normalisation methods can be used.

- The normalised matrix is:

	TRF	PRD	THK	CAP	MNT	FLX
A1	0.447	0.385	0.455	0.454	0.222	0.467
A2	0.461	0.353	0.455	0.510	0.333	0.467
B	0.433	0.513	0.415	0.340	0.667	0.350
C1	0.452	0.449	0.435	0.397	0.444	0.408
C2	0.442	0.513	0.474	0.510	0.444	0.525

- *Second step:*
  construction of the weighted normalised matrix
- Each r_{ij} is multiplied by the (normalised) weight of the criterion:

$$v_{ij} = w_j \times r_{ij}$$

- The v_{ij} are used for identifying the ideal solutions and comparing the alternatives with them.
- The weights can be defined by the decision-makers.
- In the example, we will use the same weights as in the SAW method.

- The weighted normalised matrix in [COAT] is:

	TRF	PRD	THK	CAP	MNT	FLX
	0.20	0.20	0.16	0.18	0.12	0.14
A1	0.089	0.077	0.073	0.082	0.027	0.065
A2	0.092	0.071	0.073	0.092	0.040	0.065
B	0.087	0.103	0.066	0.061	0.080	0.049
C1	0.090	0.090	0.070	0.071	0.053	0.057
C2	0.088	0.103	0.076	0.092	0.053	0.073

- *Step three:*
  determining ideals
- The criteria are classified into two subsets:
  - I^B contains the “benefit” criteria (e.g. TRF, transfer efficiency)
  - I^C contains the “cost” criteria (e.g. CAP, capital cost)

- Denoting by a^+ the PIS and by a^- the NIS, we have:

$$a^+ = \{(\max_i v_{ij} : j \in I^B), (\min_i v_{ij} : j \in I^C), i = 1, \ldots, m\}$$

$$a^- = \{(\min_i v_{ij} : j \in I^B), (\max_i v_{ij} : j \in I^C), i = 1, \ldots, m\}$$

- So the two ideal (fictitious) solutions have coordinates:

$$a^{+} = (v_1^{+}, \ldots, v_j^{+}, \ldots, v_n^{+})$$

$$a^{-} = (v_1^{-}, \ldots, v_j^{-}, \ldots, v_n^{-})$$

- a^+ represents the ideal alternative that has maximum "benefits" and minimum "costs".
- a^- represents the ideal negative alternative that has minimum "benefits" and maximum "costs".

- The matrix shows the components of a^+ in green and those of a^- in red:

	TRF	PRD	THK	CAP	MNT	FLX
	0.20	0.20	0.16	0.18	0.12	0.14
A1	0.089	0.077	0.073	0.082	0.027	0.065
A2	0.092	0.071	0.073	0.092	0.040	0.065
B	0.087	0.103	0.066	0.061	0.080	0.049
C1	0.090	0.090	0.070	0.071	0.053	0.057
C2	0.088	0.103	0.076	0.092	0.053	0.073

a+	0.092	0.103	0.076	0.061	0.027	0.073
a-	0.087	0.071	0.066	0.092	0.080	0.049

- *Step four:*
  determination of separation measures
- The distances to the PIS and NIS of each alternative (i) are determined:

$$S_{i^+} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \quad i = 1, \ldots, m$$

$$S_{i^-} = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, \quad i = 1, \ldots, m$$

- In the [COAT] example, the distances to the ideals are:

	S+	S-
A1	0.0340	0.0575
A2	0.0471	0.0440
B	0.0597	0.0443
C1	0.0359	0.0399
C2	0.0408	0.0493

- The nearer is an alternative to PIS the smaller is S_{i+} (it tends to 0).
- The nearer is an alternative to NIS the smaller is S_{i-} (it tends to 0).
- An alternative is preferable if it is near to PIS and far from NIS.
- However, an alternative that has small S_{i+} does not necessarily have large S_{i-} (see C1 in the example).

- *Fifth step:*
  determining the proximity (closeness) to the ideal solution
- The *relative closeness of* a_i to the ideal solution is defined as follows:

$$RC_i = \frac{S_{i^-}}{S_{i^+} + S_{i^-}} \quad 0 \leq RC_i \leq 1$$

- If a_i matches the PIS: $RC_i = 1$
- If a_i matches the NIS: $RC_i = 0$.

- In the [COAT] example we obtain:

	RCi
A1	0.628
A2	0.483
B	0.426
C1	0.526
C2	0.547

- The following ranking is then obtained:

$$A1 \succ C2 \succ C1 \succ A2 \succ B$$

Comment on the weights

- The TOPSIS method uses a clear analytical structure that allows the influence of different parameters on the result to be tested numerically.
- Methods for quantifying weights have been proposed that would give them an “objective” meaning.
- The aim is to diminish the (subjective) influence of the DMs on the outcome and instead give more emphasis to the measurable characteristics of the alternatives.

- Among the most popular approaches are:
  - the mean weight method (i.e. $w_j=1/n$ for all $j=1,...n$)
  - the use of a weight related to the information content provided by the alternatives in the criterion.
- The idea behind the second approach is that the importance of a criterion for the decision should be linked to its ability to discriminate between alternatives.
- If the performance of the available alternatives against a criterion g_j were all very similar, it would be of little use to support the DM's choice.
- Conversely, if the performances of the alternatives were very different for a g_j, such information would be very useful for the selection.

- A method based on these concepts is the *standard deviation* method (SDM).
- It consider the variability or uncertainty of the values $g_j(a_i)=x_{ij}$.
- This variability is quantified through the standard deviation of the performances of the alternatives.
- A criterion with high standard deviation is then considered more important than another with low standard deviation.

- The SDM method considers the normalised performance matrix $[r_{ij}]$ and calculates the standard deviation of the values provided by the alternatives with respect to the generic criterion g_j :

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{m}(r_{ij} - \overline{r_j})^2}{m}} \quad k = 1, \dots, n$$

- The weight of criterion g_j is then calculated as:

$$w_j = \frac{\sigma_j}{\sum_{k=1}^{n}\sigma_k} \quad j = 1, \dots, n$$

- In the [COAT] example, the weights can be calculated from the normalised matrix $[r_{ij}]$:

SD	0.0094	0.0654	0.0202	0.0661	0.1474	0.0595
W	0.0256	0.1777	0.0548	0.1797	0.4006	0.1616

- Applying the TOPSIS steps with the above weights gives:

$A1 \succ A2 \succ C2 \succ C1 \succ B$

	RCi
A1	0.849
A2	0.686
B	0.188
C1	0.503
C2	0.509