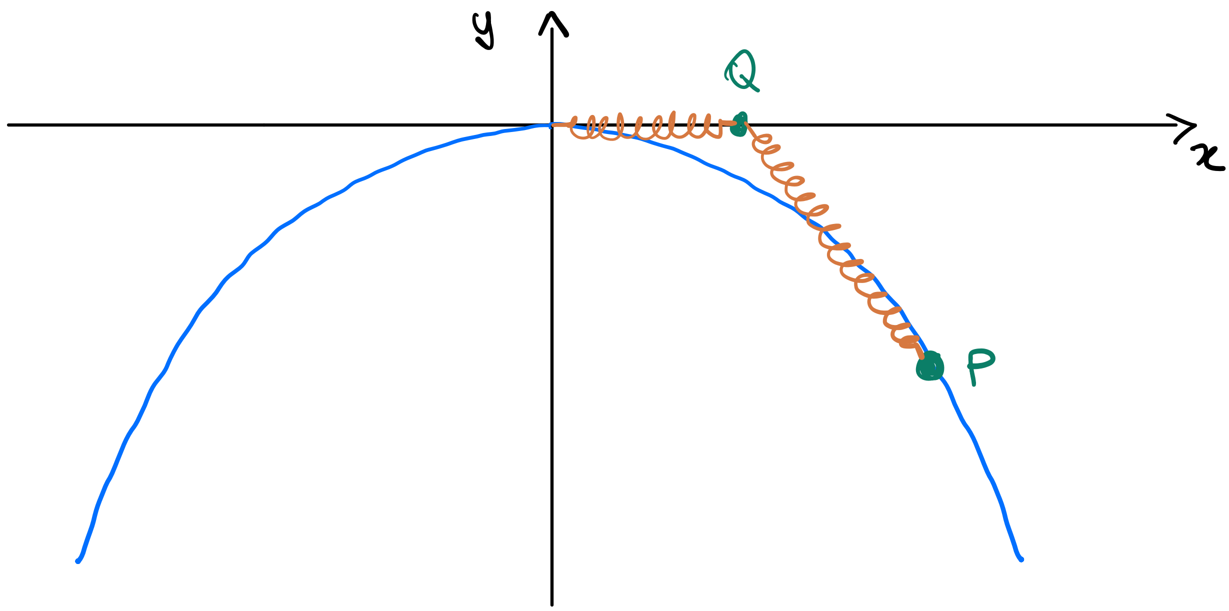

$y = -x^2/a$

$x_P = s \qquad \dot{x}_P = \dot{s}$

$y_P = -s^2/a \qquad \dot{y}_P = -\frac{2s}{a}\dot{s}$

$x_Q = u \qquad \dot{x}_Q = \dot{u}$

$y_Q = 0 \qquad \dot{y}_Q = 0$

1) $$T = \frac{m}{2}\left(\dot{s}^2\left(1 + \frac{4s^2}{a^2}\right) + \dot{u}^2 \right)$$

$$V = \frac{k}{2}u^2 + \frac{k}{2}\left((s-u)^2 + \frac{s^4}{a^2} \right) - mg\frac{s^2}{a}$$

$$L = \frac{m}{2}\left(\dot{s}^2\left(1 + \frac{4s^2}{a^2}\right) + \dot{u}^2 \right) - \frac{k}{2}u^2 - \frac{k}{2}\left((s-u)^2 + \frac{s^4}{a^2} \right) + mg\frac{s^2}{a}$$

$$Q = \begin{pmatrix} m(1+4s^2/a^2) & 0 \\ 0 & m \end{pmatrix}$$

2) Eq. Lagr. :

$$\frac{d}{dt}\frac{\partial L}{\partial \dot s} = \frac{d}{dt}\left[m\dot s\left(1+\frac{4s^2}{a^2}\right)\right] = m\ddot s\left(1+\frac{4s^2}{a^2}\right) + \frac{8ms\dot s^2}{a^2}$$

$$\frac{\partial L}{\partial s} = \frac{4}{a^2}ms\dot s^2 - k(s-u) - \frac{2k}{a^2}s^3 + 2mg\frac{s}{a}$$

$$\ddot s\left(1+\frac{4s^2}{a^2}\right) = -\frac{4}{a^2}s\dot s^2 - \frac{k}{m}(s-u) - \frac{2k}{a^2 m}s^3 + 2g\,s/a$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot u} = m\ddot u$$

$$\frac{\partial L}{\partial u} = -ku - k(u-s) \qquad\qquad \ddot u = -\frac{2k}{m}u + \frac{k}{m}s$$

3) $u, s \mapsto -u, -s$ simm. discreta $\rightarrow$ non posso applicare il teorema di Noether.

4) $$V = \frac{k}{2}u^2 + \frac{k}{2}\left((s-u)^2 + \frac{s^4}{a^2}\right) - mg\frac{s^2}{a}$$

$$= ku^2 + \left(\frac{k}{2} - \frac{mg}{a}\right)s^2 + \frac{k}{2a^2}s^4 - ksu$$

$$\begin{cases} \partial_s V = 2\left(\frac{k}{2} - \frac{mg}{a}\right)s + \frac{2k}{a^2}s^3 - ku \\ \partial_u V = 2ku - ks \end{cases}$$

$(\partial_s V, \partial_u V) = (0,0)$

$$u = s/2, \quad \left[\left(\frac{k}{2} - \frac{2mg}{a}\right) + \frac{2k}{a^2}s^2\right]s = 0$$

cioè $$\frac{2k}{a^2}s\left[s^2 - \left(\frac{mga}{k} - \frac{a^2}{4}\right)\right]$$

$s = 0 \qquad u = 0$

$s = \pm s_* \qquad u = \pm \frac{s_*}{2} \qquad$ con $\qquad s_*^2 = \frac{a^2}{4}\left(\frac{4mg}{ka} - 1\right)$

ESISTONO se $\frac{4mg}{ka} > 1$

$$\partial^2 V = \begin{pmatrix} k - \frac{2mg}{a} + 6\frac{ks^2}{a^2} & -k \\ -k & 2k \end{pmatrix}$$

$$\partial^2 V(0,0) = \begin{pmatrix} k - \frac{2mg}{a} & -k \\ -k & 2k \end{pmatrix} \qquad Tr = \frac{k}{2}\left(6 - \frac{4mg}{ka}\right) \qquad det = k^2\left(1 - \frac{4mg}{ka}\right)$$

se $\frac{4mg}{ka} < 1$ $\quad det > 0$ e $Tr > 0 \rightarrow$ STAB.

altrim. $det < 0 \Rightarrow$ un autoval. $< 0 \rightarrow$ instab.

$$\partial^2 V\left(\pm s_*, \pm \frac{s_*}{2}\right) = \begin{pmatrix} k - \frac{2mg}{a} + 3\cancel{6}\frac{k}{\cancel{a^2}}\,\frac{\cancel{a^2}}{\cancel{4}\,2}\left(\frac{4mg}{ka} - 1\right) & -k \\ -k & 2k \end{pmatrix}$$

$$= \begin{pmatrix} 10\frac{mg}{a} - 2k & -k \\ -k & 2k \end{pmatrix} \qquad Tr = 10\frac{mg}{a} > 0$$

$$det = 5 \cdot \frac{4mgk}{a} - 4k^2 - k^2 = 5k^2\left(\frac{4mg}{ka} - 1\right) > 0$$

quando esistono $\rightarrow$ STAB. quando $\exists$

5) Si ponga $\frac{4mg}{ka} = \frac{2}{15}$, cioè $\boxed{a = 30 \frac{mg}{k}}$

e si trovino freq. piccole oscillazioni attorno pto equil. stab.

$$A = Q(0,0) = m\mathbb{1}$$

$$B = \begin{pmatrix} k - \frac{2mg}{a} & -k \\ -k & 2k \end{pmatrix} = \begin{pmatrix} \frac{14}{15}k & -k \\ -k & 2k \end{pmatrix}$$

$$\det(B - \lambda A) = \det \begin{pmatrix} \frac{14}{15}k - \lambda m & -k \\ -k & 2k - \lambda m \end{pmatrix} =$$

$$= m^2 \det \begin{pmatrix} \frac{14}{15}\frac{k}{m} - \lambda & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m} - \lambda \end{pmatrix} =$$

$\frac{28}{15} - 1$

$$= m^2 \left[\left(\frac{14}{15}\frac{k}{m} - \lambda\right)\left(\frac{2k}{m} - \lambda\right) - \frac{k^2}{m^2} \right] = 0$$

$$\Longleftrightarrow \quad \lambda^2 - \frac{44}{15}\frac{k}{m}\lambda + \frac{13}{15}\frac{k^2}{m^2}$$

$15 \times 13 = 45 + 15$

$$\frac{\Delta}{4} = \frac{4 \cdot 121}{(15)^2} - \frac{13}{15} = \frac{4 \cdot 121 - 13 \cdot 15}{(15)^2} =$$

$484 - 195 = 289$

$$= \frac{484 - 195}{(15)^2} = \frac{289}{(15)^2} = \left(\frac{17}{15}\right)^2$$

$$\lambda_{1,2} = \left(\frac{22}{15} \pm \frac{17}{15} \right)\frac{k}{m} = \begin{cases} \frac{13}{5}\frac{k}{m} & = \omega_1^2 \\ \frac{k}{3m} & = \omega_2^2 \end{cases}$$

$$\frac{14}{15} - \frac{39}{15} = -\frac{25}{15} = -\frac{5}{3}$$

6) Soluzioni eq. moto linearizzate.

$$0 = (B - \lambda_1 A)\,\bar{u}_1 = m \begin{pmatrix} \frac{14}{15}\frac{k}{m} - \frac{13}{5}\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m} - \frac{13}{5}\frac{k}{m} \end{pmatrix} \bar{u}_1$$

$$\hookrightarrow \begin{pmatrix} -\frac{5}{3} & -1 \\ -1 & -\frac{3}{5} \end{pmatrix} \bar{u}_1 = 0 \iff \bar{u}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$0 = (B - \lambda_2 A)\,\bar{u}_2 = m \begin{pmatrix} \frac{14}{15}\frac{k}{m} - \frac{1}{3}\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m} - \frac{k}{3m} \end{pmatrix} \bar{u}_2$$

$$\hookrightarrow \begin{pmatrix} \frac{3}{5} & -1 \\ -1 & \frac{5}{3} \end{pmatrix} \bar{u}_2 = 0 \iff \bar{u}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} s(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + A_1 \cos\left(\sqrt{\frac{13k}{5m}}\, t + \varphi_1 \right) \begin{pmatrix} 3 \\ -5 \end{pmatrix} +$$

$$+ A_2 \cos\left(\sqrt{\frac{k}{3m}}\, t + \varphi_2 \right) \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

7) $\omega_1 = \sqrt{\frac{13}{5}\frac{k}{m}}$

$\omega_2 = \sqrt{\frac{k}{3m}}$

$\frac{\omega_1}{\omega_2} = \sqrt{\frac{39}{5}}$ irrazionale

$\Downarrow$

moto NON è periodico

(è "quasi-periodico").

8) Da $\dot{x}_P$ e $\dot{x}_Q$ abbiamo $\omega^2 s^2$ e $\omega^2 u^2$

$$\to T_{new} = \frac{m}{2}\left(1 + \frac{4s^2}{a^2}\right)\dot{s}^2 + \frac{m}{2}\dot{u}^2 + \underline{\frac{m}{2}\omega^2(s^2+u^2)}$$

$$= T - \delta V_{eff}$$

parte centripeta in L

cioè $L_{new} = T - V_{eff}$

con $V_{eff} = V + \delta V_{eff} = V - \frac{m}{2}\omega^2(s^2+u^2)$

$\partial_s V_{eff} = \partial_s V - m\omega^2 s$

$\partial_u V_{eff} = \partial_u V - m\omega^2 u$

Ponendo $s=u$ $\quad 0 = (2k - m\omega^2)s - ks = (k - m\omega^2)s = 0$

$\Rightarrow \omega^2 = \frac{k}{m} \leadsto$ messo in prima eq. $s = u = \pm\sqrt{\frac{a(ak+2mg)}{2k}}$

ESERCIZIO 1

3) $n=3 \rightsquigarrow \underline{6}$ eq. diff. al 1° ordine.

$\underset{\sim}{2}$ costanti del moto riducono le equaz da 6 a

$$\underline{6} - \underset{\sim}{2} = 4$$

4) L inv. per rotazioni attorno asse 1, cioè

$$\bar{\varphi}(q,\alpha) = \begin{pmatrix} q_1 \\ q_2 \cos\alpha - q_3 \operatorname{sen}\alpha \\ q_2 \operatorname{sen}\alpha + q_3 \cos\alpha \end{pmatrix}$$

$$\frac{\partial\bar{\varphi}}{\partial\alpha}(q,\alpha)\Big|_{\alpha=0} = \begin{pmatrix} 0 \\ -q_2 \operatorname{sen}\alpha - q_3 \cos\alpha \\ q_2 \cos\alpha - q_3 \operatorname{sen}\alpha \end{pmatrix}\Bigg|_{\alpha=0} = \begin{pmatrix} 0 \\ -q_3 \\ q_2 \end{pmatrix}$$

$$P = \frac{\partial\bar{\varphi}}{\partial\alpha}(q,0)\cdot \bar{p} \qquad \text{con } p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i$$

$$= -q_3\dot{q}_2 + q_2\dot{q}_3 = M_1 \quad //$$

6) $L = T - V$ con $V = -\frac{\alpha^2}{2}(q_1^2+q_2^2)$

$$H = \bar p \cdot \dot{\bar q} - L\Big|_{\dot q = p} = T+V = \frac{p_1^2+p_2^2+p_3^2}{2} - \frac{\alpha^2}{2}(q_1^2+q_2^2)$$

$$\{M_1, H\} = \left\{ q_2p_3 - q_3p_2 \,,\, \frac{1}{2}\sum_{i=1}^{3} p_i^2 - \frac{\alpha^2}{2}\sum_{i=1}^{2} q_i^2 \right\} =$$

$M_1 = q_2p_3 - q_3p_2$ (in coord p,q)

$$= p_3\cdot p_2 - p_2 \cdot p_3 - \left(-q_3(-\alpha^2 q_2) + q_2(-\alpha^2 q_3)\right) = 0$$

applico def.

7) Con P.P. :

$\{p_r, q_s\} = 0$

$\{p_r, p_s\} = 0$

$\{q_r, q_s\} = 0$

se $r \neq s$, perché p_i e q_i dip. da $\tilde p_i$ e $\tilde q_i$, ma non da $\tilde p_j, \tilde q_j$ con $j \neq i$ e quindi nei prodotti di derivate ho sempre un fattore zero.

$\{q_i, q_i\} = 0$

$\{p_i, p_i\} = 0$ per antisimm. di P.P.

$$\{q_1, p_1\} = \{\tilde q_1, \tilde p_1\} = 1$$

$$\{q_2, p_2\} = \left(\sqrt{\frac{2\tilde p}{\alpha}}\,\operatorname{senh}\tilde q_2\right)\left(-\sqrt{\frac{\alpha}{2\tilde p_2}}\,\operatorname{senh}\tilde q_2\right) - \left(\sqrt{\frac{1}{2\alpha\tilde p_2}}\,\cosh\tilde q_2\right)\left(-\sqrt{2\alpha\tilde p_2}\,\cosh\tilde p_2\right) = -\operatorname{senh}\tilde q_2^2 + \cosh\tilde p_2^2 = 1$$

$\{q_3, p_3\}$ analogo a ↑ $= 1$. //

Con Id. di Lie :

$$p_1 \, dq_1 + p_2 \, dq_2 + p_3 \, dq_3 =$$

$$= \tilde{p}_1 \, d\tilde{q}_1 - \sqrt{2\alpha \tilde{p}_2} \; \mathrm{senh}\, \tilde{q}_2 \left(\frac{1}{\sqrt{2\alpha \tilde{p}_2}} \cosh \tilde{q}_2 \, d\tilde{p}_2 + \sqrt{\frac{2\tilde{p}_2}{\alpha}} \, \mathrm{senh}\, \tilde{p}_2 \, d\tilde{q}_2 \right)$$

$$- \sqrt{2\alpha \tilde{p}_3} \; \mathrm{senh}\, \tilde{q}_3 \left(\frac{1}{\sqrt{2\alpha \tilde{p}_3}} \cosh \tilde{q}_3 \, d\tilde{p}_3 + \sqrt{\frac{2\tilde{p}_3}{\alpha}} \, \mathrm{senh}\, \tilde{p}_3 \, d\tilde{q}_3 \right)$$

$$= \tilde{p}_1 \, d\tilde{q}_1 - \mathrm{senh}\, \tilde{q}_2 \cosh \tilde{q}_2 \, d\tilde{p}_2 - 2\tilde{p}_2 \, \mathrm{senh}^2 \tilde{q}_2 \, d\tilde{q}_2$$

$$- \mathrm{senh}\, \tilde{q}_3 \cosh \tilde{q}_3 \, d\tilde{p}_3 - 2\tilde{p}_3 \, \mathrm{senh}^2 \tilde{q}_3 \, d\tilde{q}_3$$

$$= \tilde{p}_1 \, d\tilde{q}_1 + \tilde{p}_2 \, d\tilde{q}_2 + \tilde{p}_3 \, d\tilde{q}_3 + dF$$

$$ch^2 - sh^2 = 1$$

con $F = - \tilde{p}_2 \, \mathrm{senh}\, \tilde{q}_2 \cosh \tilde{q}_2$

Infatti $\dfrac{\partial F}{\partial \tilde{q}_2} = - \tilde{p}_2 \left(\mathrm{senh}^2 \tilde{q}_2 + \cosh^2 \tilde{q}_2 \right) =$

$$= - \tilde{p}_2 \left(2 \, \mathrm{senh}^2 \tilde{q}_2 + 1 \right)$$

$$\frac{\partial F}{\partial \tilde{p}_2} = - \mathrm{senh}\, \tilde{q}_2 \cosh \tilde{q}_2$$

8) $K_0 = 0$ perché trasf. can. indip. da t

$c = 1$ da dim. di 7)

$$\Rightarrow K(\tilde{p}, \tilde{q}) = \tilde{H}(\tilde{p}, \tilde{p}) =$$

$$= \frac{1}{2} \tilde{p}_1^2 + \frac{1}{\cancel{2}}(\cancel{2}\alpha \tilde{p}_2 \operatorname{sech}^2 \tilde{q}_2 + \cancel{2}\alpha \tilde{p}_3 \operatorname{sech}^2 \tilde{q}_2)$$

$$- \frac{\alpha^{\cancel{2}}}{\cancel{2}} \left(\frac{\cancel{2}\tilde{p}_2}{\cancel{\alpha}} \cosh^2 \tilde{q}_2 + \frac{\cancel{2}\tilde{p}_3}{\cancel{\alpha}} \cosh^2 \tilde{q}_3 \right) =$$

$$= \frac{\tilde{p}_1^2}{2} - \alpha \tilde{p}_2 - \alpha \tilde{p}_3$$

$\dot{\tilde{p}}_1 = 0 \rightarrow \tilde{p}_1(t) = \tilde{p}_1^0$

$\dot{\tilde{p}}_2 = 0 \rightarrow \tilde{p}_2(t) = \tilde{p}_2^0$

$\dot{\tilde{p}}_3 = 0 \rightarrow \tilde{p}_3(t) = \tilde{p}_3^0$

$\dot{\tilde{q}}_1 = \tilde{p}_1 \rightarrow \tilde{q}_1(t) = \tilde{p}_1^0 t + \tilde{q}_1^0$

$\dot{\tilde{q}}_2 = -\alpha \rightarrow \tilde{q}_2(t) = -\alpha t + \tilde{q}_2^0$

$\dot{\tilde{q}}_3 = -\alpha \rightarrow \tilde{q}_3(t) = -\alpha t + \tilde{q}_3^0$

$p_1(t) = \tilde{p}_1^0$ $\qquad\qquad$ $\tilde{q}_1(t) = \tilde{p}_1^0 t + \tilde{q}_1^0$ $\quad (*)$

$\Rightarrow p_2(t) = -\sqrt{2\alpha \tilde{p}_2^0} \operatorname{senh}(-\alpha t + \tilde{q}_2^0)$ $\quad q_2(t) = \sqrt{\frac{2\tilde{p}_2^0}{\alpha}} \cosh(-\alpha t + \tilde{q}_2^0)$

$p_3(t) = -\sqrt{2\alpha \tilde{p}_3^0} \operatorname{senh}(-\alpha t + \tilde{q}_3^0)$ $\quad q_3(t) = \sqrt{\frac{2\tilde{p}_3^0}{\alpha}} \cosh(-\alpha t + \tilde{q}_3^0)$

9) Espandere (*) per $t \ll 1$ $\to$ termine prop. a t cioè δp e δq.

Oppure

$$\delta p_1 = t\{p, H\} = -t\frac{\partial H}{\partial q_1} = t\alpha^2 q_1$$

$$\delta q_1 = t\{q, H\} = t\frac{\partial H}{\partial p_1} = t\, p_1$$

10) Flusso Ham. di H coniuga H in K=0

$$\Rightarrow \quad K_0(p^0, q^0) = -\tilde{H}(p^0, q^0)$$

parte dip. da t si cancella grazie a $ch^2 - sh^2 = 1$

$p(t), q(t)$ in $t=0$

$$\tilde{H}(p^0, q^0) = H(\overset{(*)}{p, q}) =$$

$$= \frac{(\tilde{p}_1^0)^2}{2} - \alpha\, \tilde{p}_2^0 - \alpha\, \tilde{p}_3^0 = H(p^0, q^0)$$

$$\Rightarrow K_0(p^0, q^0) = -H(p^0, q^0).$$