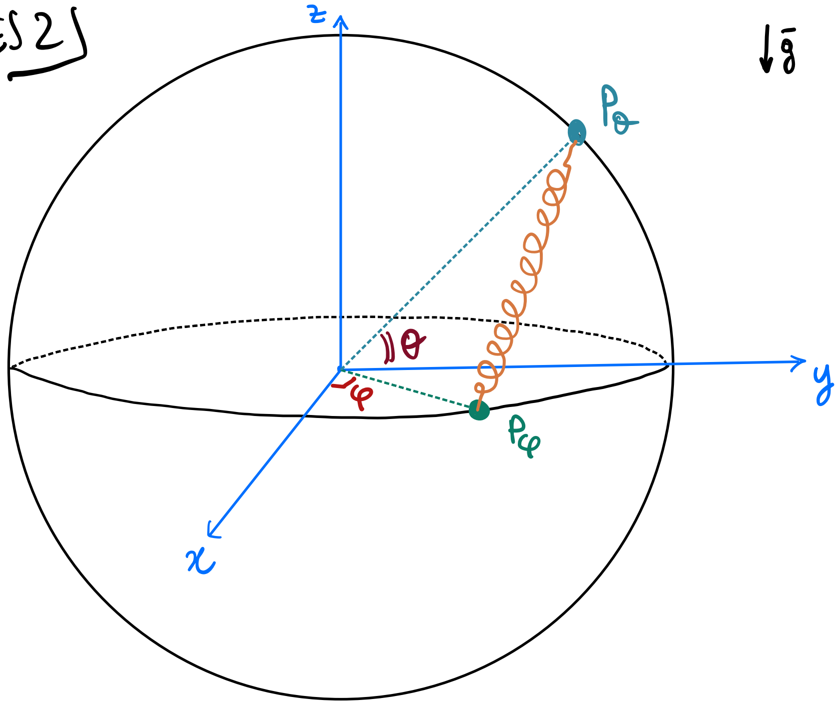
ES 2)



$x_1 = R\cos\varphi$

$y_1 = R\,\mathrm{sen}\,\varphi$

$z_1 = 0$

$\varphi_1 \equiv \varphi$

$\varphi_2 \equiv \theta$

$x_2 = 0$

$y_2 = R\cos\theta$

$z_2 = R\,\mathrm{sen}\,\theta$

$$\Delta x^2 = R^2\left[\cos^2\varphi + \mathrm{sen}^2\theta + (\mathrm{sen}\,\varphi - \cos\theta)^2\right] =$$

$$= 2R^2\left[1 - \mathrm{sen}\,\varphi\cos\theta\right]$$

1)

$$T = \frac{m}{2}R^2\left(\dot{\varphi}^2 + \dot{\theta}^2\right)$$

$$\mathcal{A} = \begin{pmatrix} mR^2 & \\ & mR^2 \end{pmatrix}$$

$$V = mgR\,\mathrm{sen}\,\theta + kR^2\left[1 - \mathrm{sen}\,\varphi\cos\theta\right]$$

$$L = \frac{mR^2}{2}(\dot{\varphi}^2+\dot{\theta}^2) - mgR\,\text{sen}\,\theta + KR^2\,\text{sen}\,\varphi\cos\theta \quad \cancel{-KR^2}$$

2) $$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = mR^2\ddot{\varphi} - KR^2\cos\varphi\cos\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = mR^2\ddot{\theta} + mgR\cos\theta + KR^2\,\text{sen}\,\varphi\,\text{sen}\,\theta = 0$$

$$\begin{cases} \ddot{\varphi} = \frac{K}{m}\cos\varphi\cos\theta \\ \ddot{\theta} = -\frac{K}{m}\,\text{sen}\,\varphi\,\text{sen}\,\theta - \frac{g}{R}\cos\theta \end{cases}$$

3)

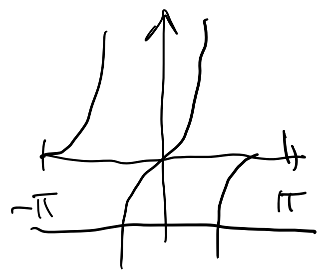
$$V = mgR\,\mathrm{sen}\,\theta + KR^2\left[1 - \mathrm{sen}\,\varphi\,\cos\theta\right]$$

$$\partial\varphi_2 : \quad mgR\cos\theta + KR^2\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta$$

$$\partial\varphi_1 : \quad -KR^2\cos\varphi\,\cos\theta$$

$$\varphi = \pm\pi/2 \longrightarrow \operatorname{tg}\theta = \mp\frac{mg}{KR}$$

$$\theta = \pm\pi/2 \hookrightarrow \varphi = 0, \pi$$



2 angoli: $\pm\theta^*$ e $\pi\pm\theta^*$ per segno

con $\operatorname{tg}\theta^* = \frac{mg}{KR}$, $0<\theta^*<\frac{\pi}{2}$

$$\partial^2 V = \begin{pmatrix} KR^2\,\mathrm{sen}\,\varphi\,\cos\theta & KR^2\cos\varphi\,\mathrm{sen}\,\theta \\ KR^2\cos\varphi\,\mathrm{sen}\,\theta & -mgR\,\mathrm{sen}\,\theta + KR^2\,\mathrm{sen}\,\varphi\,\cos\theta \end{pmatrix}$$

$$\partial^2 V\left(+\frac{\pi}{2}, \overbrace{\operatorname{arctg}\left(-\frac{mg}{KR}\right)}^{-\theta^*,\ \pi-\theta^*}\right) = \begin{pmatrix} \overset{-\theta^*}{\underset{\pi-\theta^*}{\pm}} KR^2\cos\theta^* & 0 \\ 0 & \overset{-\theta^*}{\underset{\pi-\theta^*}{\pm}} KR^2\cos\theta^*\left(1+\left(\frac{mg}{KR}\right)^2\right) \end{pmatrix}$$

$\left(\frac{\pi}{2}, -\theta^*\right) \rightarrow$ STAB.

$\left(\frac{\pi}{2}, \pi-\theta^*\right) \rightarrow$ INSTAB.

$$\partial^2 V\left(-\frac{\pi}{2}, \overbrace{\operatorname{arctg}\left(\frac{mg}{KR}\right)}^{\theta^*,\ \pi+\theta^*}\right) = \begin{pmatrix} \overset{\theta^*}{\underset{\pi+\theta^*}{\mp}} KR^2\cos\theta^* & 0 \\ 0 & \mp KR^2\cos\theta^*\left(1+\left(\frac{mg}{KR}\right)^2\right) \end{pmatrix}$$

$\left(-\frac{\pi}{2}, \theta^*\right) \rightarrow$ INSTAB.

$\left(-\frac{\pi}{2}, \pi+\theta^*\right) \rightarrow$ STAB.

$$\partial^2 V\left(0, \frac{\pi}{2}\right) = \begin{pmatrix} 0 & KR^2 \\ KR^2 & -mgR \end{pmatrix} \quad \det < 0 \quad \text{INSTAB.}$$

lo stesso per $\left(0, -\frac{\pi}{2}\right), \left(\pi, \frac{\pi}{2}\right), \left(\pi, -\frac{\pi}{2}\right)$.

4) Entrambi i stabili hanno stessa Hessiana. Per entrambi:

$$B = kR^2\cos\theta^* \begin{pmatrix} 1 & 0 \\ 0 & 1+\left(\frac{mg}{KR}\right)^2 \end{pmatrix} \quad A = mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(B-\lambda A) = mR^2 \begin{pmatrix} \frac{k}{m}\cos\theta^* - \lambda & \\ & \frac{k}{m}\left(1+\left(\frac{mg}{KR}\right)^2\right)\cos\theta^* - \lambda \end{pmatrix}$$

$$\det(B-\lambda A)=0 \to \begin{cases} \lambda_1 = \frac{k}{m}\cos\theta^* \\ \lambda_2 = \frac{k}{m}\cos\theta^*\left(1+\left(\frac{mg}{KR}\right)^2\right) \end{cases}$$

soluz. attorno ai 2 pti di equil stab.

5)
$$\varphi(t) = \frac{\pi}{2} + A_1\cos(\sqrt{\lambda_1}\, t + c_1) \qquad \varphi(t) = -\frac{\pi}{2} + A_1\cos(\sqrt{\lambda_1}\, t + c_1)$$

$$\theta(t) = -\theta^* + A_2\cos(\sqrt{\lambda_2}\, t + c_2) \qquad \theta(t) = \pi + \theta^* + A_2\cos(\sqrt{\lambda_2}\, t + c_2)$$

6) Quando mg/KR è tc.

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{\lambda_2}{\lambda_1}} = \sqrt{1+\left(\frac{mg}{KR}\right)^2} \in \mathbb{Q}$$

per es. $\frac{mR}{KR} = \frac{3}{4}$

$$\to \frac{\omega_2}{\omega_1} = \sqrt{1+\frac{9}{16}} = \frac{5}{4} \in \mathbb{Q}.$$

7) $x_1 = R\cos\varphi \qquad x_2 = R\cos\theta\cos\omega t$

$y_1 = R\,\mathrm{sen}\,\varphi \qquad y_2 = R\cos\theta\,\mathrm{sen}\,\omega t$

$z_1 = 0 \qquad z_2 = R\,\mathrm{sen}\,\theta$

$$\dot{x}_2 = -R\dot{\theta}\,\mathrm{sen}\,\theta\cos\omega t - R\omega\cos\theta\,\mathrm{sen}\,\omega t$$

$$\dot{y}_2 = -R\dot{\theta}\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\omega t + R\omega\cos\theta\cos\omega t$$

$$\dot{z}_2 = R\dot{\theta}\cos\theta$$

$$T_{new} = \frac{m}{2}R^2\left(\dot{\varphi}^2 + \overbrace{\dot{\theta}^2 + \omega^2\cos^2\theta}\right) = T + \frac{m}{2}R^2\omega^2\cos^2\theta$$

$$V_{new} = mgR\,\mathrm{sen}\,\theta + \frac{KR^2}{2}\left[(\cos\varphi - \cos\theta\,\mathrm{sen}\,\omega t)^2 + (\mathrm{sen}\,\varphi - \cos\theta\cos\omega t)^2 + \mathrm{sen}^2\theta\right]$$

$$= mgR\,\mathrm{sen}\,\theta + KR^2\left(1 - \cos\theta\,\mathrm{sen}(\varphi + \omega t)\right)$$

$$L = T_{new} - V_{new} = T - V_{eff}$$

L'energia non si conserva perché $\frac{\partial L}{\partial t} \neq 0$

con

$$V_{eff} = mgR\,\mathrm{sen}\,\theta - KR^2\,\mathrm{sen}(\varphi + \omega t)\cos\theta - \frac{m\omega^2}{2}R^2\cos^2\theta$$

8) $\varphi' = \varphi + \omega t$

$\dot{\varphi}' = \dot{\varphi} + \omega$

DERIVATA TOT.

$$L' = \frac{m}{2}R^2\left(\dot{\varphi}'^2 + \cancel{2\omega\dot{\varphi}'} + \cancel{\omega^2} + \dot{\theta}^2\right) - mgR\,\mathrm{sen}\,\theta + KR^2\,\mathrm{sen}\,\varphi'\cos\theta + \frac{m\omega^2}{2}R^2\cos^2\theta$$

9) $V_{eff} = mgR\,\text{sen}\,\theta - KR^2\,\text{sen}\,\varphi'\cos\theta - \frac{m\omega^2}{2}R^2\cos^2\theta$

$\partial_{\varphi'} V_{eff} = -KR^2\cos\varphi'\cos\theta \quad \leftarrow \varphi'=0 \Rightarrow \cos\theta=0 \quad \theta=\pm\frac{\pi}{2}$

$\partial_\theta V_{eff} = mgR\cos\theta + KR^2\,\text{sen}\,\varphi'\,\text{sen}\,\theta + m\omega^2R^2\cos\theta\,\text{sen}\,\theta$

$\varphi'=0 \rightarrow \omega^2 = -\frac{mgR\cos\theta}{mR^2\cos\theta\,\text{sen}\,\theta} = -\frac{g}{R\,\text{sen}\,\theta} \underset{\uparrow}{=} \frac{g}{R}$

$\theta=-\frac{\pi}{2}$ (unico che dà pto stab., cioè $\omega^2>0$).

10) $\theta_e = 0 \xrightarrow{(z_{P_0}=0)\Rightarrow \varphi'_e=\pm\pi/2}$ non è soluz. per ω finito.

Vediamo cosa succede per $|\theta_e| \ll 1$ e poi prendiamo lim.

Per $\theta_e \ll$ seconda eq. diventa

$mgR \pm KR^2\theta_e + m\omega^2R^2\theta_e$

$$\rightarrow \omega^2 = \frac{-mgR \mp KR^2\theta_e}{mR^2\theta_e} \quad >0 \text{ solo per } \theta_e<0$$

$$= \frac{\frac{g}{R} \mp \frac{K}{m}(-\theta_e)}{-\theta_e} = \frac{g}{R}\frac{1}{-\theta_e} \mp \frac{K}{m} \xrightarrow[\theta_e\to 0^-]{} \infty$$

Ci vuole velocità angolare ∞ (cioè forza centrifuga infinita) per far rimanere P_0 a $z=0$.

$$\frac{1}{2}\left[\cos^2\theta \quad P_1^2 + P_2^2 \quad + \quad \operatorname{sen}^2\theta\left(Q_1^2 + Q_2^2\right) + \operatorname{sen}\theta\cos\theta\left(P_1Q_2 + P_2Q_1\right)\right]$$

ES 1)

6) $\{q_1, p_1\} = \cos^2\alpha + \sin^2\alpha = 1$

$\{q_2, p_2\}$ //

$\{q_1, q_2\} = \cos\alpha \sin\alpha - \cos\alpha \sin\alpha = 0$

$\{q_1, p_2\} = 0 = \{q_2, p_1\}$ poiché uso $\{\tilde{q}_i, \tilde{p}_i\}$

$\{p_1, p_2\} = -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 0$

7) $G(\tilde{p}, \tilde{q}) = \tilde{p}_1\tilde{p}_2 + \tilde{q}_1\tilde{q}_2$

$$\delta p_1 = -\alpha\tilde{q}_2 \qquad \delta p_2 = -\alpha\tilde{q}_1$$
$$\delta q_1 = +\alpha\tilde{p}_2 \qquad \delta q_2 = \alpha\tilde{p}_1$$

$\Leftarrow$ Espandendo (1) in $\alpha \ll 1$ ($\delta x = x - \tilde{x}$)

$$\delta p_1 = \alpha\{\tilde{p}_1, \tilde{p}_1\tilde{p}_2 + \tilde{q}_1\tilde{q}_2\} = \alpha\tilde{q}_2\{\tilde{p}_1, \tilde{q}_1\} = -\alpha\tilde{q}_2 \quad \checkmark$$

$$\delta q_1 = \alpha\{\tilde{q}_1, \quad '' \quad \} = \alpha\tilde{p}_2\{\tilde{q}_1, \tilde{p}_1\} = \alpha\tilde{p}_2 \quad \checkmark$$

$$\delta p_2 = \alpha\{\tilde{p}_2, \quad '' \quad \} = \alpha\tilde{q}_1\{\tilde{p}_2, \tilde{q}_2\} = -\alpha\tilde{q}_1 \quad \checkmark$$

$$\delta q_2 = \alpha\{\tilde{q}_2, \quad '' \quad \} = \alpha\tilde{p}_1\{\tilde{q}_2, \tilde{p}_2\} = \alpha\tilde{p}_1 \quad \checkmark$$

8) Inverto :

$$p_1 = \frac{\tilde{P}_1}{\cos\alpha} - q_2 \operatorname{tg}\alpha = \frac{\partial F_2}{\partial q_1} \qquad \tilde{q}_1 = \frac{q_1}{\cos\alpha} - \tilde{P}_2 \operatorname{tg}\alpha = \frac{\partial F_2}{\partial \tilde{P}_1}$$

$$p_2 = \frac{\tilde{P}_2}{\cos\alpha} - q_1 \operatorname{tg}\alpha = \frac{\partial F_2}{\partial q_2} \qquad \tilde{q}_2 = \frac{q_2}{\cos\alpha} - \tilde{P}_1 \operatorname{tg}\alpha = \frac{\partial F_2}{\partial \tilde{P}_2}$$

$$p_1 = \dots \rightarrow \quad F_2 = \frac{\tilde{P}_1 q_1}{\cos\alpha} - q_1 q_2 \operatorname{tg}\alpha + f(\tilde{P}_1, \tilde{P}_2, q_2)$$

$$p_2 = \dots \rightarrow \quad \cancel{-q_1 \operatorname{tg}\alpha} + \frac{\partial f}{\partial q_2} = \frac{\tilde{P}_2}{\cos\alpha} \cancel{- q_1 \operatorname{tg}\alpha} \rightarrow f = \frac{q_2 \tilde{P}_2}{\cos\alpha} + g(\tilde{P}_1, \tilde{P}_2)$$

$$F_2 = \frac{\tilde{P}_1 q_1 + \tilde{P}_2 q_2}{\cos\alpha} - q_1 q_2 \operatorname{tg}\alpha + g(\tilde{P}_1, \tilde{P}_2)$$

$$q_1 = \dots \rightarrow \quad -\tilde{P}_2 \operatorname{tg}\alpha = \frac{\partial g}{\partial \tilde{P}_1} \rightarrow g = -\tilde{P}_1 \tilde{P}_2 \operatorname{tg}\alpha + h(\tilde{P}_2)$$

$$q_2 = \dots \rightarrow \quad h = \text{cost.}$$

$$F_2(q, \tilde{P}) = \frac{\tilde{P}_1 q_1 + \tilde{P}_2 q_2}{\cos\alpha} - (\tilde{P}_1 \tilde{P}_2 + q_1 q_2) \operatorname{tg}\alpha$$