Computer Vision and Pattern Recognition

Course ID: 554SM – Fall 2020

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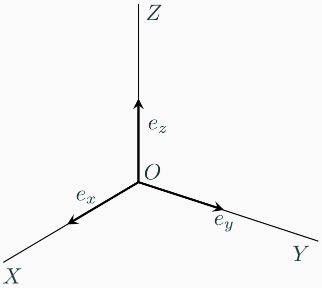
Rigid Geometric Transformations

Reference systems

Cartesian coordinates

Most part of the following material is taken from Tomasi (2019), but any of the recommended textbooks contains an adequate treatment of the subject.

- A *Cartesian reference system* for three-dimensional space consists of a point called the *origin* and three mutually perpendicular oriented lines through the origin called the *axes*.
- The order in which the axes are listed is fixed and is part of the definition. The plane through the origin perpendicular to the first (resp. second, or third) is the *first (resp. second, or third) reference plane)*.
- The positive directions of the axes are typically marked by specifying a point at unit distance from the origin. These points are called the *unit points*.
- A Cartesian reference system is *right-handed* if the smallest rotation that brings the first unit point to the second is counterclockwise when viewed from the third unit point. It is *left-handed* otherwise.



A right-handed Cartesian reference system. The vectors e_x, e_y and e_z specify the unit points.

Cartesian coordinates (cont.)

- The *Cartesian coordinates* of a point in three-dimensional space are the signed distances of the point from the first, second, and third reference plane.
- The sign of the i-th coordinate is positive if the point belongs to the half-space, delimited by the i-th reference plane, containing the i-th unit point.
- It follows that
  - the coordinates of the origin are $t = [0 \;\; 0 \;\; 0]^\top$;
  - the coordinates of the unit points are $e_x = [1 \;\; 0 \;\; 0]^\top$, $e_x = [0 \;\; 1 \;\; 0]^\top$, $e_x = [0 \;\; 0 \;\; 1]^\top$;
  - the vector $p = [x \;\; y \;\; z]^\top$ of coordinates of a generic point P in space can be written as:

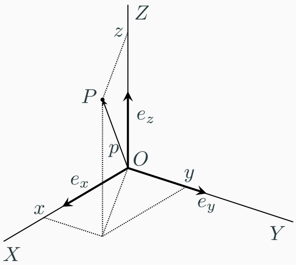
$$p = xe_x + ye_y + ze_z.$$

  - the distance of P from the origin O is (by applying twice the Pythagorean Theorem):

$$d(O, P) = \sqrt{x^2 + y^2 + z^2}$$

and corresponds to the Euclidean norm of the vector p:

$$d(O, P) = \|p\| .$$



Dot product

We recall that the *dot product* (or *scalar product* or *inner product*) between the vectors a and b has the following geometrical meaning:

$$a \cdot b = \|a\| \, \|b\| \cos \vartheta$$

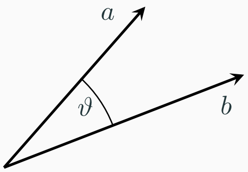
where ϑ is the acute angle between the two arrows that represent a and b. When the vectors are treated as single-column matrices, the dot product is obtained by a matrix product as follows:

$$a \cdot b = a^\top b = b^\top a.$$

Two vectors a and b are said to be *orthogonal* if their dot product is zero:

$$a^\top b = 0.$$

If this is the case, the arrows representing a and b are mutually perpendicular.



Orthogonal projection

Given two vectors a and b, the *orthogonal projection* of a onto b is the vector $p \in \text{span}\{b\}$ closest to a, i.e.

$$p = \underset{p \in \text{span}\{b\}}{\text{argmin}} \|p - a\|$$

It can be shown that p can be obtained by a suitable projection matrix, precisely

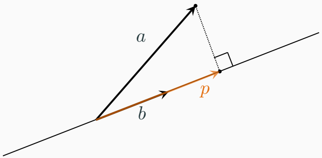
$$p = P_b a,$$

where P_b is the following square, symmetric, rank-1 matrix:

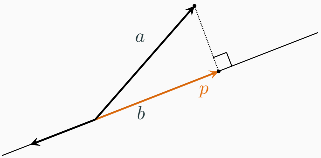
$$P_b = \frac{bb^\top}{b^\top b}.$$

The *signed magnitude* of the orthogonal projection is the scalar

$$\tilde{p} = \frac{b^\top a}{\|b\|} = \|p\| \,\text{sign}(b^\top a).$$



Orthogonal projection of a onto b.
The signed magnitude is positive.



Orthogonal projection of a onto b.
The signed magnitude is negative.

Orthogonal projection (cont.)

For example, the orthogonal projection matrix relative to e_x is

$$P_{e_x} = \frac{e_x e_x^\top}{e_x^\top e_x} = \frac{1}{\|e_x\|^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the orthogonal projection of $p = [x \;\; y \;\; z]^\top$ onto e_x is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

and x is the signed magnitude of the projection. The following property holds true and will be useful when dealing with transformations between different reference systems.

Property

The coordinates of a point in space are the signed magnitudes of the orthogonal projections of the vector of coordinates of the point onto the three unit vectors that define the coordinate axes.

Multiple reference systems

In the following we will employ multiple reference systems. Notation must be introduced to avoid possible ambiguities.

- Reference systems will be identified by natural numbers;
- the number zero is reserved for a privileged system called the *world reference system*;
- a left superscript will denote the reference system that a vector or a transform is expressed in;
- a left superscript of zero can be optionally omitted.

The above rules imply that:

- ${}^2p$ is the vector of the coordinates of the point P in reference frame 2. The same point has world coordinates ${}^0p$ or simply p;
- since the origin T of a reference system has always zero coordinates in that reference system, we have for all i:

$$
{}^i t_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};
$$

Multiple reference systems (cont.)

- if we denote by ${}^q i_q, {}^q j_q, {}^q k_q$ the unit vectors of the q-th reference system, we have for all q

$$\begin{bmatrix} {}^q i_q & {}^q j_q & {}^q k_q \end{bmatrix} = \begin{bmatrix} i_0 & j_0 & k_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Rotations

Rotations

A rotation is a transformation between two Cartesian reference systems of equal origin and handedness (i.e. both right-handed or both left-handed).

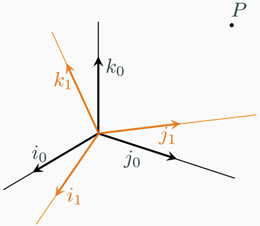
Let S_0 and S_1 be two reference systems of equal origin:

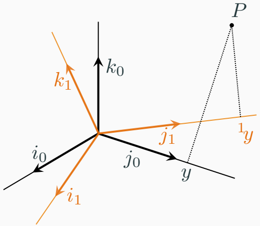
$$t_0 = t_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and let i_0, j_0, k_0 and i_1, j_1, k_1 denote the respective unit points, as expressed with respect to S_0.

Let the point P be represented, in S_0, by the vector of coordinates

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xi_0 + yj_0 + zk_0.$$





Rotations (cont.)

The same triplet of coordinates in S_0 can be obtained as

$$p = {}^1x i_1 + {}^1y j_1 + {}^1z k_1$$

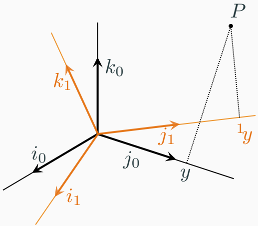
where ${}^1x$, ${}^1y$, ${}^1z$ are the orthogonal projections of p onto i_1, j_1, k_1 respectively:

$${}^1x = i_1^\top p, \quad {}^1y = j_1^\top p, \quad {}^1z = k_1^\top p.$$

By defining ${}^1p = [{}^1x \;\; {}^1y \;\; {}^1z]^\top$, the last three equations can be written in compact form as:

$${}^1p = R_1 p \quad \text{where} \quad R_1 = {}^0R_1 = \begin{bmatrix} i_1^\top \\ j_1^\top \\ k_1^\top \end{bmatrix},$$

where the 3×3 matrix ${}^0R_1$ is a *rotation matrix*.

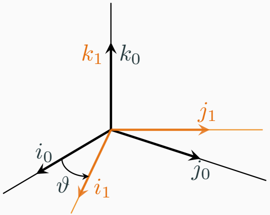


In general, given any two Cartesian reference systems S_a and S_b having the same origin,

$$ {}^{b}p = {}^{a}R_b\, {}^{a}p \quad \text{where} \quad {}^{a}R_b = \begin{bmatrix} {}^{a}i_b^\top \\ {}^{a}j_b^\top \\ {}^{a}k_b^\top \end{bmatrix}. $$

Thus we can recover the vector ${}^{b}p$ of the coordinates of P with respect to S_b, by pre-multiplying by ${}^{a}R_b$ the vector ${}^{a}p$ of the coordinates of P with respect to S_a.

The rows of ${}^{a}R_b$ are the coordinates of the unit points of S_b expressed in the reference S_a.



For example, if S_1 is obtained by rotating S_0 about its third axis of an angle ϑ, the resulting rotation matrix is

$$^0R_1 = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The third row of $^0R_1$ states that the third coordinate remains unchanged.

Rotations (cont.)

Since a rotation is a reversible transformation, any rotation matrix ${}^aR_b$ must have an inverse, i.e. a matrix ${}^bR_a$ such that

$$ {}^ap = {}^bR_a \, {}^bp. $$

By applying to ${}^ap$ the forward (from S_a to S_b) and the backward (from S_b to S_a) transformations we get ${}^ap$ itself:

$$ {}^ap = {}^bR_a \left({}^aR_b \, {}^ap \right) = \underbrace{\left({}^bR_a \, {}^aR_b \right)}_{I} {}^ap. $$

Theorem

The inverse R^{-1} of a rotation matrix R is its transpose:

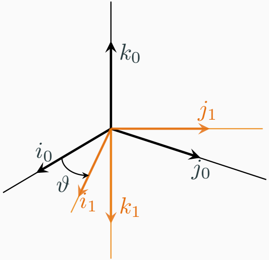
$$ R^\top R = R R^\top = I. \tag{1} $$

Equations (1) guarantee that R is an *orthogonal matrix* (its columns are mutually orthogonal and unitary, as well as its rows). Thus, rotation matrices are orthogonal matrices. Notice that the converse is not true, as stated by the following Lemma.

Lemma

A matrix R is a rotation matrix if and only if $R^\top R = RR^\top = I$ and $\det(R) = 1$.

If a matrix R is orthogonal, its determinant can be either 1 or -1. In the latter case, R is not a rotation matrix because it implies *inversions* (or *mirror flips*) i.e. it changes the direction of some of the axes as in the following figure.



The transformation from S_0 to S_1 in figure cannot be achieved by a rotation: it corresponds to a rotation followed by the inversion of the third axis.

Coordinate transformations

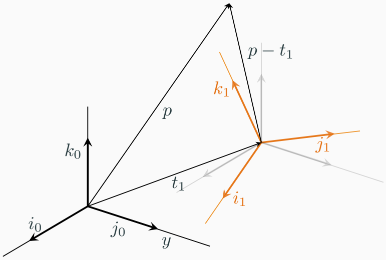
Coordinate transformations

Let S_0 be the world reference system, having origin $t_0 = [0 \;\; 0 \;\; 0]^\top$ and unit vectors $i_0 = [1 \;\; 0 \;\; 0]^\top$, $j_0 = [0 \;\; 1 \;\; 0]^\top$, $k_0 = [0 \;\; 0 \;\; 1]^\top$.

A right-handed reference system S_1 can differ from S_0 by a translation of the origin from t_0 to t_1 and a rotation of the axes from the unit points i_0, j_0, k_0 to the unit points i_1, j_1, k_1.

Take a point with coordinates $p = [x \;\; y \;\; z]^\top$ in S_0. We want to find the coordinates ${}^1p = [{}^1x \;\; {}^1y \;\; {}^1z]^\top$ of the same point w.r.t. S_1. Suppose that S_0 undergo the following transformations:

1. first, the origin of S_0 is translated to t_1. We obtain an intermediate reference frame (represented in gray in the figure) and the new coordinates $p - t_1$ for the point;
2. then, the intermediate frame is rotated by R_1, obtaining the reference frame S_1.



Coordinate transformations (cont.)

Thus, the coordinates in the reference S_1 are expressed by:

$$ {}^1p = R_1(p - t_1). \tag{2} $$

Solving (2) for p yields the inverse transformation:

$$ p = R_1^\top {}^1p + t_1, \tag{3} $$

corresponding to a rotation of $R_1^\top$ followed by a translation of $-t_1$. Eq. (3) can be written as

$$ p = R_1^\top \left({}^1p - (-R_1 t_1)\right) \tag{4} $$

thus, by comparing with (2) we obtain

$$ {}^1R_0 = {}^0R_1^\top \qquad \text{and} \qquad {}^1t_0 = -{}^0R_1\,{}^0t_1 $$

More generally, if

$$ {}^bp = {}^aR_b({}^ap - {}^at_b) $$

then

$$ {}^ap = {}^bR_a({}^bp - {}^bt_a) \quad \text{where} \quad {}^bR_a = {}^aR_b^\top \quad \text{and} \quad {}^bt_a = -{}^aR_b\,{}^at_b. $$

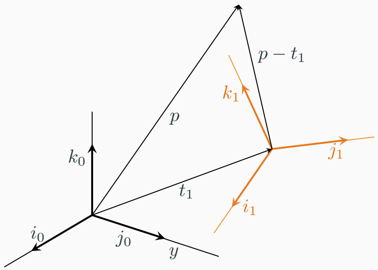
Consider again the coordinate transform equation:

$$ {}^{b}p = {}^{a}R_b({}^{a}p - {}^{a}t_b). $$

A compact representation of the coordinate transform can be obtained by observing that

$$ {}^{b}p = {}^{a}R_b({}^{a}p - {}^{a}t_b) = {}^{a}R_b\,{}^{a}p - {}^{a}R_b\,{}^{a}t_b $$

$$ = [\underbrace{{}^{a}R_b}_{3\times 3} \quad \underbrace{-{}^{a}R_b\,{}^{a}t_b}_{3\times 1}] \begin{bmatrix} {}^{a}p \\ 1 \end{bmatrix}. $$



The previous can be interpreted as first rotating and then translating (the translation vector is thus expressed w.r.t. the intermediate, rotated frame). Clearly we have:

$$ \begin{bmatrix} {}^{b}p \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{a}R_b & -{}^{a}R_b\,{}^{a}t_b \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} {}^{a}p \\ 1 \end{bmatrix} $$

where $0^\top = [0 \quad 0 \quad 0]^\top$.

We now introduce the *homogeneous representation* of a generic vector p as the vector $\tilde{p}$ obtained by adding a fourth unit component

$$\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}.$$

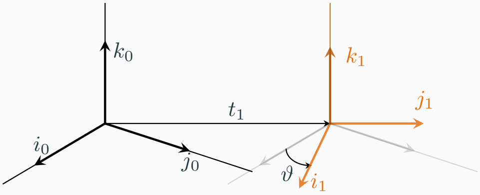
In terms of homogeneous vectors, the coordinate transform equation becomes:

$${}^{b}\tilde{p} = \begin{bmatrix} {}^{a}R_{b} & -{}^{a}R_{b}\,{}^{a}t_{b} \\ 0^{\top} & 1 \end{bmatrix} {}^{a}\tilde{p} = {}^{a}A_{b}\,{}^{a}\tilde{p}.$$

Thus, the rigid transformation can be expressed as a multiplication by a suitable *homogeneous transformation matrix* ${}^{a}A_{b}$ which is determined by ${}^{a}R_{b}$ and ${}^{a}t_{b}$.

Notice that, in general, the homogeneous transformation matrices are not orthogonal:

$$A^{-1} \neq A^{\top}.$$



As an example, suppose that S_1 is obtained by translating the origin of S_0 to $t_1 = [-a\sin\vartheta \quad a\cos\vartheta \quad 0]^\top$ and then performing a rotation of ϑ about the third axis. We have

$${}^0R_1 = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad -{}^0R_1 t_1 = -\begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a\sin\vartheta \\ a\cos\vartheta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$$

thus the homogeneous transformation matrix is $${}^0A_1 = \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 & 0 \\ -\sin\vartheta & \cos\vartheta & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

References

Tomasi, C. (2019). Rigid Geometric Transformations. Accessed: October 7, 2019.

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Rigid Geometric Transformations

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