Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA
PER IL SISTEMA TERRA

LINEAR SYSTEMS

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Green's function

Green's function (GF) is a basic solution to a linear differential equation, a building block that can be used to construct many useful solutions.

If one considers a linear differential equation written as:

$$L(x)u(x)=f(x)$$

where L(x) is a linear, self-adjoint differential operator, u(x) is the unknown function, and f(x) is a known non-homogeneous term, the GF is a solution of:

$$L(x)\underbrace{u(x,s)}_{G(x,s)}=\delta(x-s)$$

Why GF is important?

If such a function G can be found for the operator L, then if we multiply the second equation for the Green's function by f(s), and then perform an integration in the s variable, we obtain:

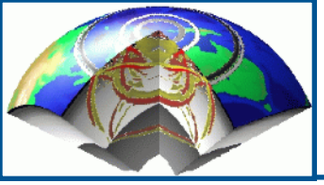
$$\int L(x)G(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x) = Lu(x)$$

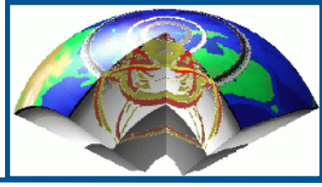
$$L\int G(x,s)f(s)ds = Lu(x)$$

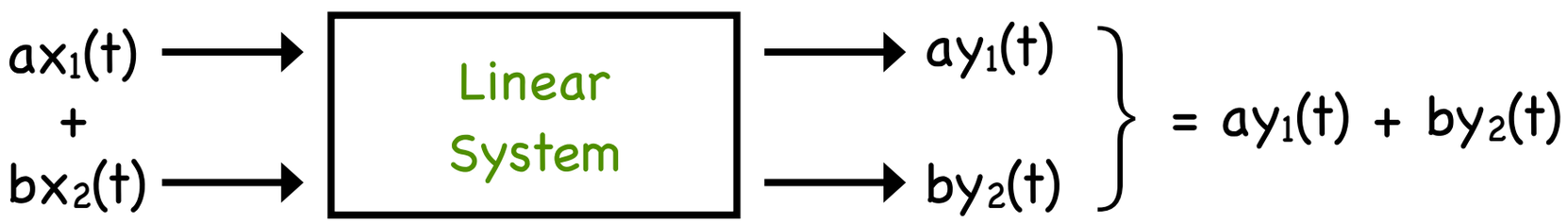
$$\boxed{u(x) = \int G(x,s)f(s)ds}$$

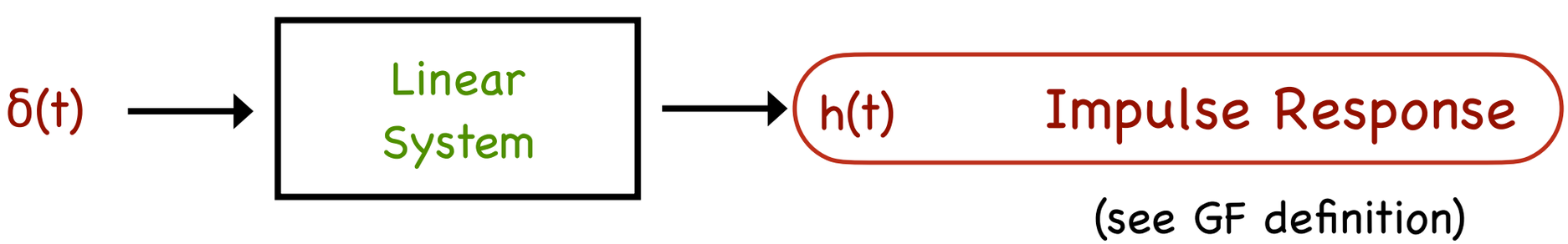
Thus, we can obtain the function u(x) through knowledge of the Green's function, and the source term. This process has resulted from the linearity of the operator L.

Linear (time invariant) Systems





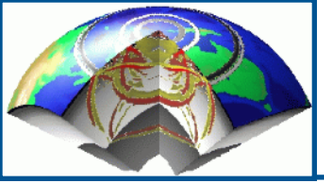




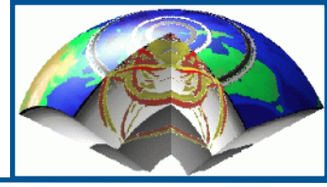
(see GF definition)

Since any input x(t) can be written as:

$$x(t) = \int x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \int x(\tau)h(t-\tau)\,d\tau = x * h$$



Convolution

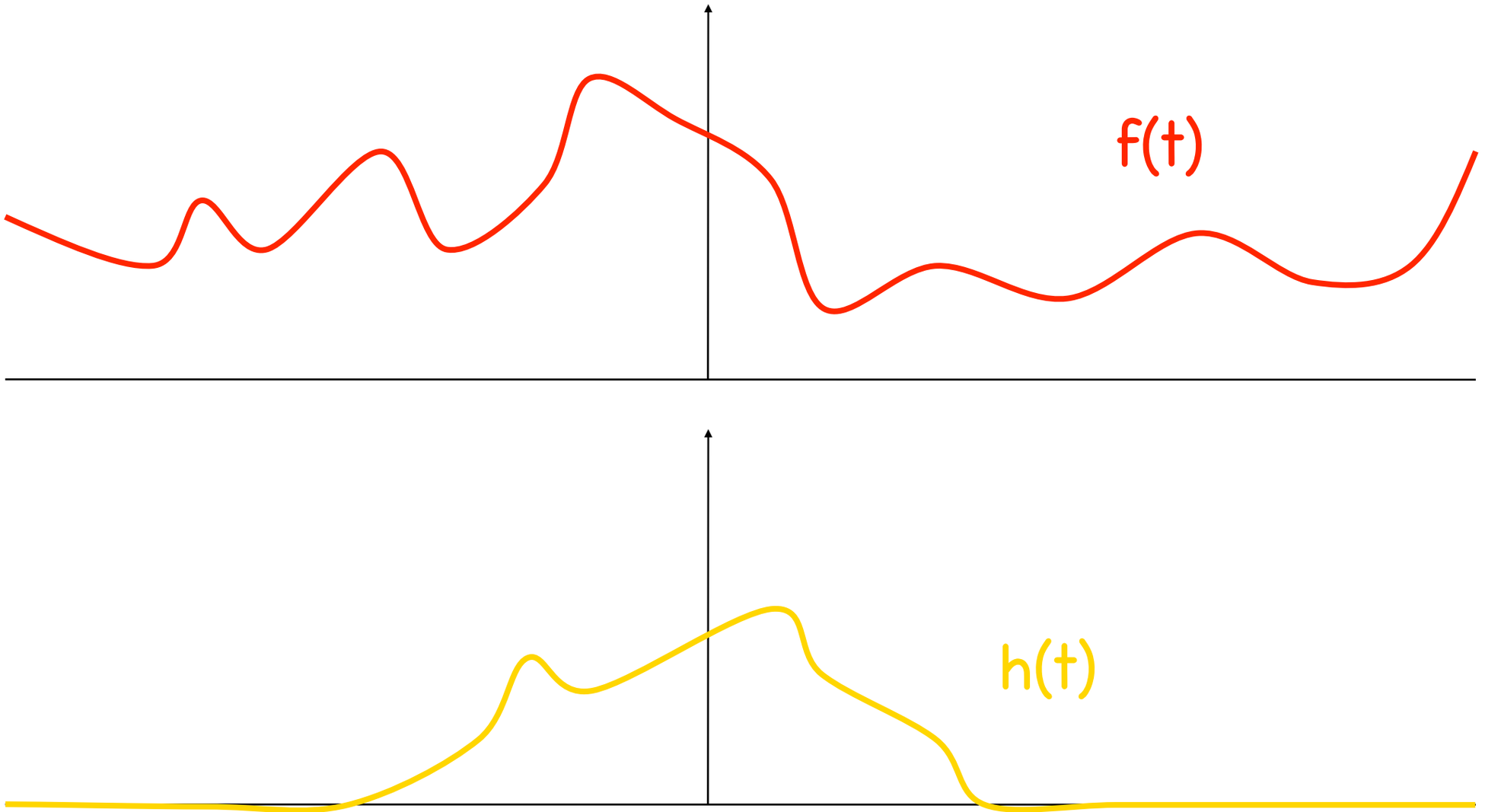


- Definition:

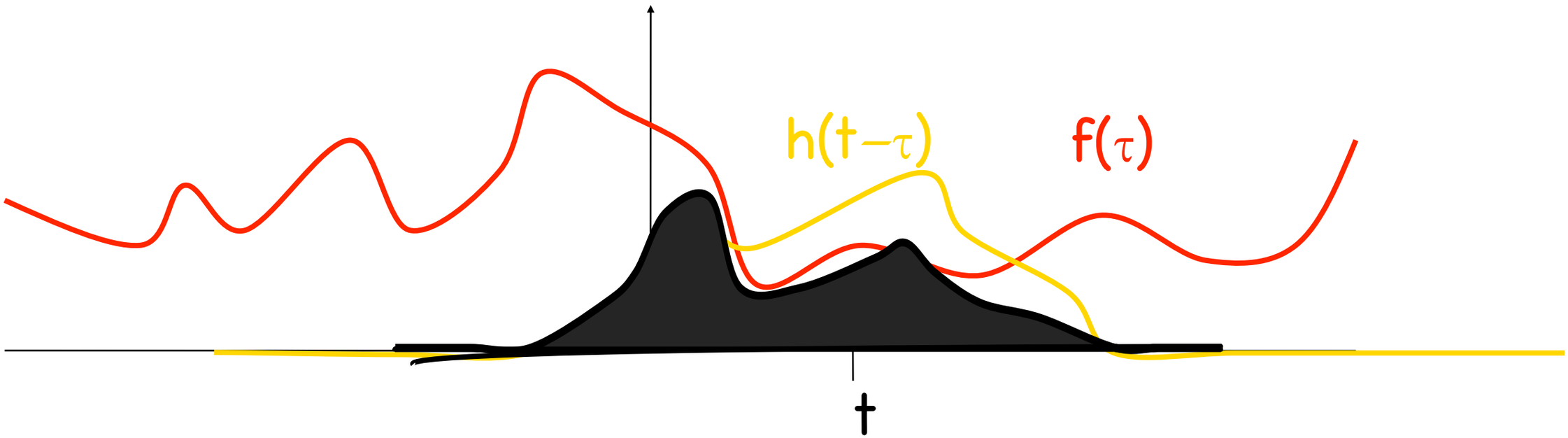
$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Convolution

- Definition: $$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$
- Pictorially



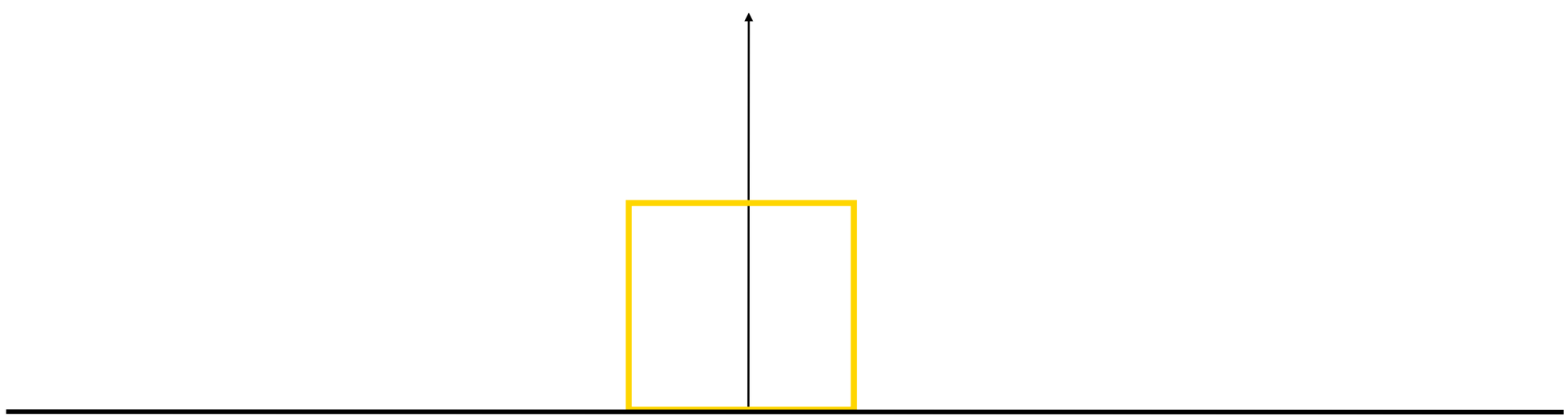
Convolution



Convolution

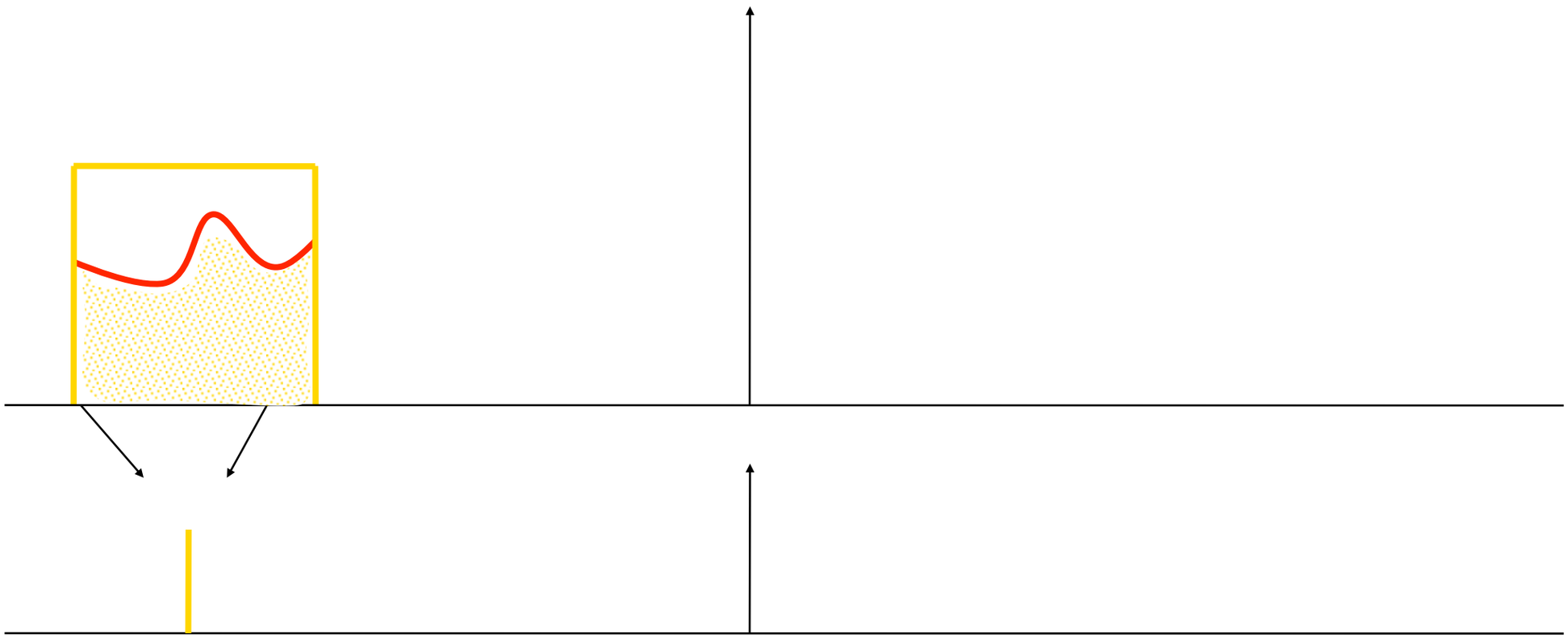
- Consider the boxcar function (box filter):

$$h(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$



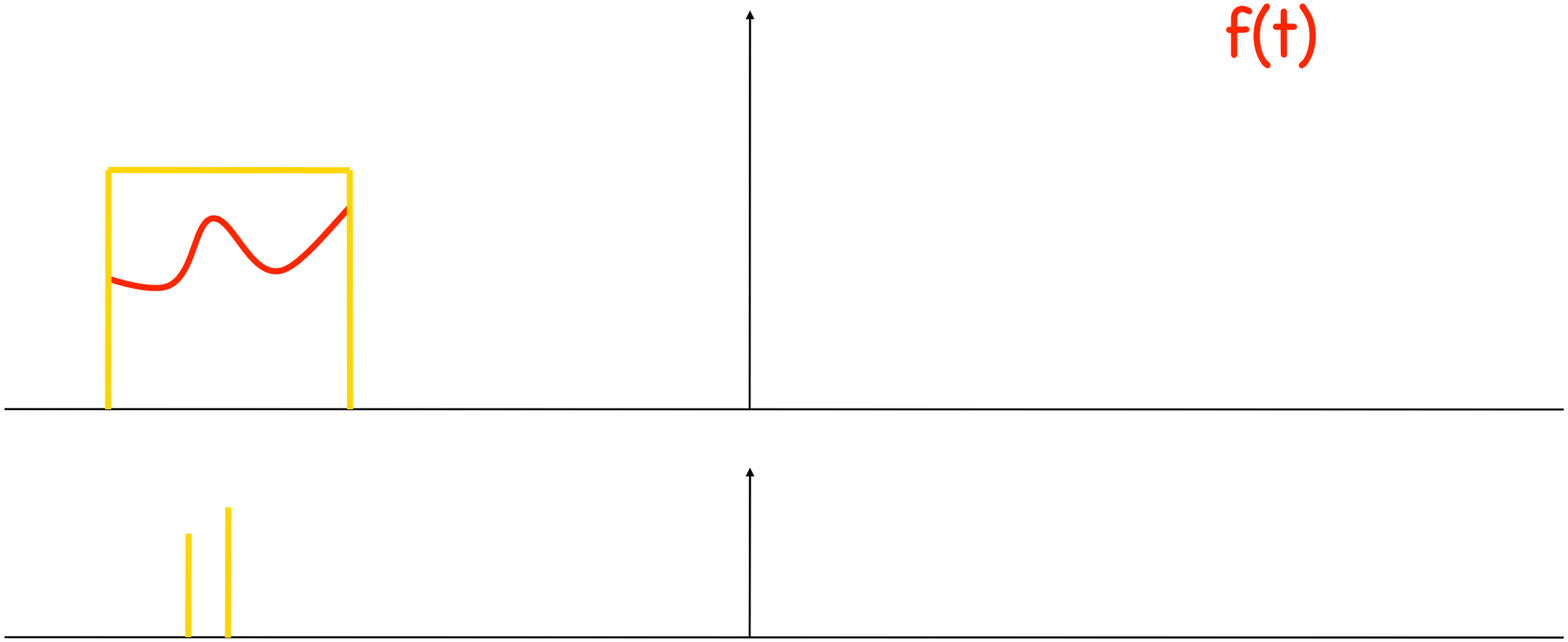
Convolution

- This function windows our function f(t)



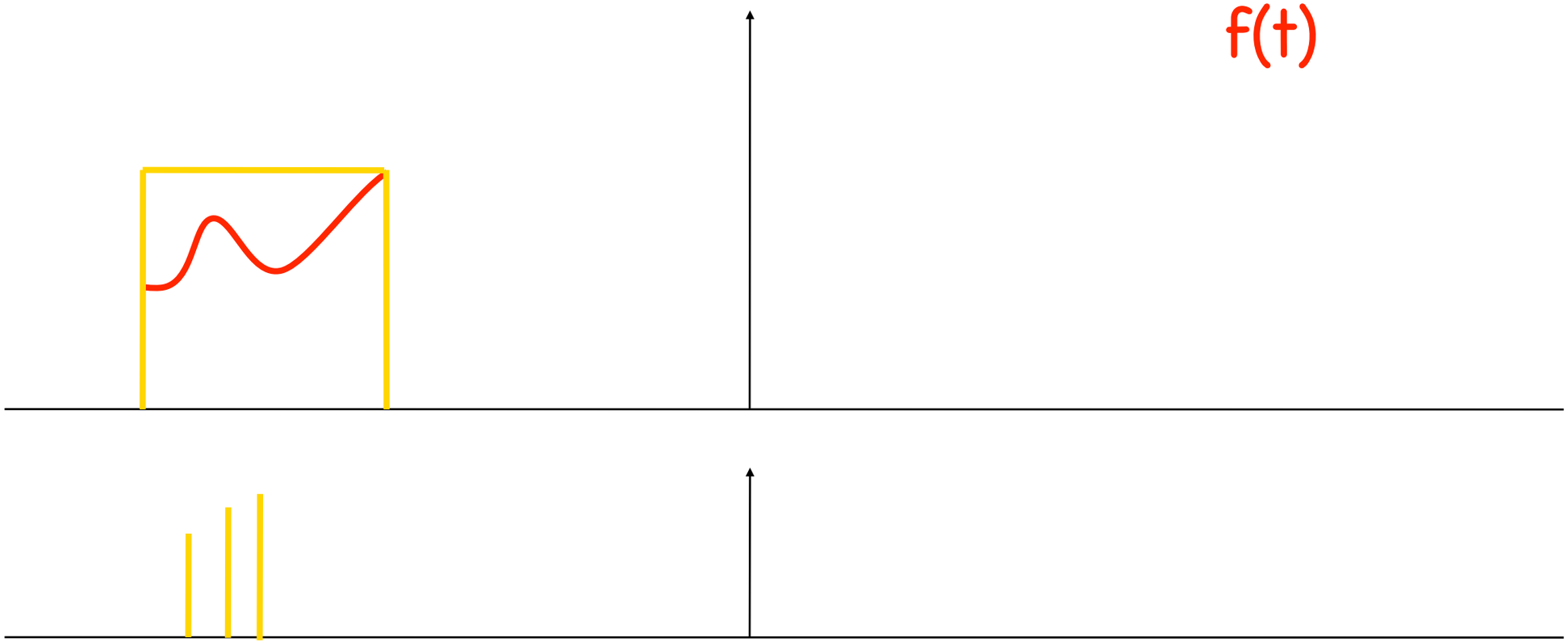
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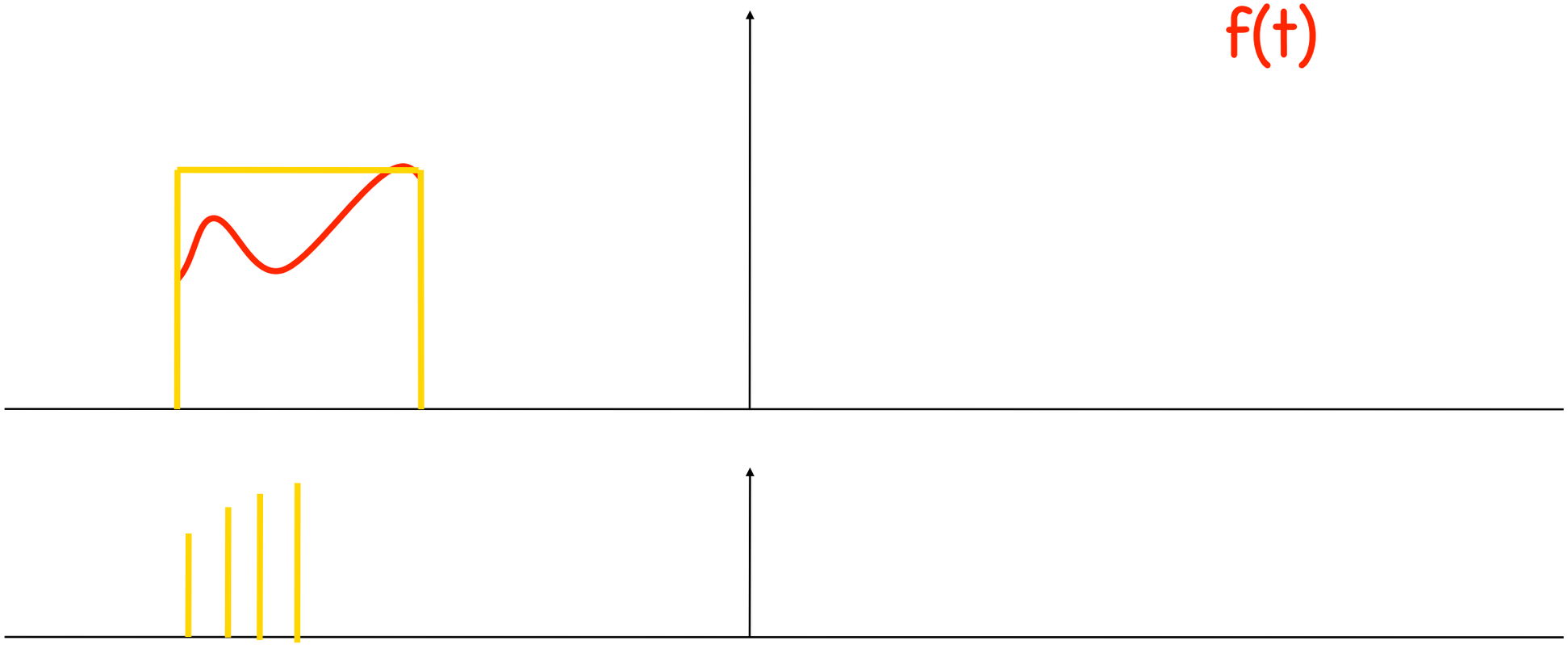
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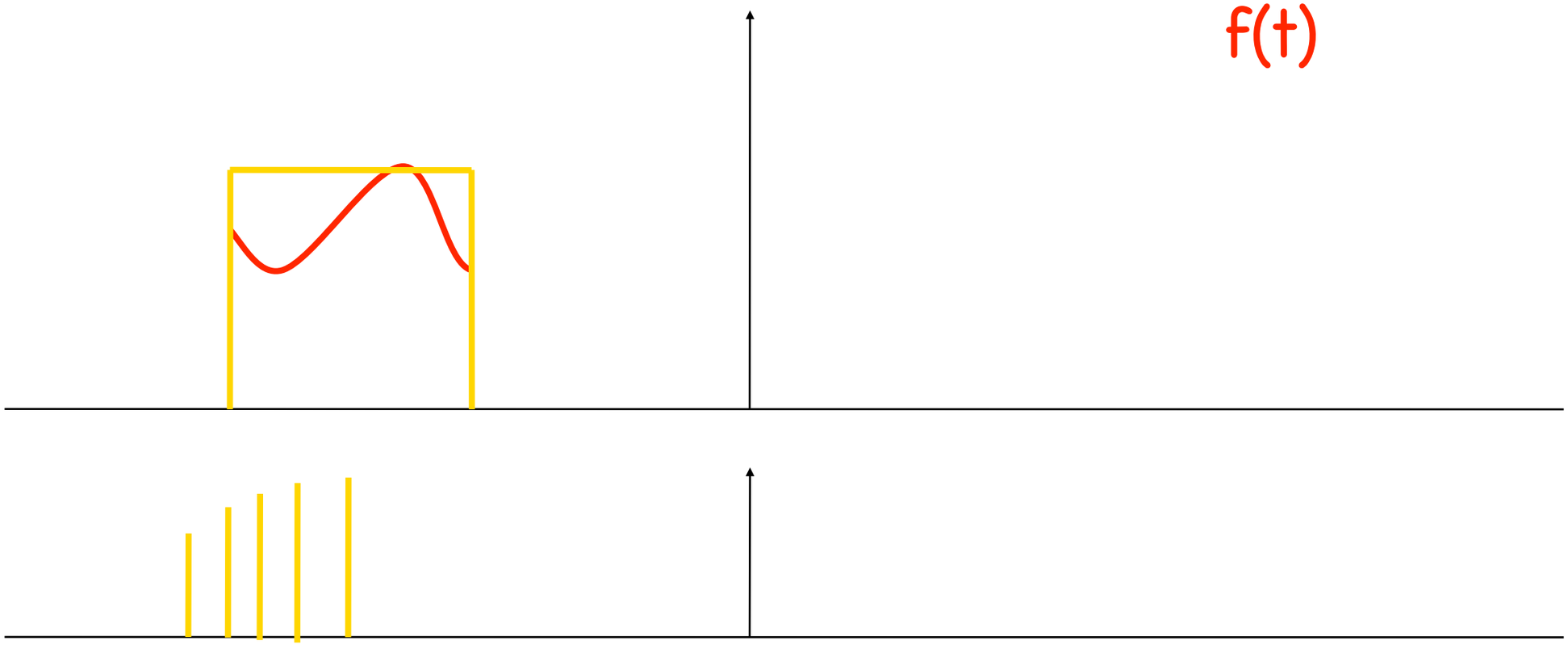
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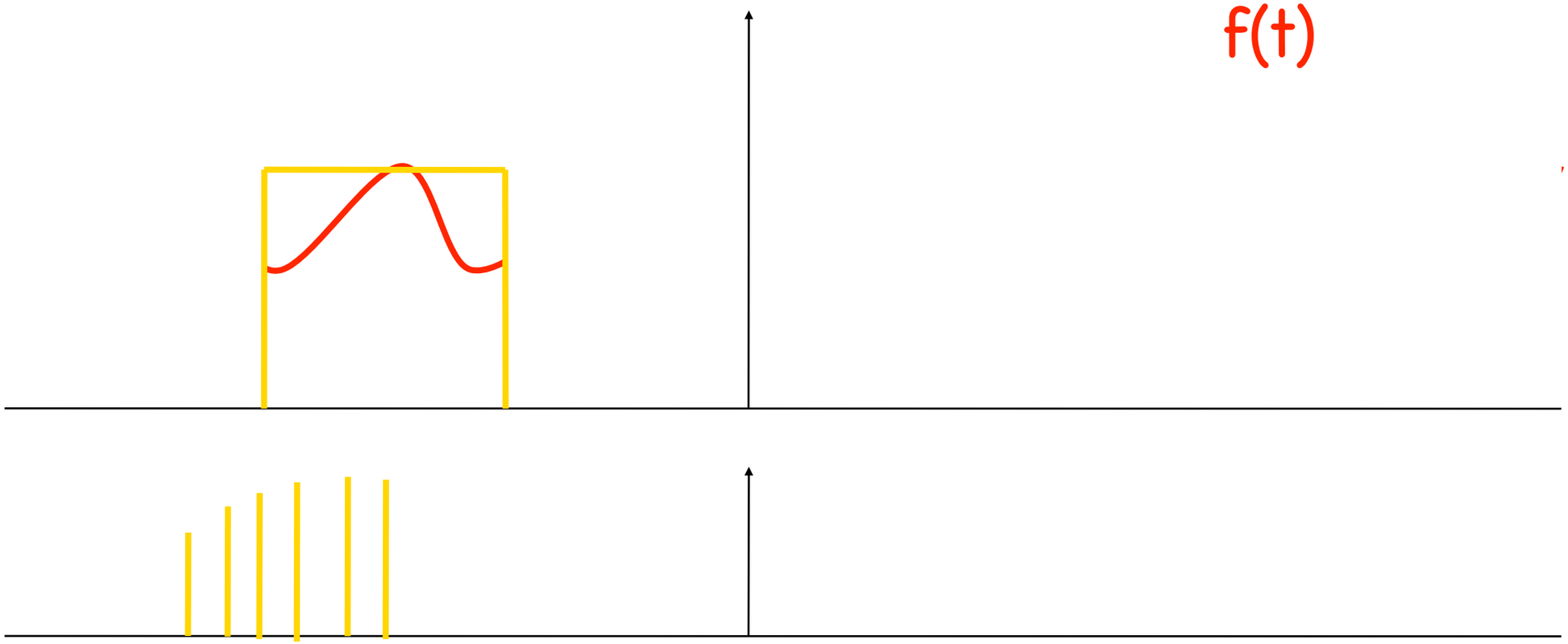
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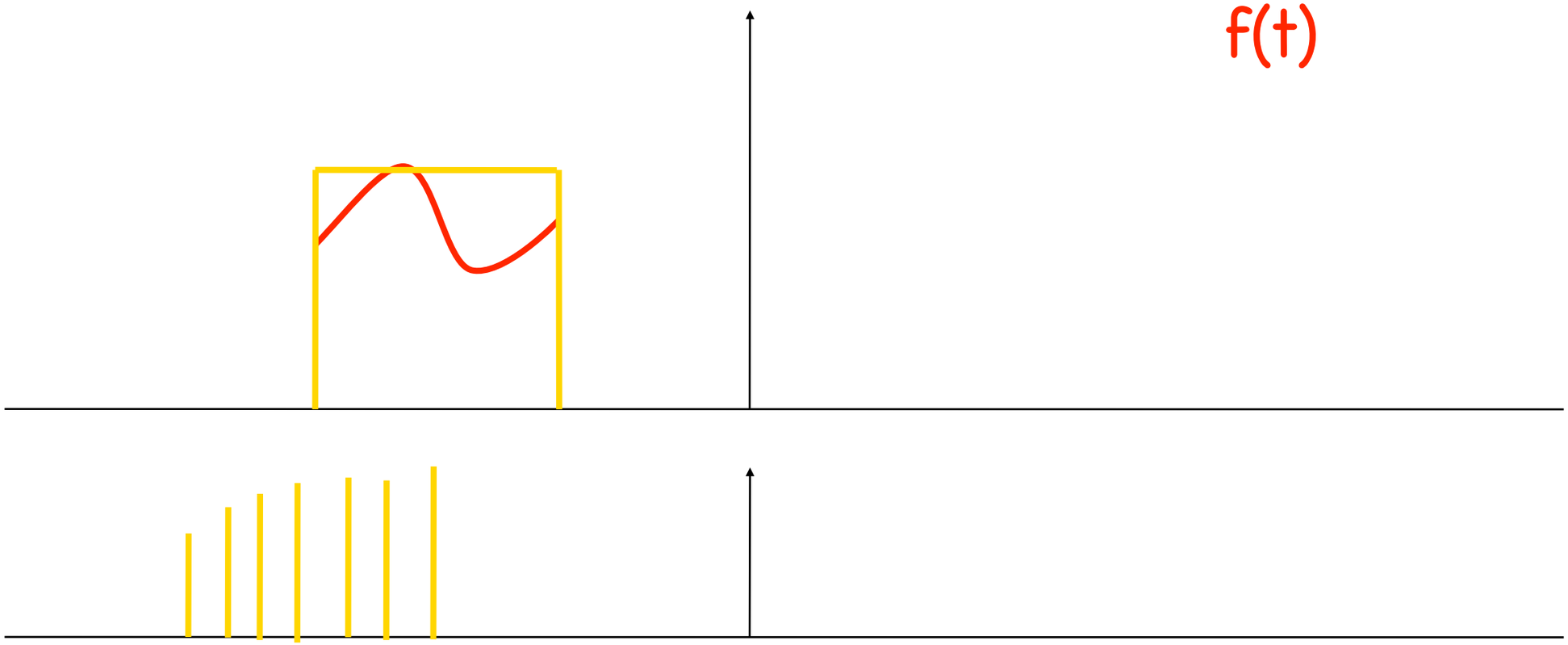
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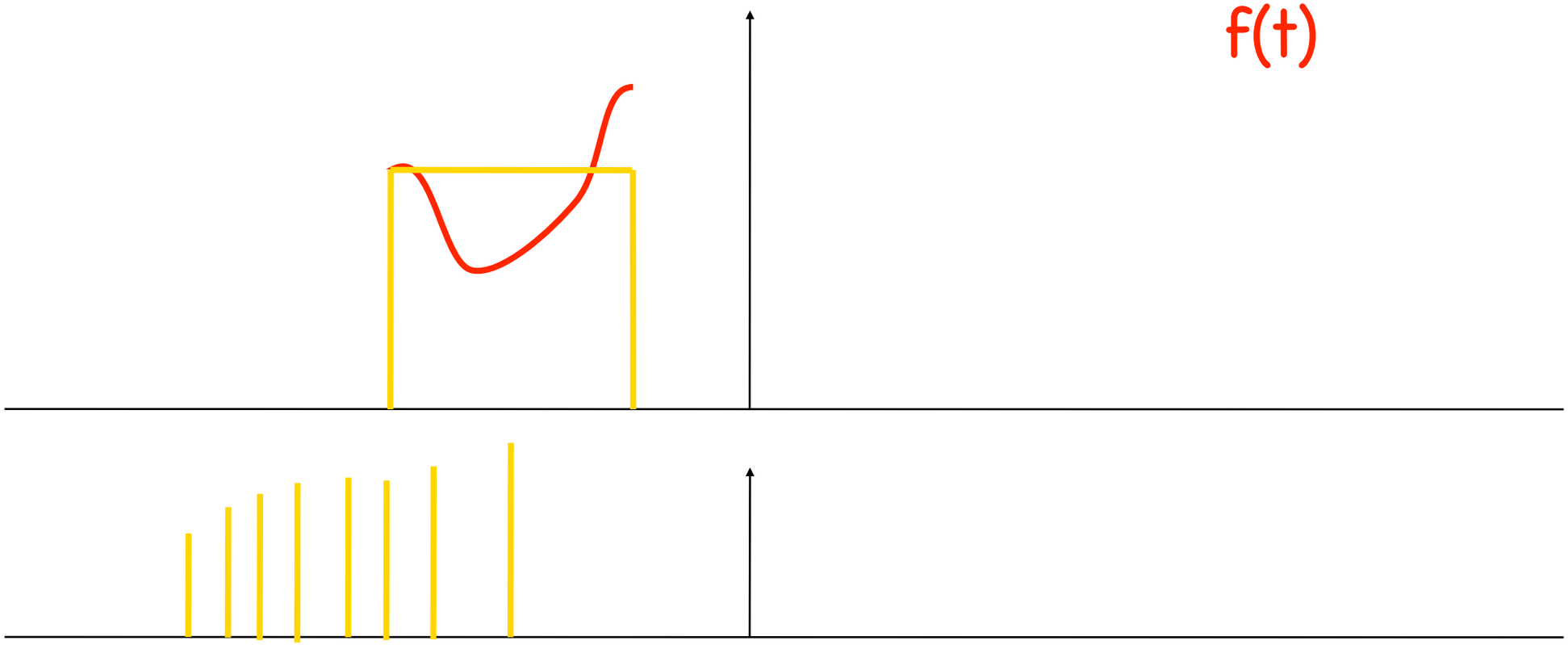
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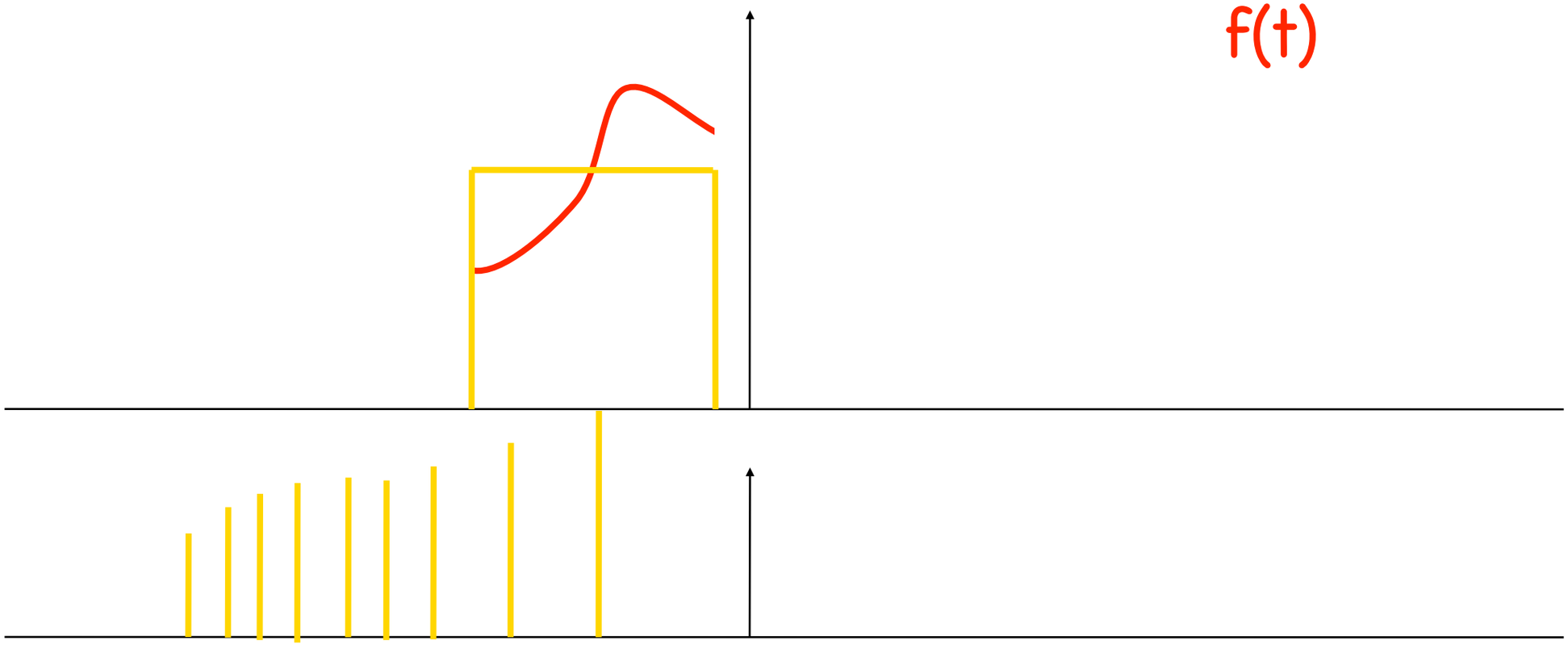
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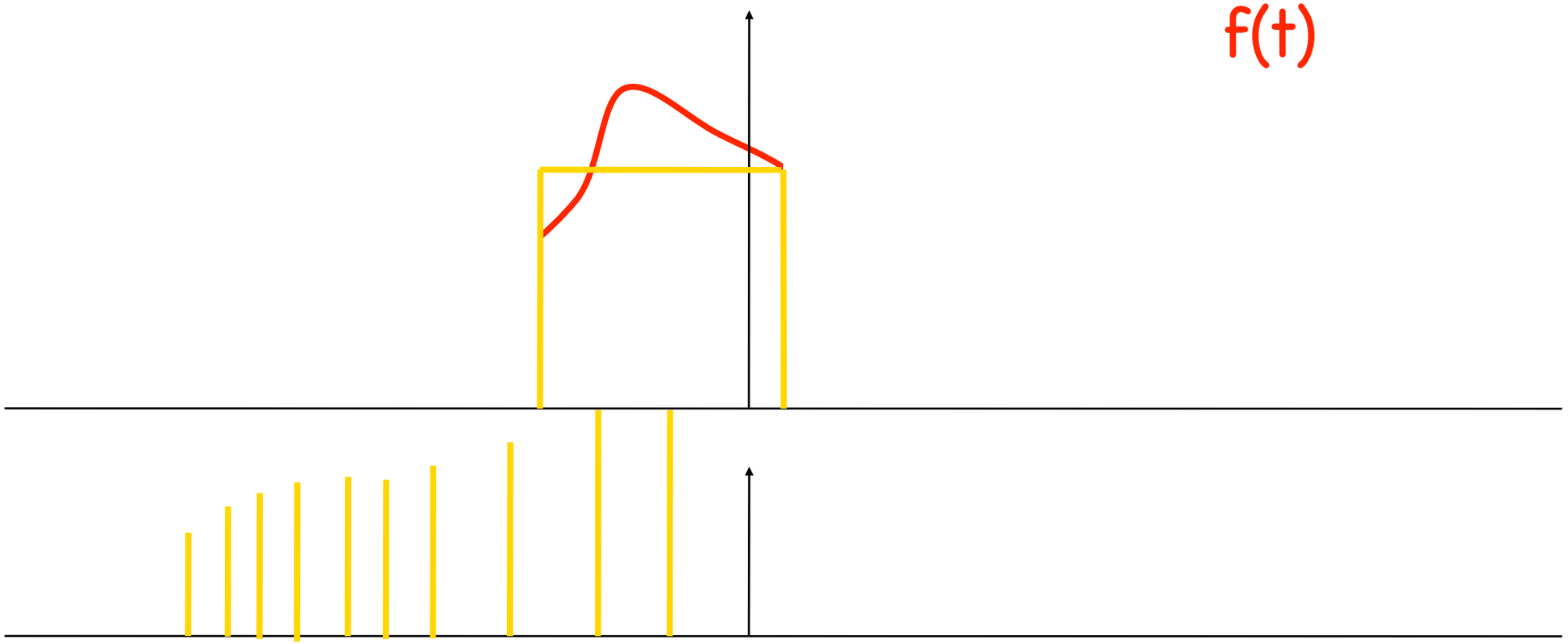
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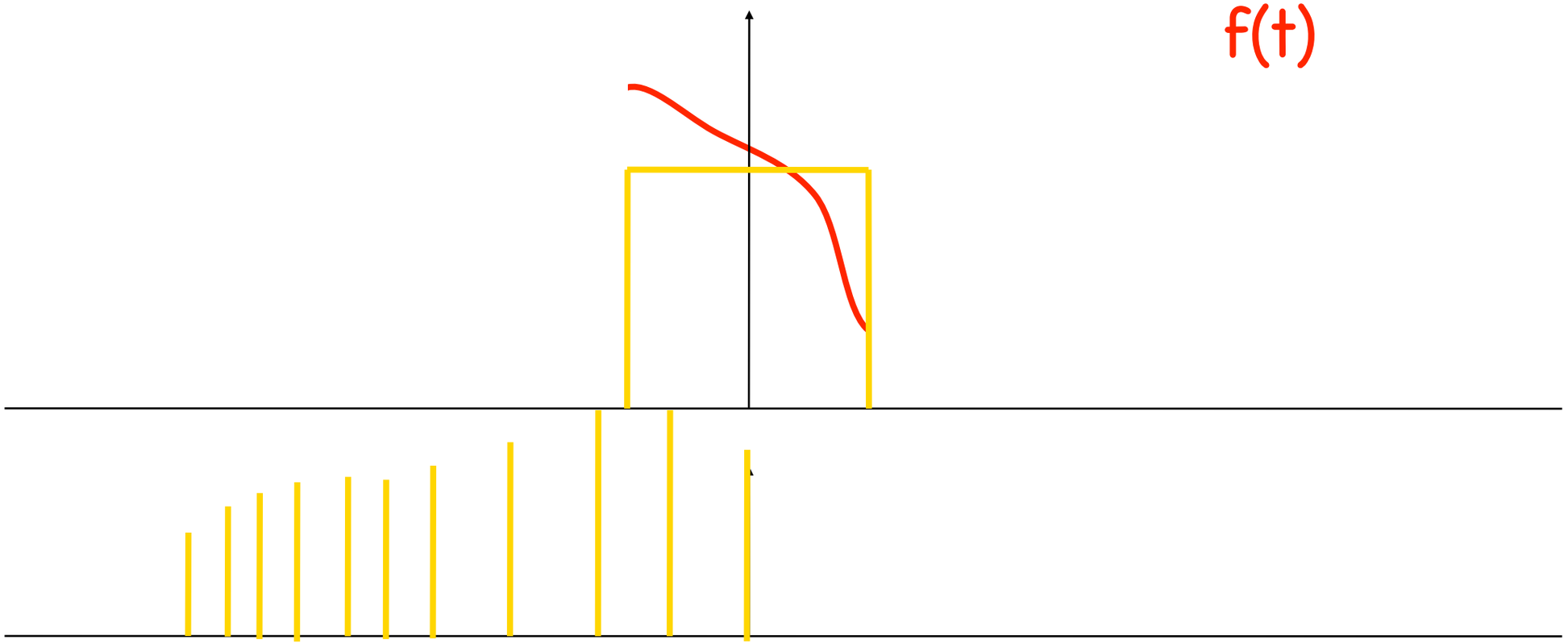
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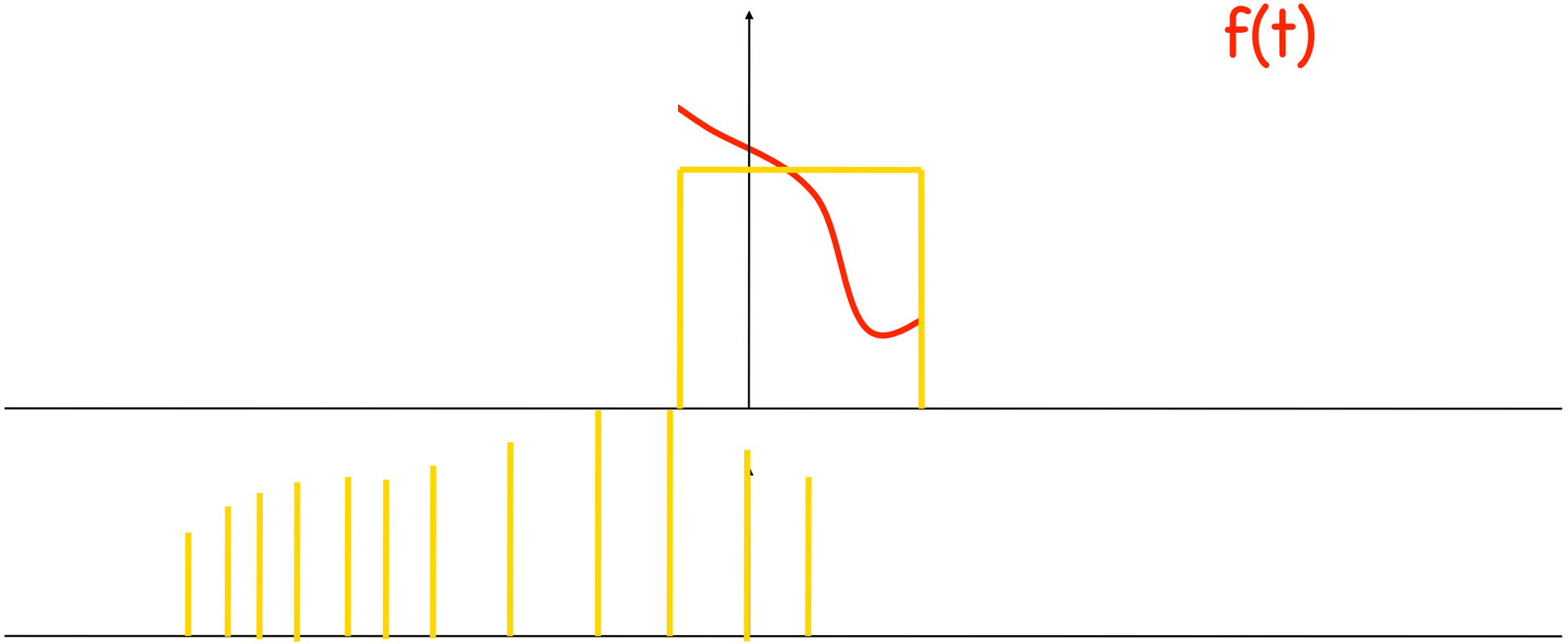
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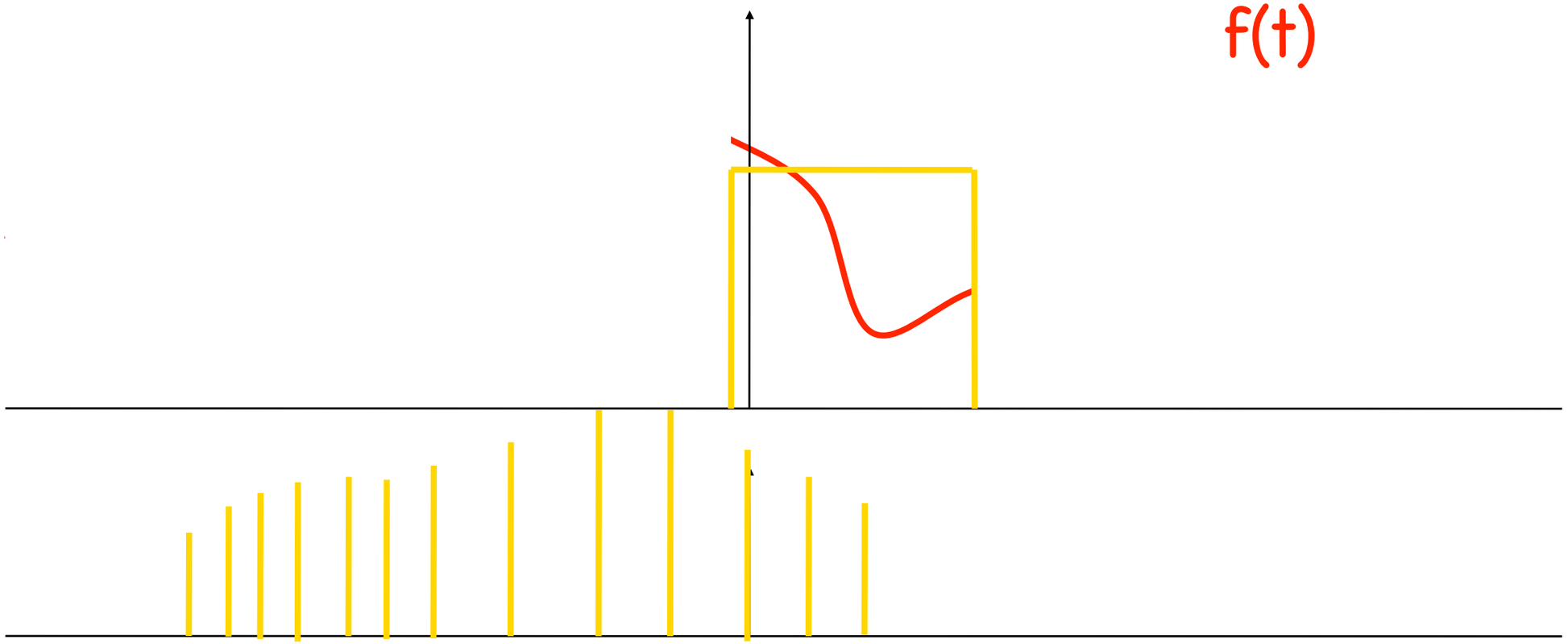
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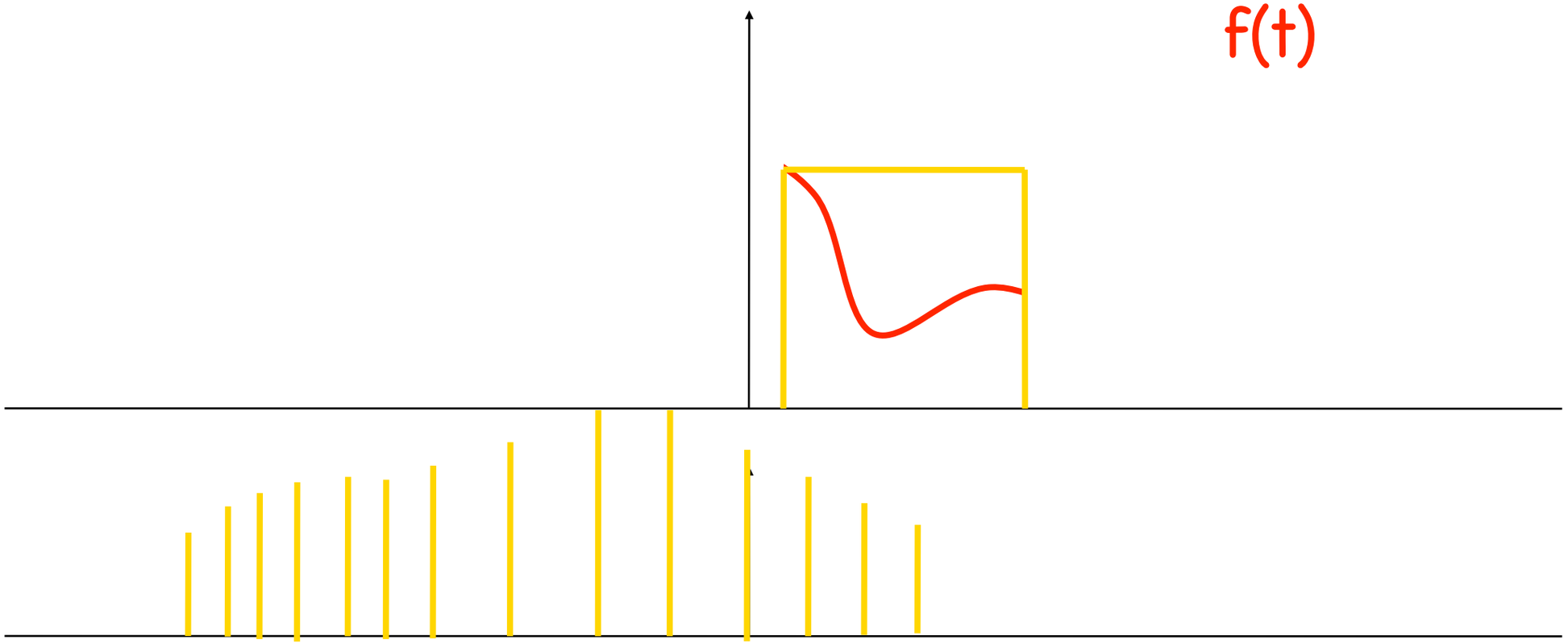
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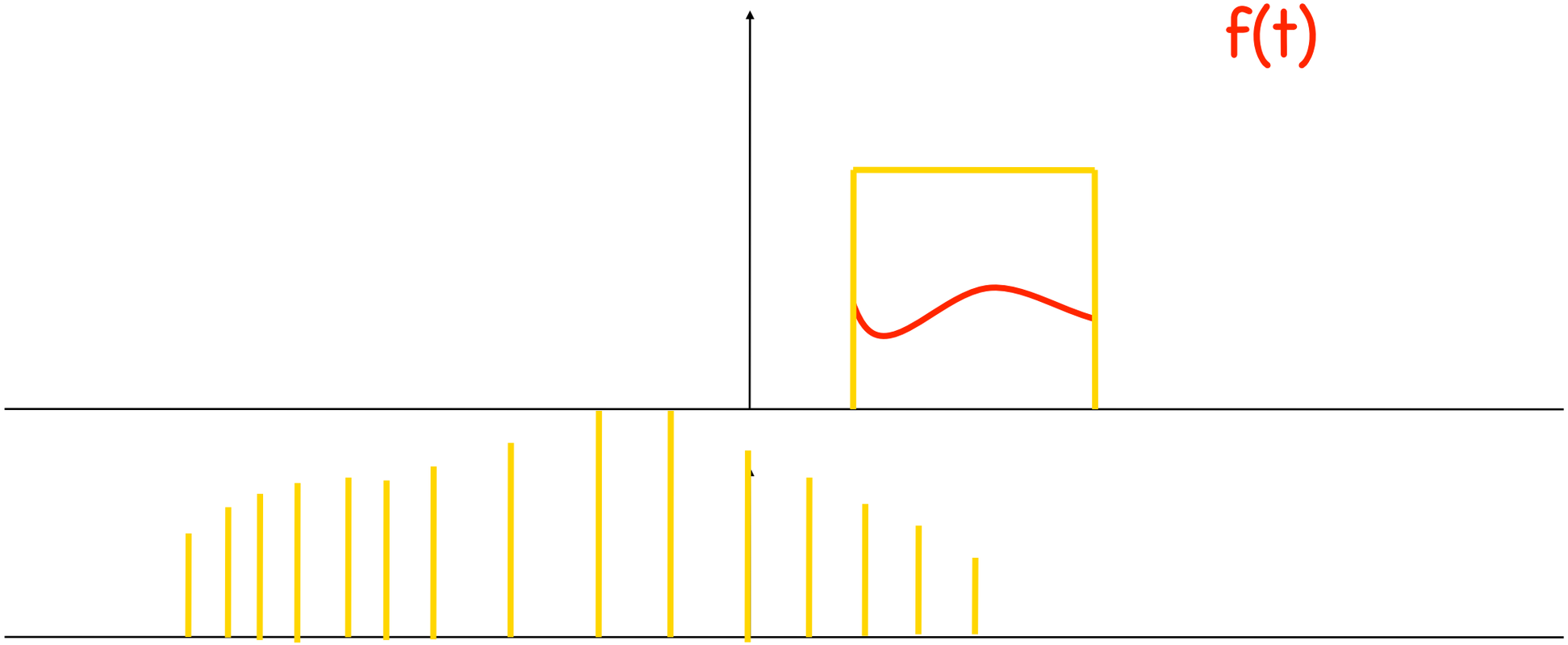
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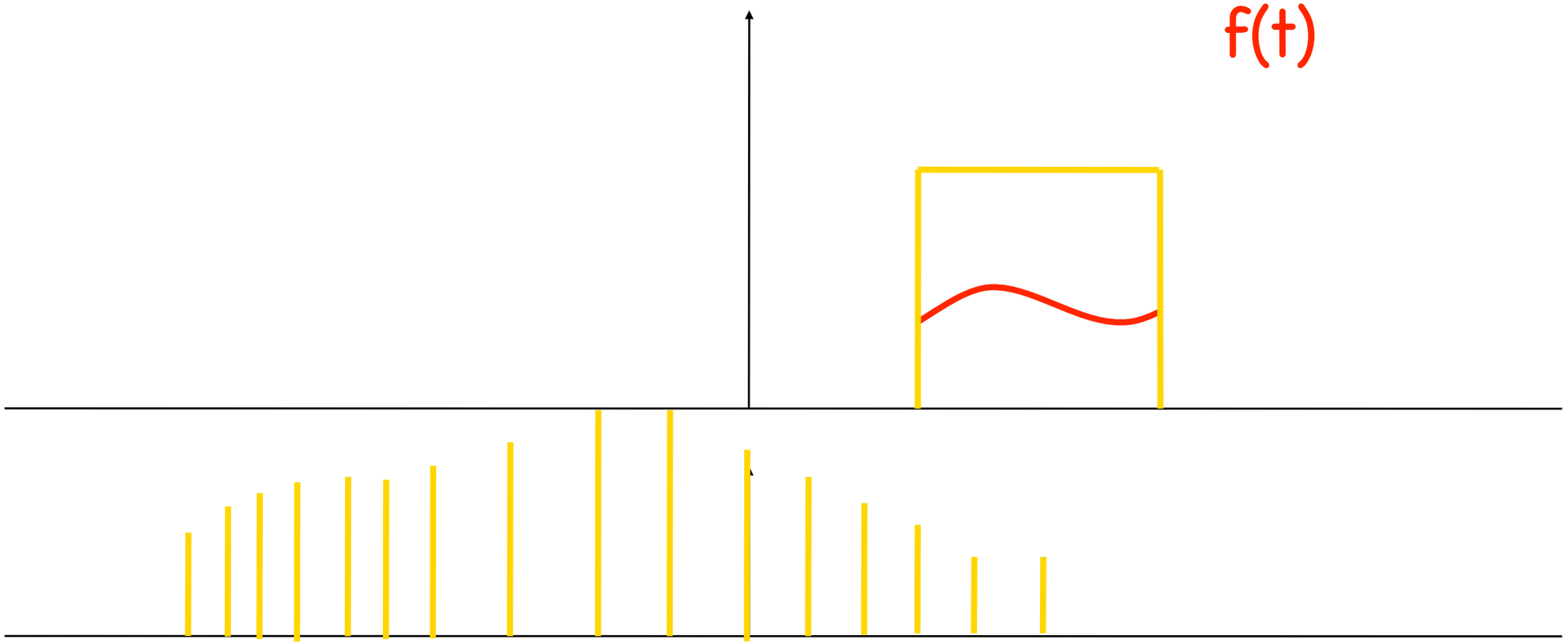
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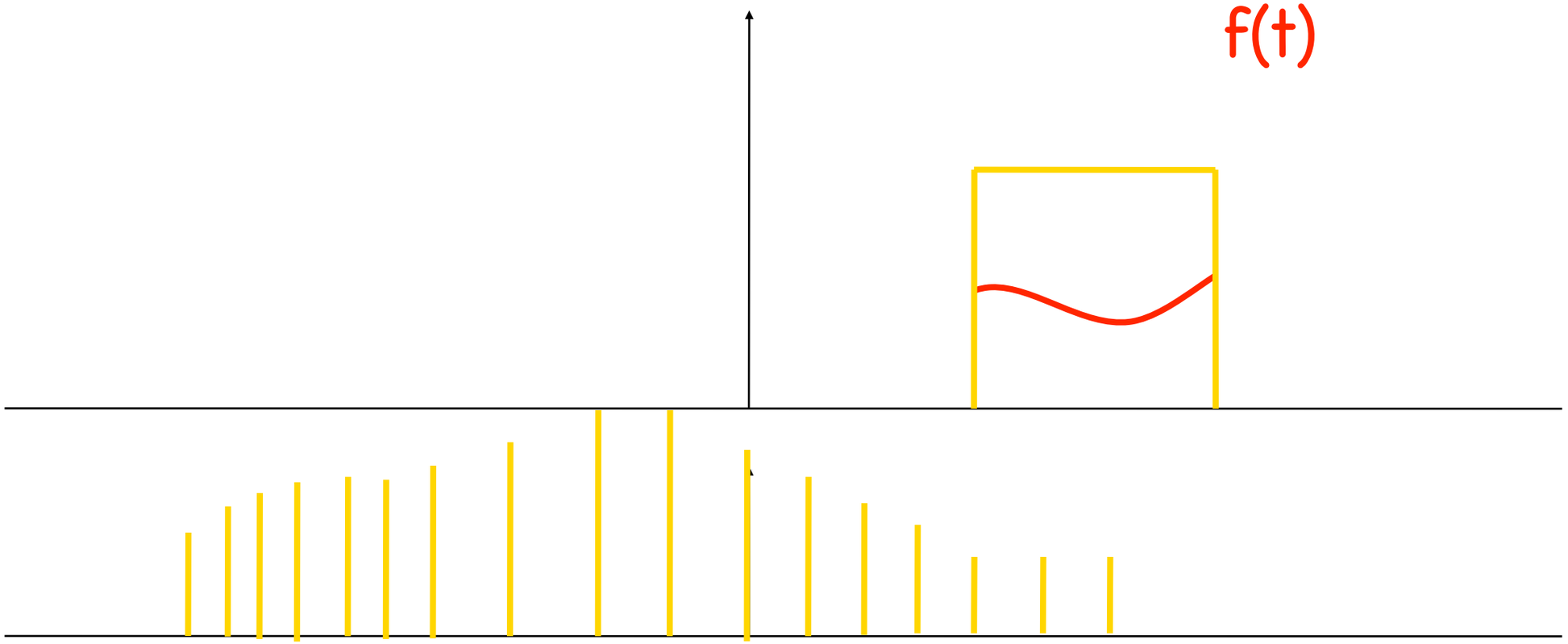
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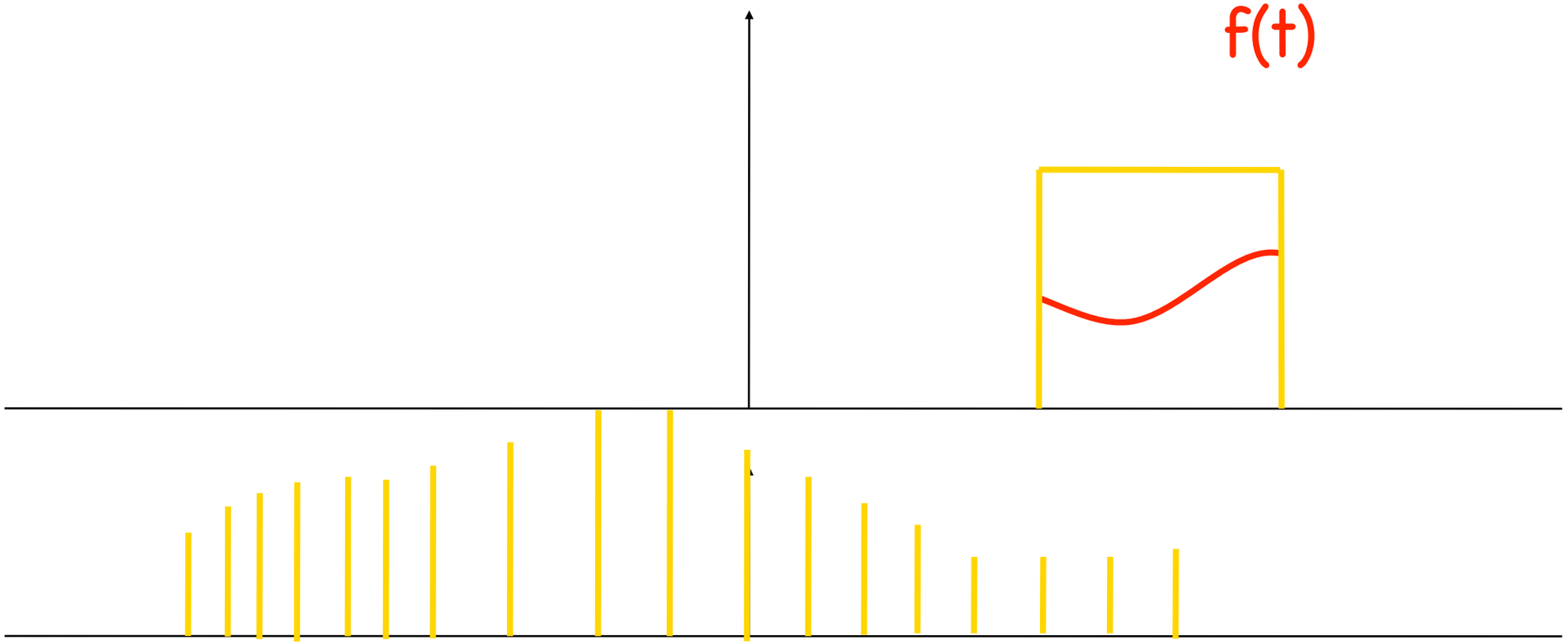
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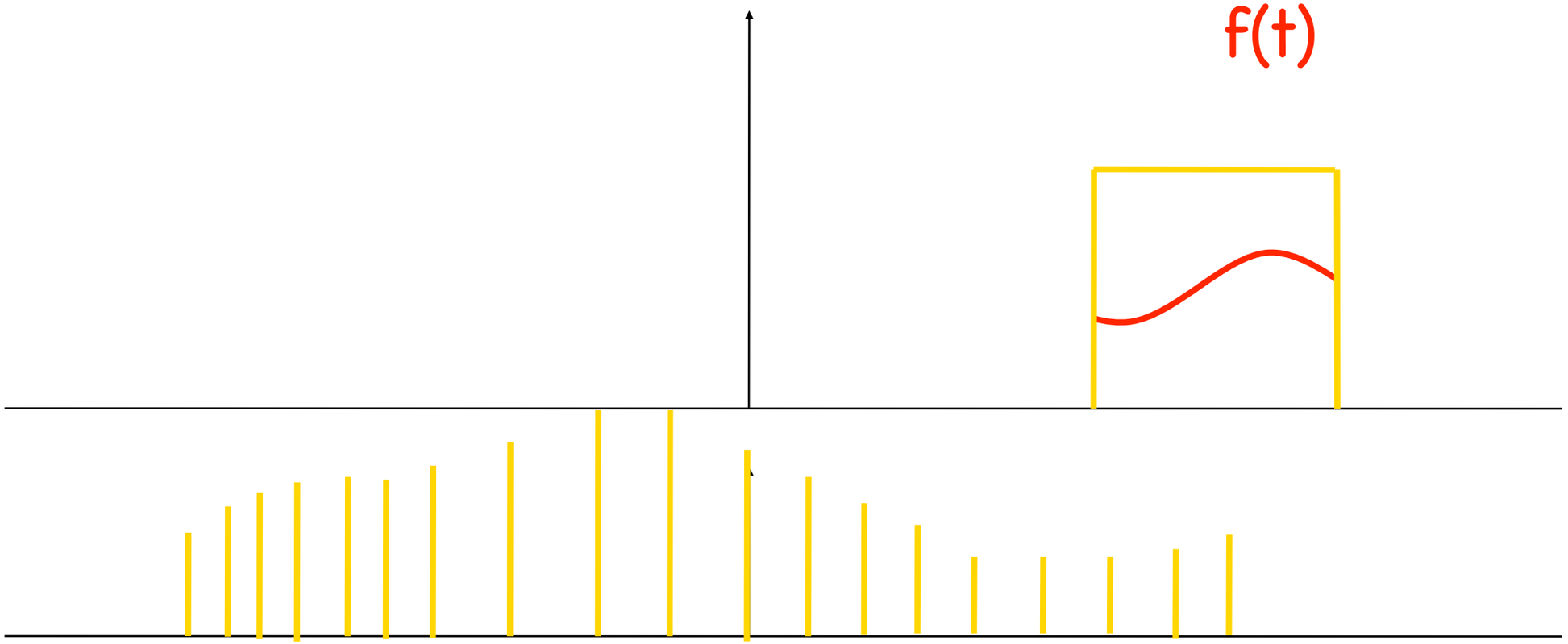
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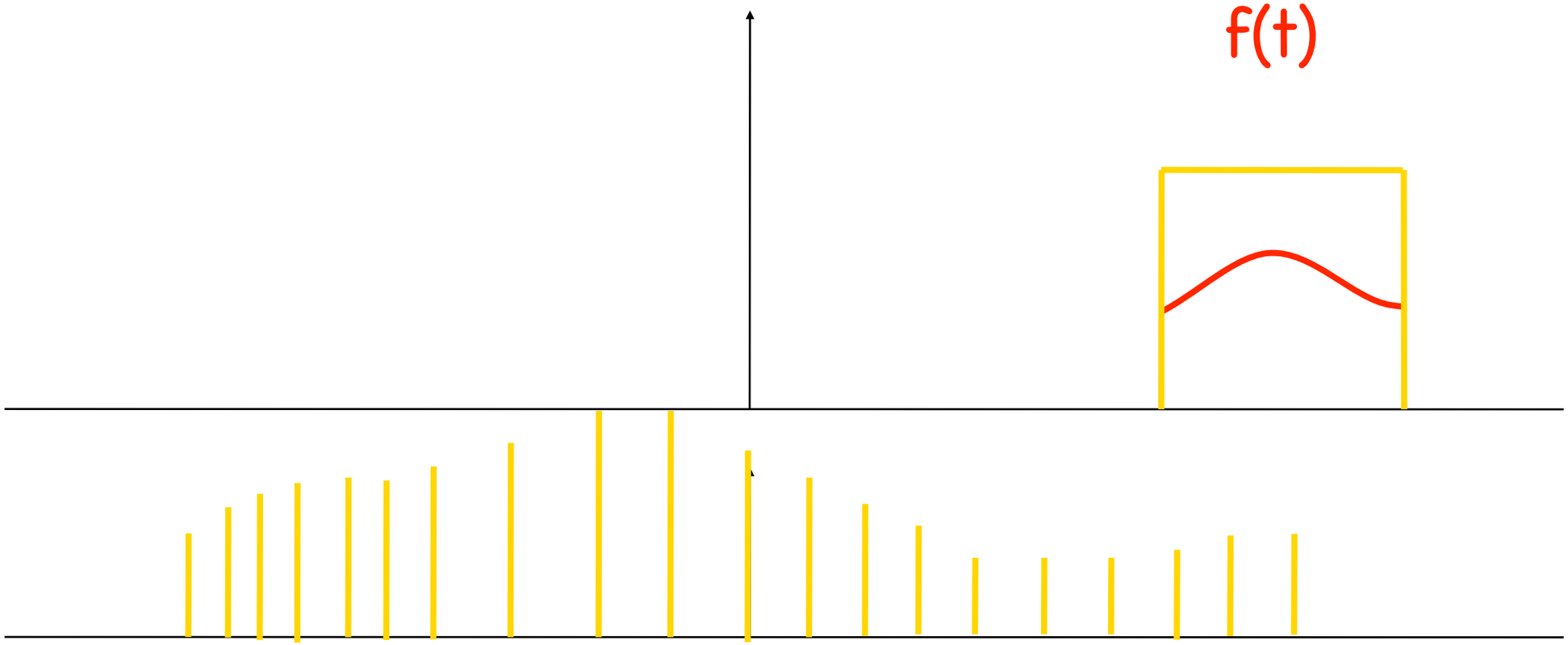
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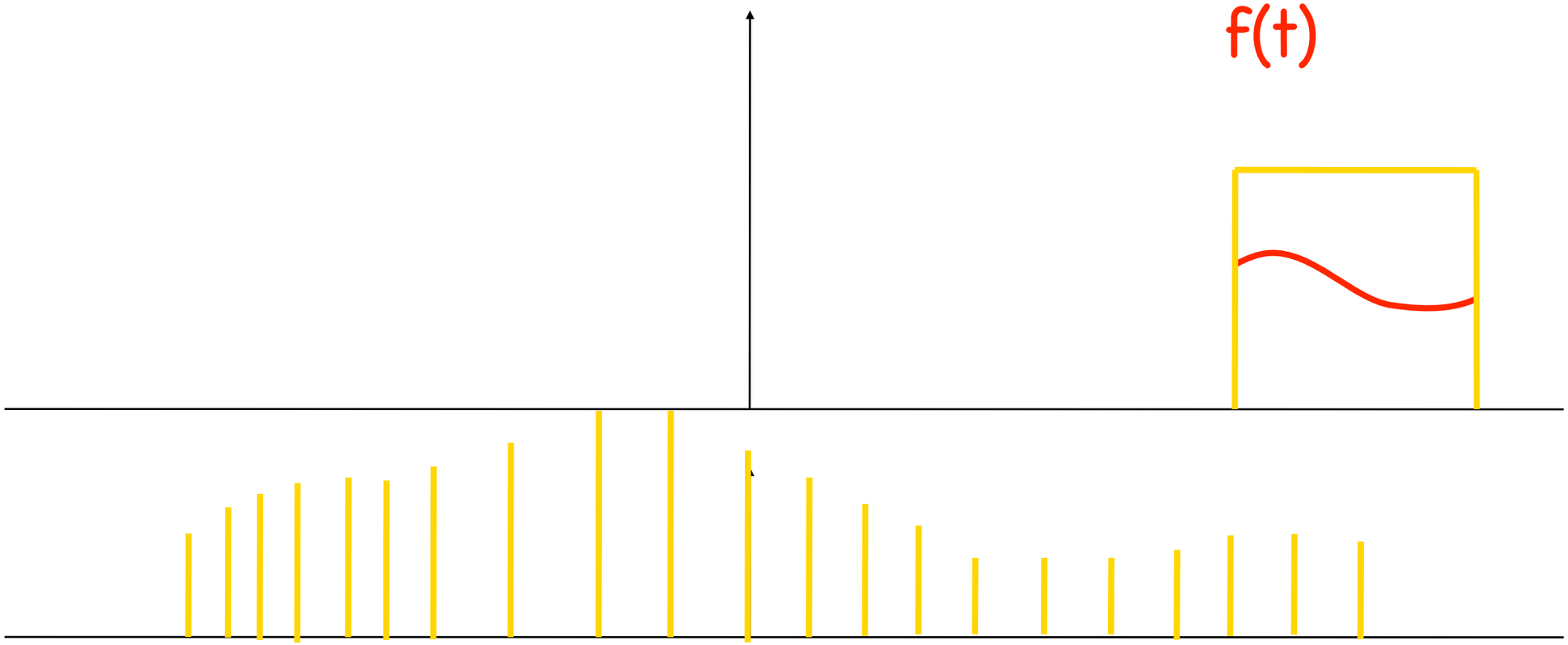
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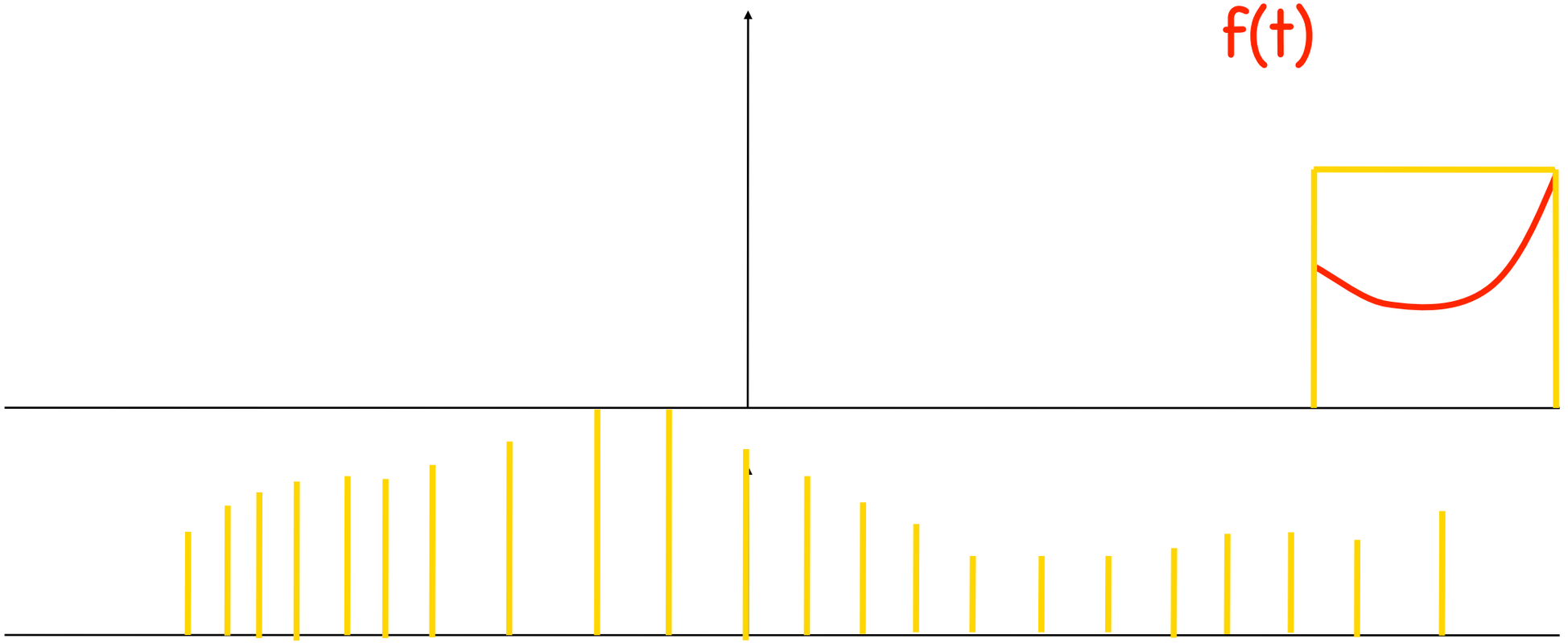
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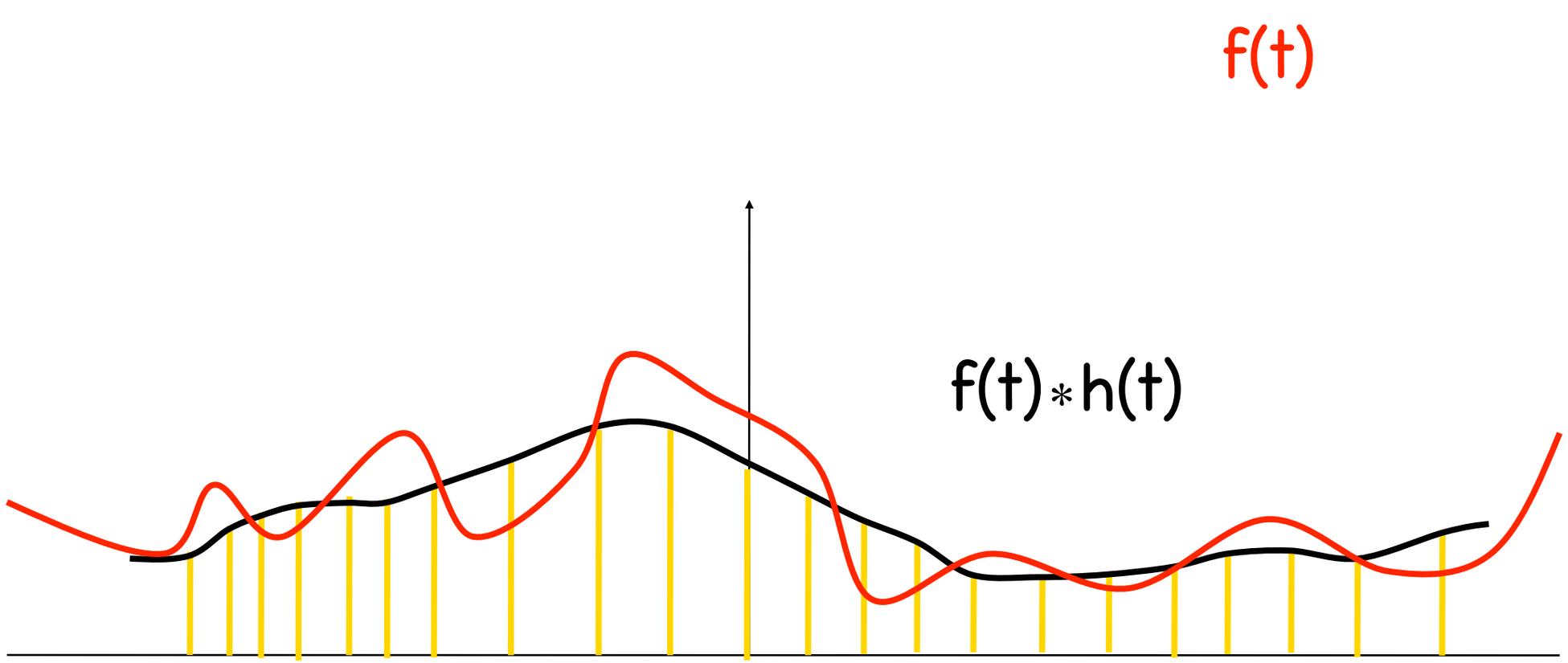
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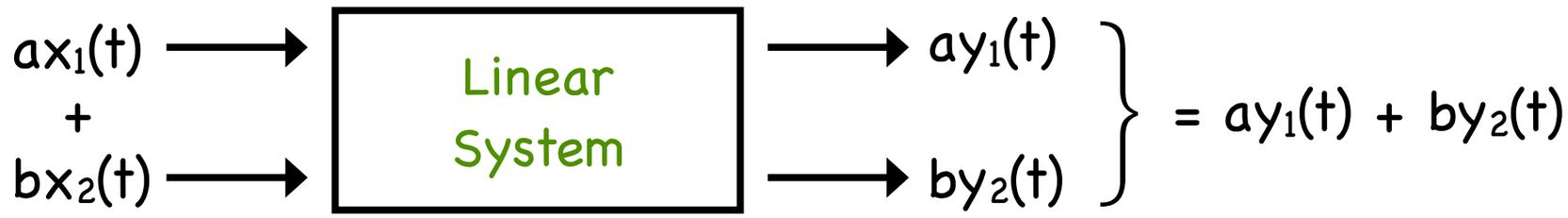


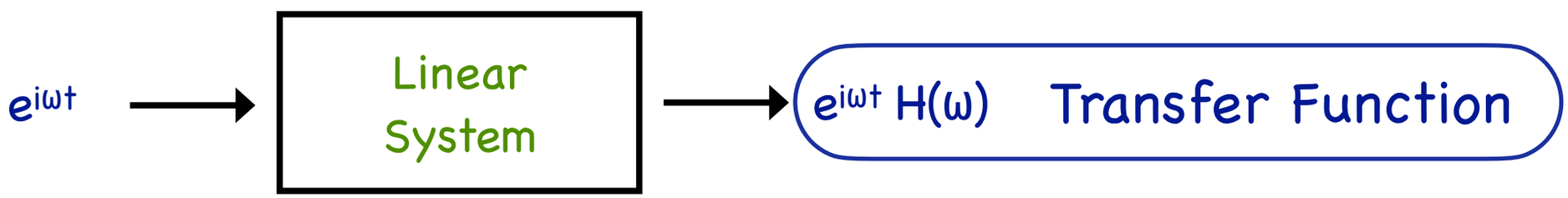
Convolution

- This particular convolution smooths out some of the high frequencies in f(t).



Linear Systems - FT





$$\int e^{i\omega\tau} h(t-\tau)\, d\tau = \int e^{i\omega(t-\tau)} h(\tau)\, d\tau = e^{i\omega t} \int e^{-i\omega\tau} h(\tau)\, d\tau$$

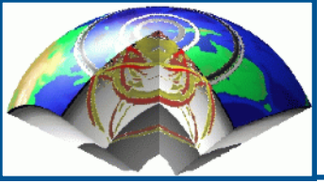
Defining the Fourier Transform of x(t) as: $\mathrm{FT}[x(t)] = X(\omega) = \int x(t) e^{-i\omega t}\, dt$

$$X(\omega) = \int x(t) e^{-i\omega t}\, dt \longrightarrow X(\omega) \cdot H(\omega)$$

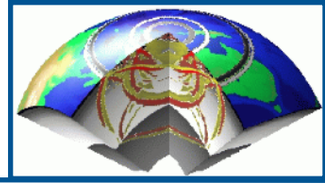
Any input x(t) can be written as a superposition of exponentials (Inverse FT):

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{i\omega t}\, d\omega \longrightarrow \frac{1}{2\pi} \int X(\omega) H(\omega) e^{i\omega t}\, d\omega = \mathrm{IFT}[X(\omega)H(\omega)] = y(t)$$

$$\mathrm{FT}[x(t) * h(t)] = X(\omega) \cdot H(\omega)$$



Convolution Theorem

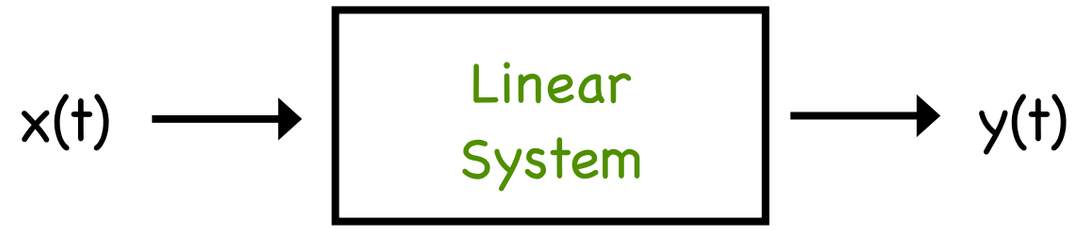


- The convolution theorem states that convolution in the time domain is equivalent to multiplication in the frequency domain, and viceversa.

$$f(t) * g(t) \Leftrightarrow F(\omega) \cdot G(\omega)$$

$$f(t) \cdot g(t) \Leftrightarrow F(\omega) * G(\omega)$$

Poles and Zeroes



$e^{i\omega t} \longrightarrow H(\omega)e^{i\omega t}$ $\qquad H(\omega)=Y(\omega)/X(\omega)$

Differential Equation:

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \ldots + b_1 \frac{dx}{dt} + b_0 x$$

$$H(\omega) = \frac{\sum_{j=0}^{m} b_j (i\omega)^j}{\sum_{k=0}^{n} a_k (i\omega)^k}$$

Zeroes

Poles

Transfer function of the forced oscillator:

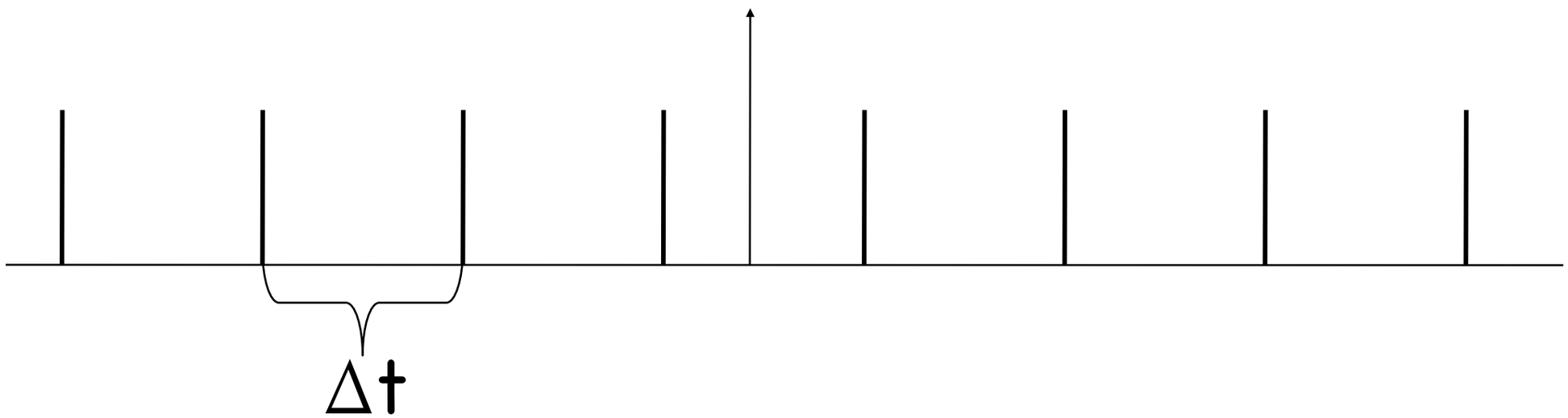
$$H(\omega) = \frac{Y(\omega)}{U(\omega)} = \frac{-(i\omega)^2}{(i\omega)^2 + 2\gamma(i\omega) + (\omega_0)^2}$$

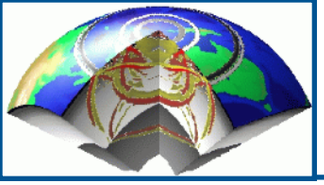
Sampling Function

- A Sampling Function or Impulse Train is defined by:

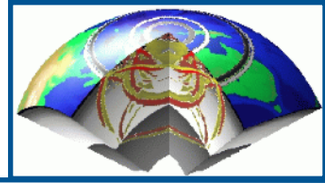
$$S_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t)$$

where Δt is the sample spacing.





Sampling Function



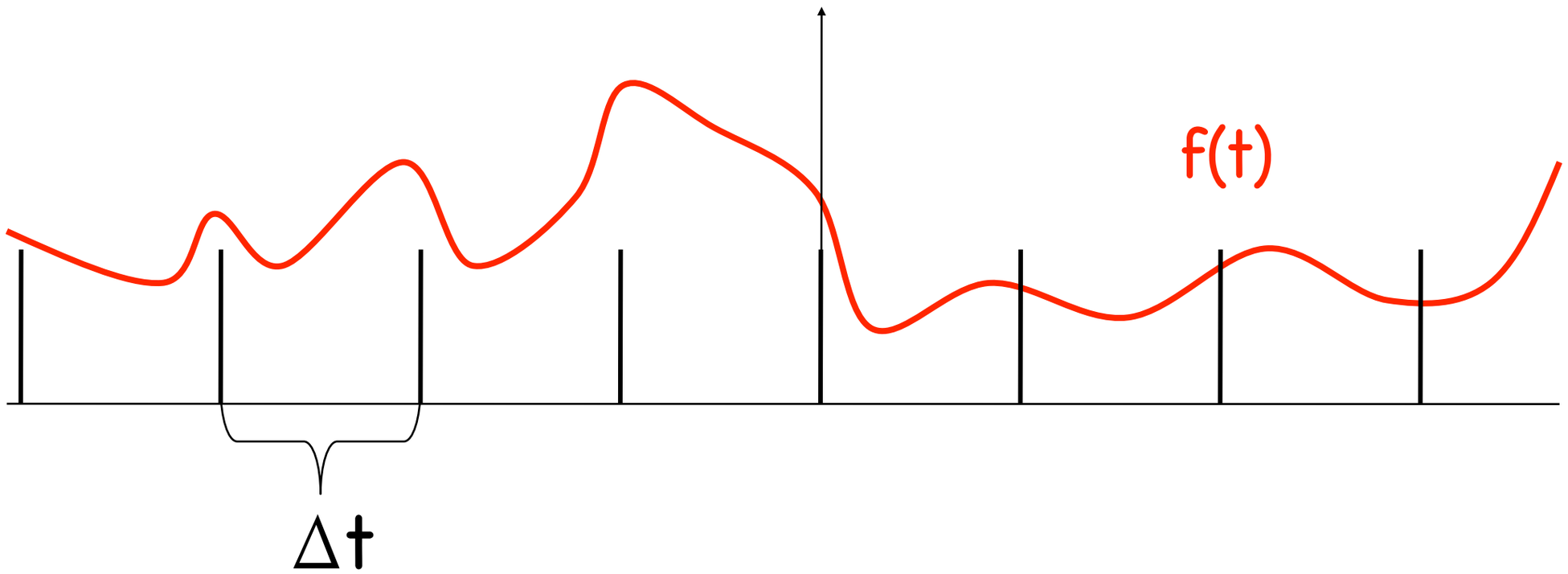
- The Fourier Transform of the Sampling Function is itself a sampling function.
- The sample spacing is the inverse.

$$S_{\Delta t}(t) \Leftrightarrow S_{\frac{1}{\Delta t}}(\omega)$$

Convolution Theorem

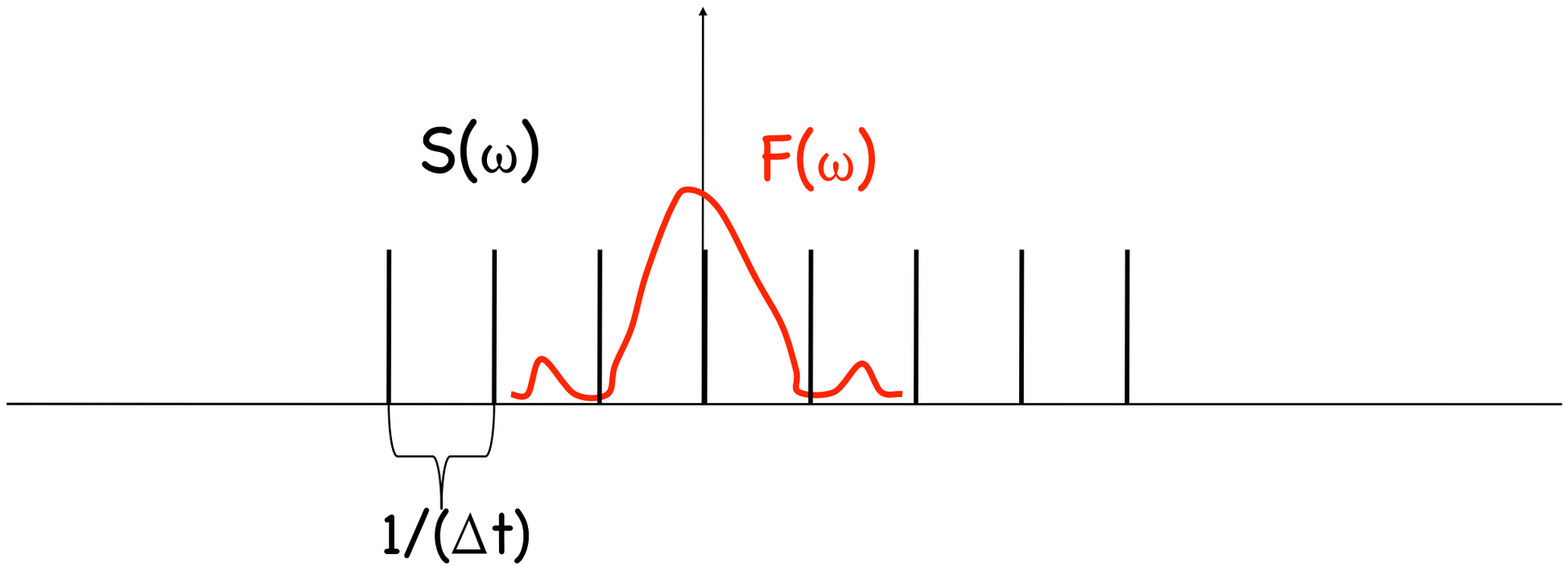
- This powerful theorem can illustrate the problems with our point sampling and provide guidance on avoiding aliasing.

- Consider: $f(t) \cdot S_{\Delta t}(t)$



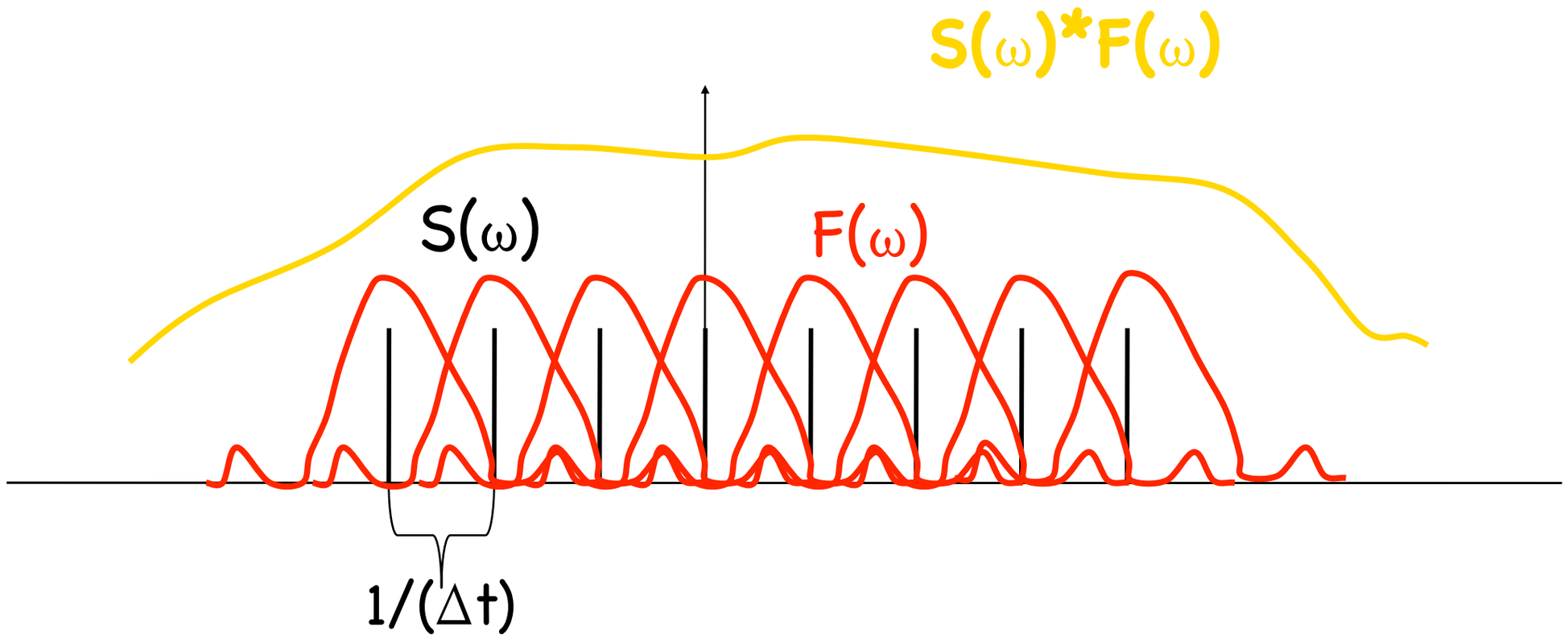
Convolution Theorem

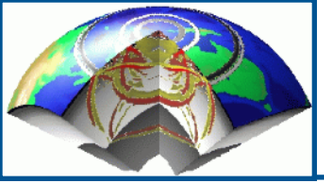
- What does this look like in the Fourier domain?



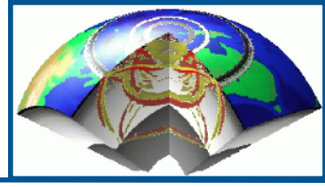
Convolution Theorem

- In Fourier domain we would convolve

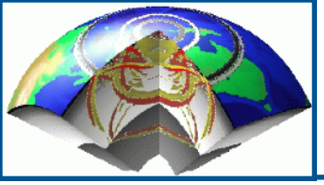




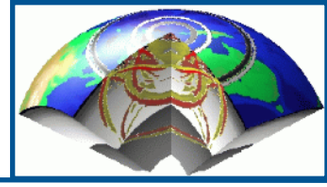
Aliasing



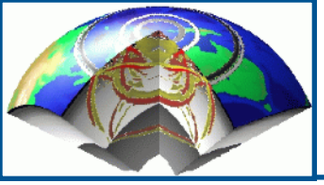
- What this says, is that any frequencies greater than a certain amount will appear intermixed with other frequencies.
- In particular, the higher frequencies for the copy at $1/\Delta t$ intermix with the low frequencies centered at the origin.



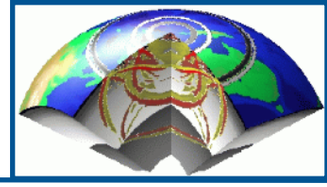
Aliasing and Sampling



- Note, that the sampling process introduces frequencies out to infinity.
- We have also lost the function f(t), and now have only the discrete samples.
- This brings us to our next theorem



Sampling Theorem



- The Shannon Sampling Theorem:

  A band-limited signal f(t), with a cutoff frequency of λ, that is sampled with a sampling spacing of Δt may be perfectly reconstructed from the discrete values $f[n\Delta t]$ by convolution with the sinc(t) function, provided the Nyquist limit: $\lambda < 1/(2\Delta t)$

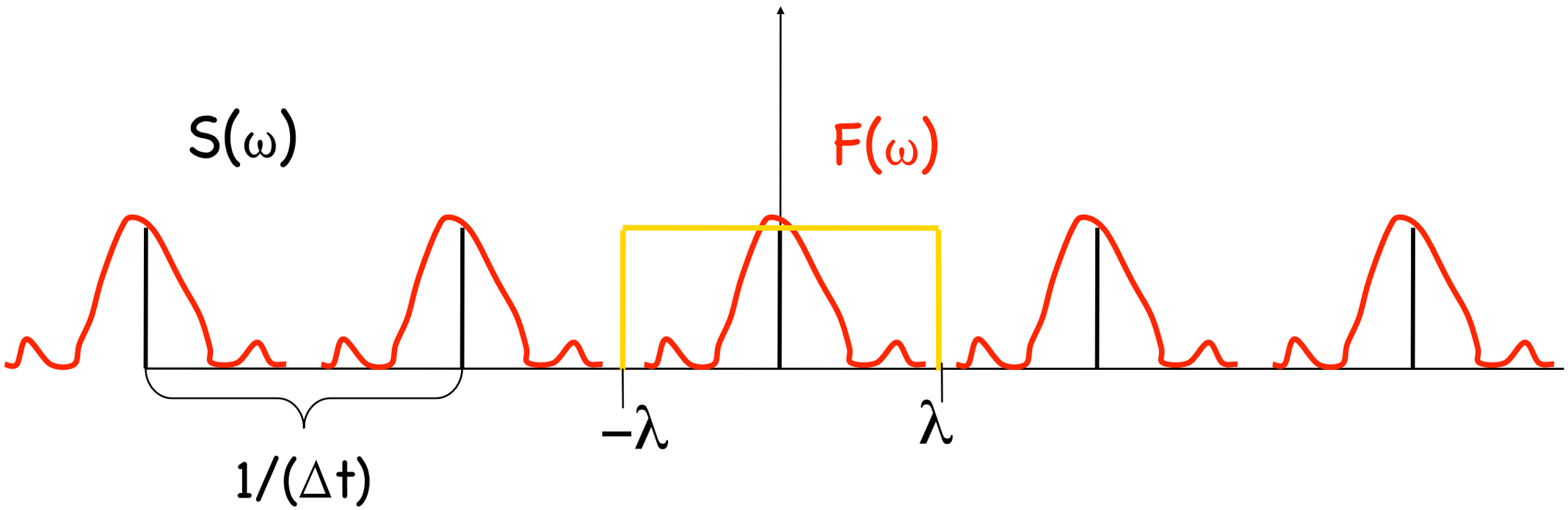
- Why is this?

  The Nyquist limit will ensure that the copies of $F(\omega)$ do not overlap in the frequency domain.

  We can completely reconstruct or determine f(t) from $F(\omega)$ using the Inverse Fourier Transform.

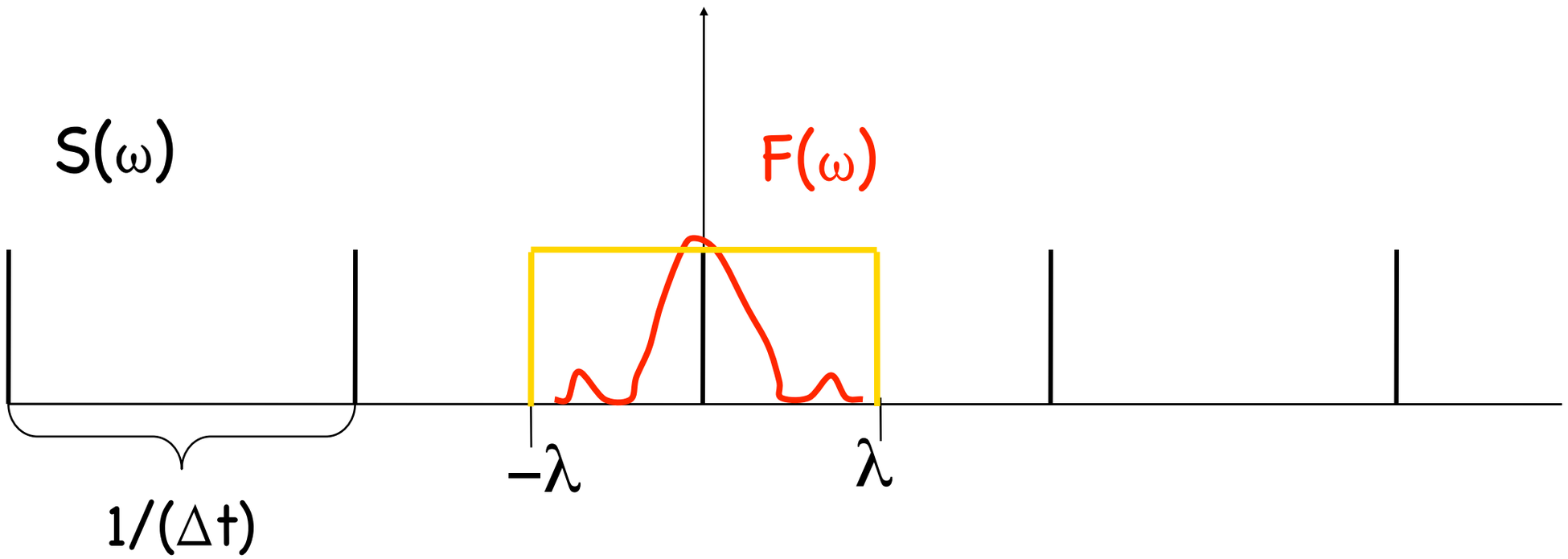
Sampling Theory

- In order to do this, we need to remove all of the shifted copies of $F(\omega)$ first.
- This is done by simply multiplying $F(\omega)$ by a box function of width 2λ.



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Sampling Theory

- So, given $f[n\Delta t]$ and an assumption that $f(t)$ does not have frequencies greater than $1/(2\Delta t)$, we can write the formula:

$$f[n\Delta t] = f(t) \cdot S_{\Delta t}(t) \Leftrightarrow F(\omega) * S_{\Delta t}(\omega)$$

$$F(\omega) = (F(\omega) * S_{\Delta t}(\omega)) \cdot \mathrm{Box}_{1/(2\Delta t)}(\omega)$$

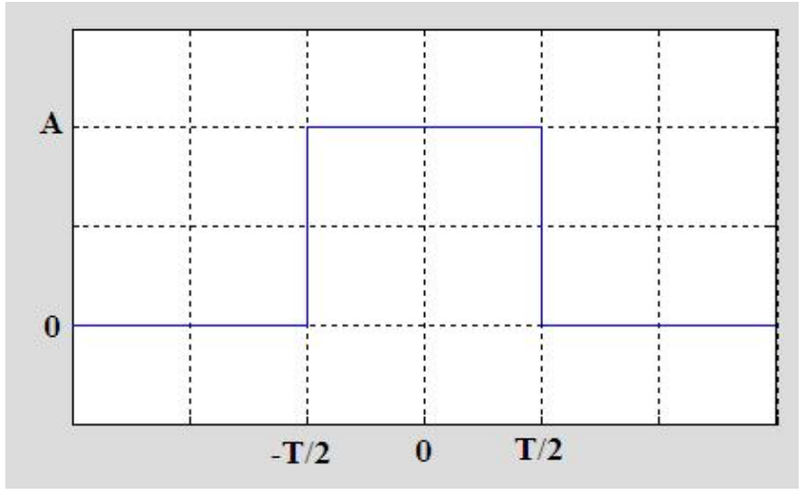
therefore,

$$f(t) = f[n\Delta t] * \mathrm{sinc}(t)$$

http://www.thefouriertransform.com/pairs/box.php

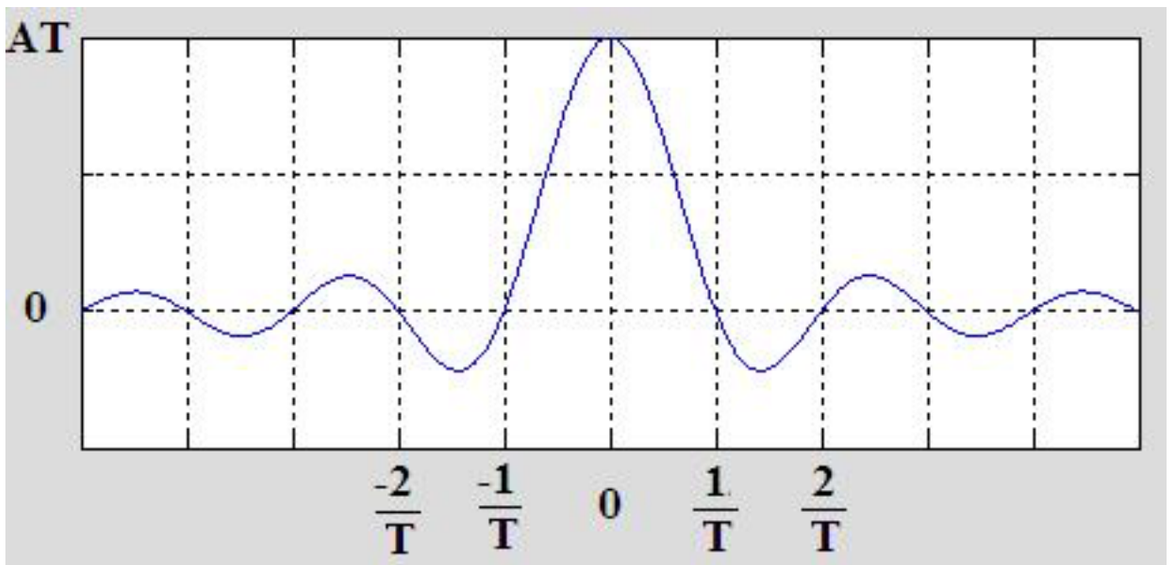
http://webapps.chem.uoa.gr/efs/applets/AppletNyquist/Appl_Nyquist2.html

FT of Boxcar

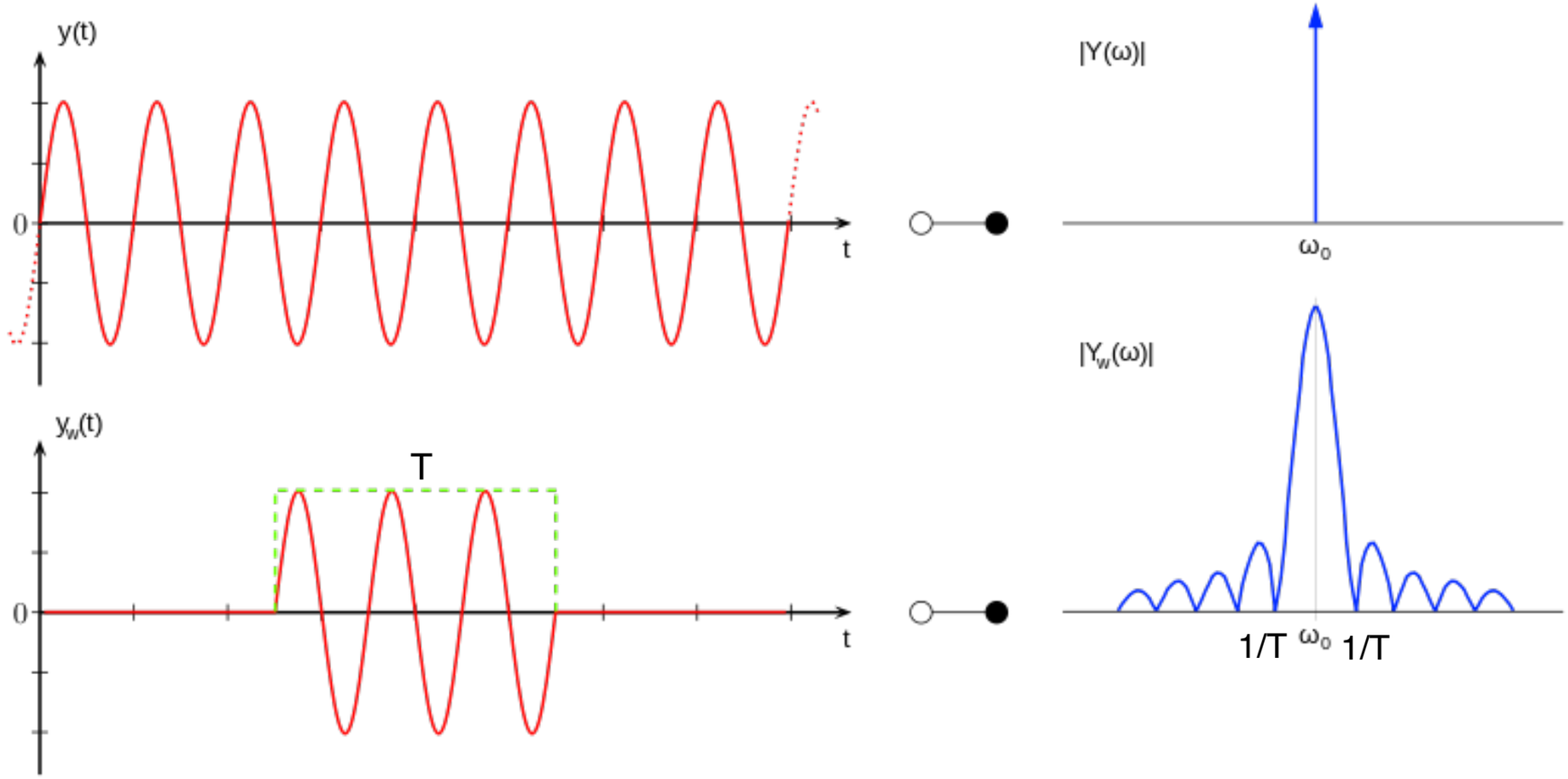


$$FT(B_T) = \int_{-\infty}^{+\infty} B_T(t) e^{-i\omega t} dt = \int_{-T/2}^{+T/2} e^{-i\omega t}\, dt \propto \frac{\sin(\pi f T)}{\pi f T} =$$

$$= \mathrm{sinc}(\pi f T) = \mathrm{sinc}(\omega T / 2)$$



Spectral leakage



- Resolving power in frequency domain is related to maximum duration in time domain:

$$\Delta f \geq \frac{1}{T}\left(= \frac{1}{N\Delta t}\right)$$

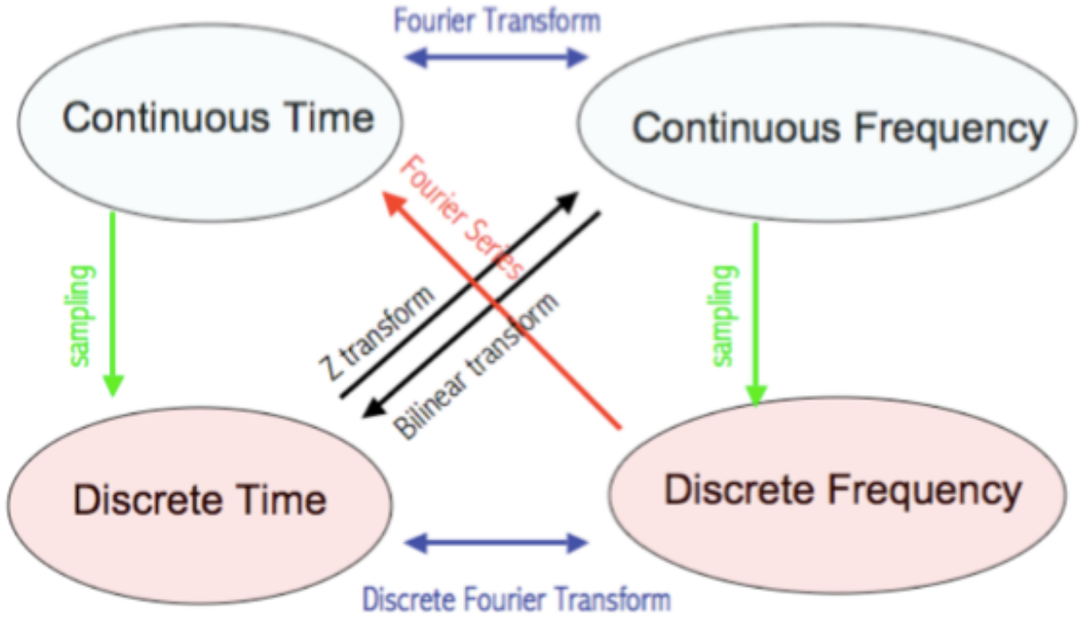
- Resolving power in time domain decides maximum resolvable frequency:

$$\Delta f \geq \frac{1}{2\Delta t}$$

$$\Delta f \Delta t \geq \frac{1}{4\pi}$$

https://www.youtube.com/watch?v=MBnnXbOM5S4

Various spaces and transforms



Signal type	Continuous time	Discrete time	Transform Domain
Finite duration	Laplace	z	Continuous complex frequency (s-plane)
Finite duration	Fourier	Discrete-time Fourier (DTFT)	Continuous real frequency
Periodic	Fourier Series	Discrete Fourier Series (DFS)	Discrete real frequency