5. (In)equalities of first and second degree (part one)

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First-degree equations

An *equation of the first degree* is an equation of the form

$$ax+b=0 \Leftrightarrow ax=-b \quad (a,b\in\mathbb{R}), \tag{1}$$

where x is the unknown.

The following cases may occur:

1. $a\neq 0$: There is a unique solution of (1), namely $x=-\frac{b}{a}$;
2. $a=0$: Two subcases may occur:
   - 2.1 $b=0$: Every number $x\in\mathbb{R}$ is a solution of the equation (1);
   - 2.2 $b\neq 0$: There is no solution of the equation (1).

Example

Consider the following *equation of the first degree*:

$$\frac{x^2 - 3}{x + 1} = x + \frac{2x}{x + 1}.$$

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Example

Consider the following *equation of the first degree*:

$$\frac{x^2-3}{x+1} = x + \frac{2x}{x+1}.$$

We must impose the condition $x+1 \neq 0 \Leftrightarrow x \neq -1$.

Multiplying both sides by $x+1 \neq 0$ we get

$$x^2 - 3 = x^2 + x + 2x \Leftrightarrow 3x = -3 \Leftrightarrow x = -1.$$

Since $x = -1$ is not allowed, there is no solution to the equation.

Second-degree equation

A *quadratic* (or *second-degree*) *equation* is an equation of the form

$$ax^2 + bx + c = 0 \quad (a, b, c \in \mathbb{R}, a \neq 0). \tag{2}$$

We look for the solutions of (2) in the unknown x.

The number of the solutions depends on the real number $\Delta = b^2 - 4ac$ which is said to be the *discriminant* of (2).

We have that

$$\begin{aligned} ax^2 + bx + c = 0 \quad &\Leftrightarrow a\left(x^2 + \tfrac{b}{a}x + \tfrac{c}{a}\right) = 0 \\ &\Leftrightarrow x^2 + 2\tfrac{b}{2a}x + \tfrac{b^2}{4a^2} - \tfrac{b^2}{4a^2} + \tfrac{c}{a} = 0 \\ &\Leftrightarrow \left(x + \tfrac{b}{2a}\right)^2 = \tfrac{b^2-4ac}{4a^2} \\ &\Leftrightarrow (x + \tfrac{b}{2a})^2 = \tfrac{\Delta}{4a^2}. \end{aligned}$$

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The following cases may occur

1. $\Delta < 0$: There is no real solution of the equation (2);
2. $\Delta = 0$: There is a unique solution $\bar{x} = -\frac{b}{2a}$ of the equation (2), which can be now written in the form

$$a(x-\bar{x})^2 = 0;$$

3. $\Delta > 0$: There are two distinct solutions

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} < \frac{-b+\sqrt{\Delta}}{2a} = x_2$$

of the equation (2), which can be now written in the form

$$a(x-x_1)(x-x_2) = 0.$$

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Example

Let us consider the following equation:

$$\frac{8-x^2}{4}+\frac{x(x-1)}{3}-1=0.$$

or, equivalently,

$$\frac{3(8-x^2)+4x(x-1)-12}{12}=0 \quad \Leftrightarrow \quad x^2-4x+12=0.$$

We then compute the discriminant, which is given by

$$\Delta=-32<0:$$

being negative, the equation has no real solutions.

Example

Let us consider now the second-degree equation

$$4(5-3x)^2 = 0 \quad \Leftrightarrow \quad (5-3x)^2 = 0$$

which is satisfied if and only if $5-3x=0$, that is

$$x = \frac{5}{3},$$

the unique real solution of the equation.

Remark

In particular, whenever a second-degree equation can be written in the form $a(x-\bar{x})^2$, for some $a \in \mathbb{R}$, $a \neq 0$, and $\bar{x} \in \mathbb{R}$, its unique real solution is given by $x = \bar{x}$.

Example

Let us consider the second-degree equation

$$3x^2 - 3(1+\sqrt{2})x + \sqrt{18} = 0$$

or equivalently

$$3\Big(x^2 - (1+\sqrt{2})x + \sqrt{2}\Big) = 0.$$

The discriminant is

$$\Delta = (1+\sqrt{2})^2 - 4\sqrt{2} = (\sqrt{2}-1)^2 > 0.$$

Being positive, the equation has two distinct real solutions, $x_1 = 1, \quad x_2 = \sqrt{2}$.

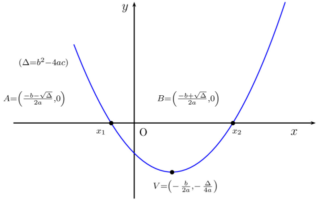
Then,

$$a(x-x_1)(x-x_2) = 3(x-1)(x-\sqrt{2}) = 0\,.$$

Quadratic inequalities

We refer to a *quadratic inequality* (or *inequality of the second-degree*) as an inequality belonging to one of the following types (with $a, b, c \in \mathbb{R}$, $a \neq 0$):

1. $ax^2 + bx + c > 0$;
2. $ax^2 + bx + c \geq 0$;
3. $ax^2 + bx + c < 0$;
4. $ax^2 + bx + c \leq 0$.



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By using the graphical interpretation of the parabola, we have

Solutions to $ax^2+bx+c>0$			
$ax^2+bx+c>0$	$\Delta<0$	$\Delta=0$	$\Delta>0$
$a>0$	$\mathbb{R}$	$\mathbb{R}\setminus\{\bar{x}\}$	$]-\infty,x_1[\cup]x_2,+\infty[$
$a<0$	$\emptyset$	$\emptyset$	$]x_1,x_2[$

Solutions to $ax^2+bx+c\geq 0$			
$ax^2+bx+c\geq 0$	$\Delta<0$	$\Delta=0$	$\Delta>0$
$a>0$	$\mathbb{R}$	$\mathbb{R}$	$]-\infty,x_1]\cup[x_2,+\infty[$
$a<0$	$\emptyset$	$\{\bar{x}\}$	$[x_1,x_2]$

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Example

Consider the following inequality

$$x - \sqrt{1 - x^2} \geq 0 \Leftrightarrow x \geq \sqrt{1 - x^2}.$$

We must first impose the condition $1 - x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1$.

Observe that we cannot "square" the inequality $x \geq \sqrt{1 - x^2}$ since it is not guaranteed that both sides are positive.

If $0 \leq x \leq 1$, then we square both sides in order to get

$$x^2 \geq 1 - x^2 \Leftrightarrow (x^2 \geq \frac{1}{2}) \Leftrightarrow \frac{1}{\sqrt{2}} \leq x \leq 1\,.$$