Mixed-Strategy Nash Equilibrium

Example: Matching Pennies

- Each player has a penny and must choose whether to display it with Tail or Head.
  - If the two pennies match then player 2 pays a penny to player 1;
  - if the pennies do not match, then player 2 receives a penny from player 1.

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- No Nash equilibrium (in pure strategies) i.e. there is no pair of strategies where players 1 and player 2 do not want to change:
  - If players' strategies match - (Head, Head) or (Tail, Tail) then Player 2 prefers to switch (she/he has to pay)
  - If players' strategies do not match (Head, Tail) or (Tail, Head) then Player 1 prefers to switch (she/he has to pay)

- The characteristic of Matching Pennies is that each player wants to outguess the other.
- There are other similar situations where each player wants to outguess the other(s): poker, football, battle,……
  - Poker: how often to bluff
  - Football: penalty, kick right, center or left
  - Tennis: serve's direction
  - Battle: attackers want to surprise the defenders, defenders want to anticipate the attack.
- In situations where players want outguess the other, there is no Nash equilibrium in pure strategies

Definition of mixed strategy

- A **mixed strategy** of player *i* is a probability distribution over the strategies in S_i
- The strategies in S_i are called *pure strategies*

  Note: in static games of complete information strategies are the actions the player could take.

Definition of mixed strategy

Example 1: Matching Pennies

- S_i = {*Head*, *Tail*}
- $(q, 1-q)$ is a mixed strategy where:
  - q is the probability to play *Head* and
  - $1-q$ is the probability to play *Tail* where $0 \leq q \leq 1$
- Note: $(0, 1)$ is the pure strategy *Tail and* $(1, 0)$ is the pure strategy *Head*

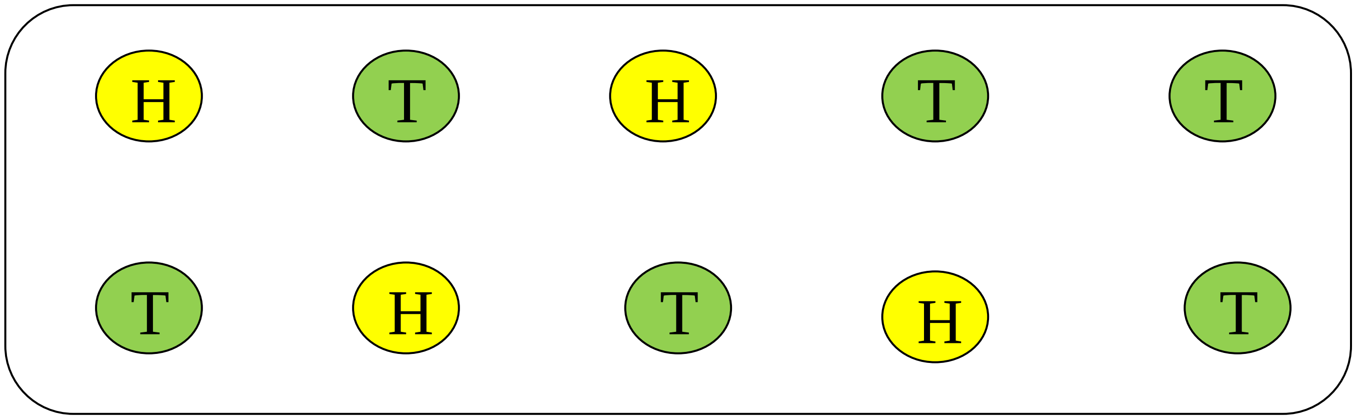
- But what means to play a mixed strategy?

Suppose that Player 1 wants to play:

- Head by probability 0.4
- Tail by probability 0.6

i.e. the mixed strategy $p_1 = (0.4, 0.6)$

The action he will play, it is chosen at random according to the distribution $(0.4, 0.6)$, for example choosing a ball from a box where 4 balls are marked by H (Head) and 6 are marked by T(tail)



		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

- $S_2 = \{L, C, R\}$
- $p_2 = (p_{2L}, p_{2C}, p_{2R})$ is a mixed strategy of Player 2 where:
  - p_{2L} is the probability to play *L*,
  - p_{2C} is the probability to play *C* and
  - p_{2R} is the probability to play *R*
- $p_2 = (q, r, 1 - q - r)$
- $0 \leq q \leq 1; 0 \leq r \leq 1; 0 \leq q + r \leq 1$
- Note: *(0, 0, 1)* is the pure strategy *R*

		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

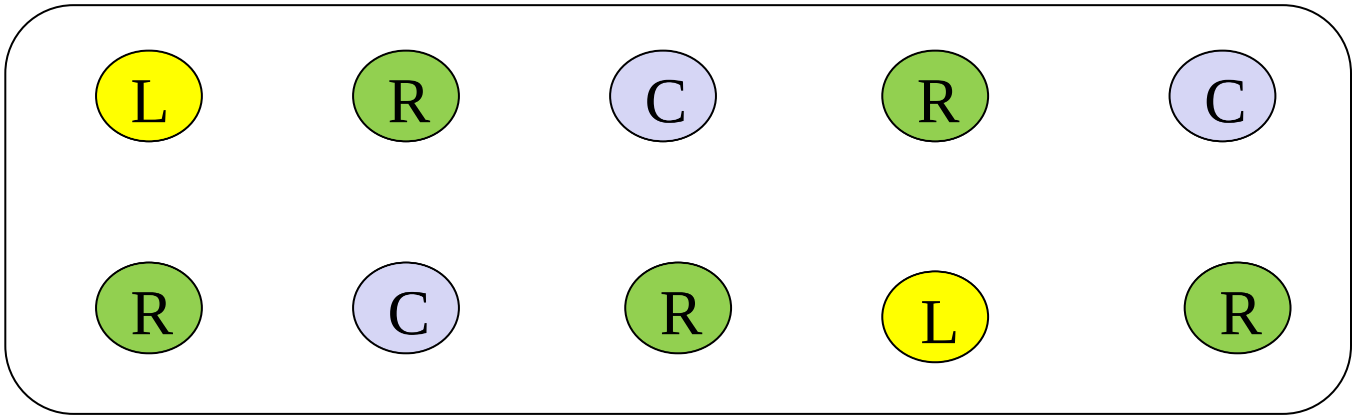
- $S_1 = \{T, Y, Z, B\}$
- $p_1 = (p_{1T}, p_{1Y}, p_{1Z}, p_{1B})$ is a mixed strategy of Player 1 where:
  - p_{1T} is the probability to play *T*,
  - p_{1Y} is the probability to play *Y*
  - p_{1Z} is the probability to play *Z*
  - p_{1B} is the probability to play *B*
- $p_1 = (q, r, z, 1 - q - r - z)$
- $0 \leq q, r, z \leq 1;\ 0 \leq q + r + z \leq 1$
- Note: *(0, 0, 1, 0)* is the pure strategy *Z*

Suppose that Player 2 wants to play:

- L by probability 0.2
- C by probability 0.3
- R by probability 0.5

i.e. the mixed strategy $p_2 = (0.2, 0.3, 0.5)$

The action he will play, it is chosen at random according to the distribution$(0.2, 0.3, 0.5)$, for example choosing a ball from a box where 2 balls are marked by L, 3 are marked by C and 5 are marked by R



Mixed strategy for player *i* in the normal form game $G = \{S_1, \dots \dots S_n;\ u_1 \dots u_n\}$

- Suppose $S_i = \{s_{i1}, \dots s_{ij}, \dots\ s_{iK}\}$ (player *i* has *K* strategies)
- A mixed strategy for player *i* is a probability distribution

$$p_i = (p_{i1}, p_{i2}, \dots, piK)$$

where p_{ij} is the probability that player *i* will play strategy s_{ij}, $j \in \{1, 2, \dots, K\}$

i. $0 \leq p_{ik} \leq 1, k \in \{1, 2, \dots \dots K\}$

ii. $p_{i1} + p_{i2} + \cdots + p_{iK} = 1$

Mixed strategies and dominated strategies

- If a strategy s_i is strictly dominated, then
  - there is no player i's belief such that to play s_i is optimal.

- <u>The converse is true only if we allow for mixed strategies</u>:
  - if there are no beliefs such that for player i is optimal to play s_i then
  - there exists another strategy that strictly dominates s_i.

Consider the following game:

		Player 2	
		L	R
Player 1	T	3, -	0, -
	M	0, -	3, -
	B	1, -	1, -

Considering only pure strategies:

B is not dominated and never is a best response:

If player 1 believes that player 2 will play L, the best response is T

If player 1 believes that player 2 will play R, the best response is M

- Here strategy B is dominated by a mixed strategy

		Player 2	
		L	R
Player 1	T	3, -	0, -
	M	0, -	3, -
	B	1, -	1, -

Now we allow for mixed strategies;

(q, 1- q) denotes the belief that player 1 holds about the player 2's play:

Player 1 believes that Player 2 plays:

- L by probability q and
- R by probability $1-q$

Given these beliefs, player 1's expected values are:

$$E_1(T) = 3q;\ E_1(M) = 3(1-q);\ E_1(B) = 1$$

		Player 2	
		L	R
Player 1	T	3, -	0, -
	M	0, -	3, -
	B	1, -	1, -

$$E_1(T) = 3q;\ E_1(M) = 3(1-q);\ E_1(B) = 1$$

for $q \geq 0.5$ the player 1's best response is T

$$E_1(T) = 3q \geq 1.5; E_1(M) = 3(1-q) \leq 1.5; E_1(B) \ = \ 1$$

for $q \leq 0.5$ the player 1's best response is M

$$E_1(T) = 3q \leq 1.5; E_1(M) = 3(1-q) \geq 1.5; E_1(B) \ = \ 1$$

Yet B is not strictly dominated by T or M

The key is that ***strategy B is dominated by a mixed strategy***:

$$p_1 = (p_{1T}, p_{1M}, p_{1B}) = (0.5, 0.5, 0)$$

$$E_1(p_1) = 0.5 \cdot 3 \cdot q + 0.5 \cdot 3 \cdot (1-q) = 1.5 > 1$$

The following game show that a pure strategy can be a best response to a mixed strategy even if the pure strategy is not a best response to a pure strategy

		Player 2	
		L	R
Player 1	T	3, -	0, -
	M	0, -	3, -
	B	2, -	2, -

B is never a best response to a **pure strategy** of player 2.

But is a best response to a player 2's mixed strategy $p_2 = (p_{2L}, p_{2R}) = (q, 1 - q)$ where $1/3 \leq q \leq 2/3$

But is a best response to a player 2's mixed strategy $p_2 = (p_{2L}, p_{2R}) = (q, 1-q)$ where $1/3 \leq q \leq 2/3$

Given p_2, player 1's expected values are:

$E_1(B) = 2 \quad E_1(T) = 3 \cdot q \; E_1(M) = 3 \cdot (1-q)$

B is a best response if

$E_1(B) \geq E_1(T)$ i.e. $2 \geq 3 \cdot q \rightarrow q \leq \frac{2}{3}$

and

$E_1(B) \geq E_1(M)$ i.e. $2 \geq 3 \cdot (1-q) \rightarrow q \geq \frac{1}{3}$

Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

$p_1 = (r, 1 - r)$ *where r is the probability that player 1 chooses Head,*

$p_2 = (q, 1 - q)$ *where q is the probability that player 2 chooses Head*

Player 1's expected payoff is:

$$E_1(r, 1 - r) =$$

$$= rq - r(1 - q) - (1 - r)q + (1 - r)(1 - q) =$$

$$= r(4q - 2) + 1 - 2q$$

Player 1's expected payoff is:

$$E_1(r, 1-r) = r(4q-2) + 1 - 2q$$

It is increasing in r if $(4q - 2) > 0$ *i.e.* $q > 0.5$

- *In this case the best response of player 1 is* $p_1 = (1,0)$

It is decreasing in r if $(4q - 2) < 0$ *i.e.* $q < 0.5$

- *In this case the best response of player 1 is* $p_1 = (0,1)$

It is equal *0* and constant for $q = 0.5$

- *In this case the best response of player 1 is*

$$p_1 = (r, 1-r)\ \forall r[0, 1]$$

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- *r:* Probability that 1 chooses Head
- *q:* Probability that 2 chooses Head

$$r(q) = \begin{cases} 1 & \text{if } q > 1/2; \\ 0 & \text{if } q < 1/2; \\ [0,1] & \text{if } q = 1/2 \end{cases}$$

$$q(r) = \begin{cases} 0 & \text{if } r > 1/2; \\ 1 & \text{if } r < 1/2 \\ [0,1] & \text{if } r = 1/2 \end{cases}$$

$$r(q) = \begin{cases} 1 & \text{if } q > 1/2; \\ 0 & \text{if } q < 1/2; \\ [0,1] & \text{if } q = 1/2 \end{cases}$$

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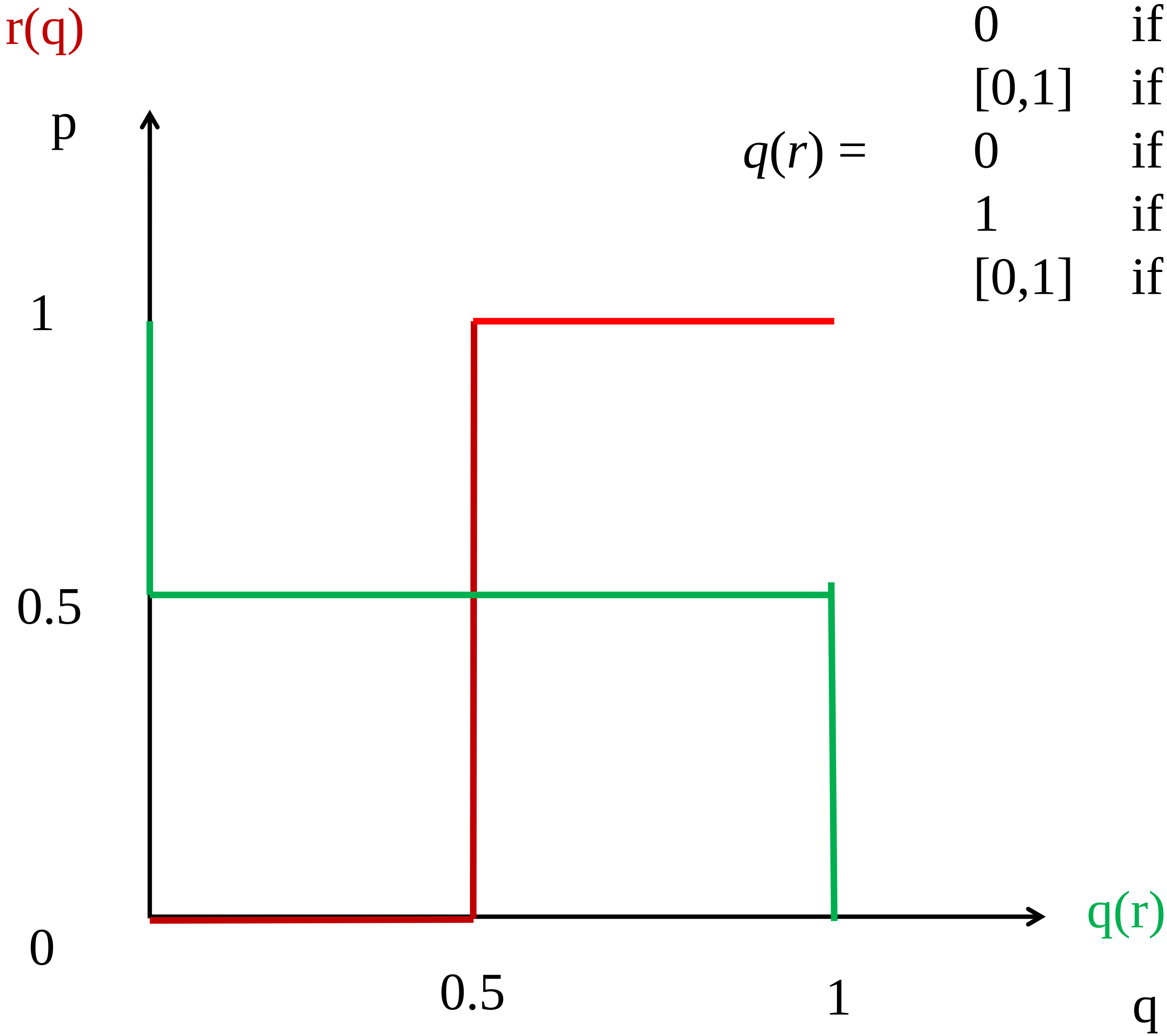


Note that player 1's strategy (0.5, 0.5) is a best response to the player 2' strategy (0.5, 0.5) and

player 2's strategy (0.5, 0.5) is a best response to the player 1's strategy (0.5, 0.5)

Then player 1 plays (0.5, 0.5) and player 2 plays (0.5, 0.5) is a Nash equilibrium in mixed strategies

Definition:

In a normal form game $G = (S_1, \ldots S_n; u_1, \ldots, u_n\}$ the mixed strategies $(p_1^*, \ldots p_n^*)$ are a Nash equilibrium if each player's mixed strategy is a best response to the other players' strategies.

Battle of the Sexes

		Player 2	
		Ball	Theatre
Player 1	Ball	2,1	0,0
	Theatre	0,0	1,2

$p_1 = (r, 1 - r)$ *where r is the probability that player 1 chooses Ball*

$p_2 = (q, 1 - q)$ *where q is the probability that player 2 chooses Ball*

Player 1's expected payoff is:

$E_1(r, 1 - r) = 2\, r\, q + (1 - r)(1 - q) = r\,(3q - 1) + 1 - q$

It is increasing in r if $(3\, q - 1) > 0$ *i.e.* $q > 1/3$ $\rightarrow BR_1$ *is* $(1, 0)$

It is decreasing in r if $(3\, q - 2) < 0$ *i.e.* $q < 1/3$ $\rightarrow BR_1$ *is* $(0, 1)$

It is equal 0 and constant for $q = 1/3$ $\rightarrow BR_1$ *is* $(r, 1 - r) \forall r \in [0,1]$

		Player 2	
		Ball	Theatre
Player 1	Ball	2,1	0,0
	Theatre	0,0	1,2

Consider player 2

$E_2(q, 1 - q) = q\ (3r - 2) + 2 - 2r$

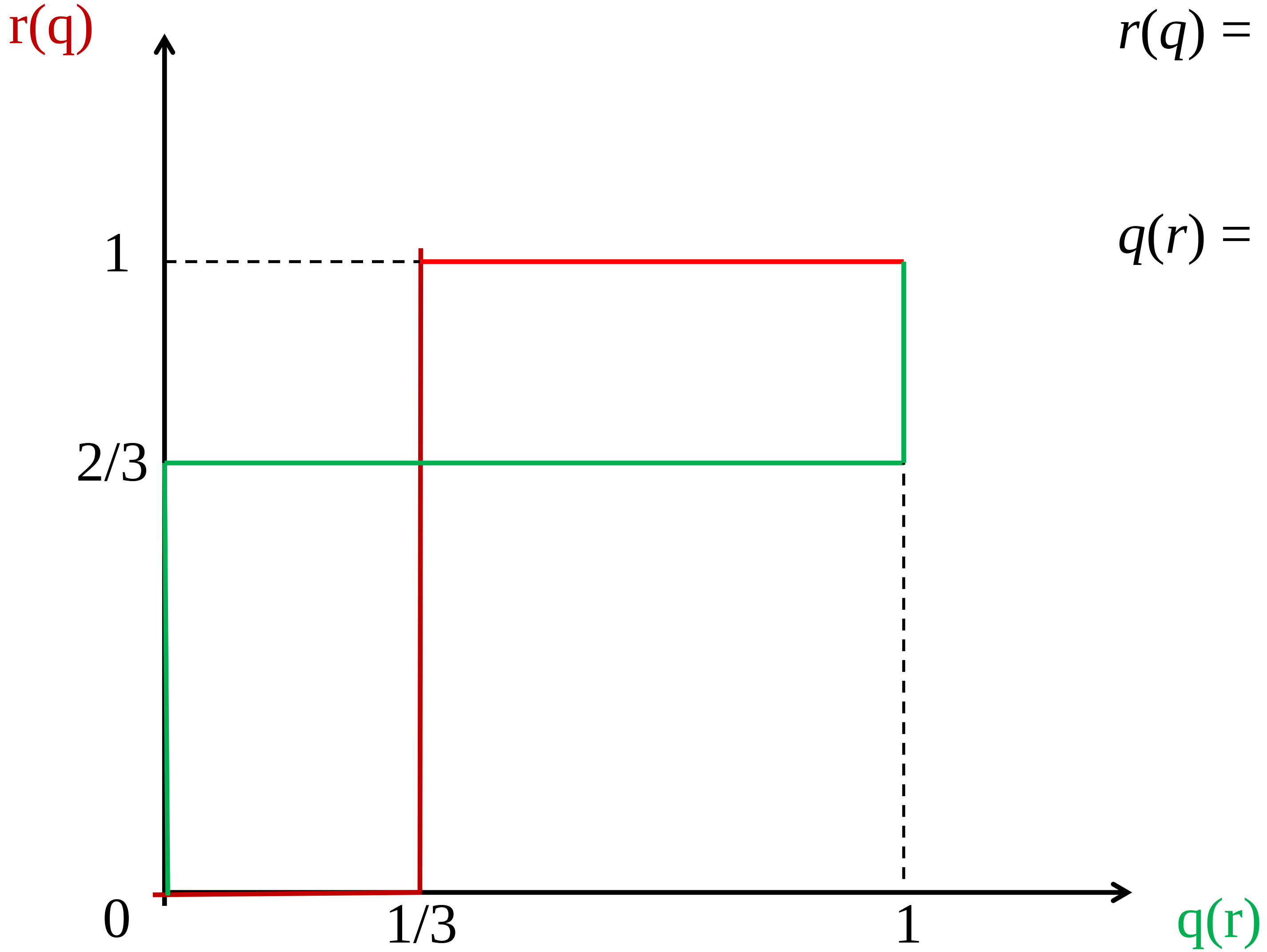
It is increasing in q if $(3r - 2) > 0$ i.e. $r > 2/3$ → BR_2 *is (1, 0)*

It is decreasing in q if $(3r - 2) < 0$ i.e. $r < 2/3$ → BR_2 *is (0, 1)*

It is equal *0* and constant for $r = 2/3$ → BR_2 *is* $(q, 1 - q) \forall q \in [0,1]$

$r(q) =$	1	if $q > 1/3$;
	0	if $q < 1/3$;
	[0,1]	if $q = 1/3$

$q(r) =$	1	if $r > 2/3$;
	0	if $r < 2/3$;
	[0,1]	if $r = 2/3$



$$r(q) = \begin{cases} 1 & \text{if } q > 1/3; \\ 0 & \text{if } q < 1/3; \\ [0,1] & \text{if } q = 1/3 \end{cases}$$

$$q(r) = \begin{cases} 1 & \text{if } r > 2/3; \\ 0 & \text{if } r < 2/3; \\ [0,1] & \text{if } r = 2/3 \end{cases}$$

Equilibria:

$(r, q) = (1,1) = $ (Ball, Ball), $\quad p_1 = (1,0), p_2 = (1,0)$

$(r, q) = (0,0) = $ (Theatre, Theatre) $\quad p_1 = (0,1), p_2 = (0,1)$

$(r, q) = (2/3,1/3) \quad p_1 = (2/3,1/3), p_2 = (1/3,2/3)$

Characterization of mixed-strategy Nash equilibria

Proposition: $(p_1*,... p_n*)$ is a mixed-strategy Nash equilibrium. If and only if the following conditions are satisfied:

1) each action s_i that is played by i with strictly positive probability according to p_i* yields **the same expected payoff** to i as strategy p_i*
2) every action s_i' that is played by i with probability 0 according to p_i* yields **at most the same expected payoff** to i as strategy p_i*

assuming, in both cases, that other players play as predicted in the Nash equilibrium $(p_1*,... p_n*)$

Useful tips for finding mixed-strategy Nash equilibria

1) Consider a player i, take a subsets S_i' of its strategies and assume that only these strategies are played by a strictly positive probability
2) Look for the other players' strategies that allow to satisfy conditions 1) and 2), i.e.
   a) The expected payoffs to play each one of the strategies in S_i' are equal to each other:
   $$E_i(s_j) = E_i(s_w) \; \forall s_j, s_w \in S_i'$$
   b) The expected payoffs to play each one of the strategies that are not in S_i' are not greater than the expected payoff of the strategies in S_i':
   $$E_i(s_j) \leq E_i(s_w) \; \forall s_j \in S_i / S_i', s_w \in S_i'$$
3) Repeat this procedure for all possible strategies' subsets of player *i*
4) Repeat for all players

		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,4
	B	1,5	4,1

No equilibrium in pure strategies.

There is no equilibrium where player 1 chooses B with strictly positive probability. T strictly dominates B, so whatever player 2 does, player 1 can increase its expected payoff by playing T instead of B. Then $p_{1B} = 0$.

That leaves player 1 choosing among T and M.

		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,4
	~~B~~	~~1,5~~	~~4,1~~

That leaves player 1 choosing among T and M.

Let be $p_{1T} = t$ and $p_{2L} = l$

To play T and M, both with strictly positive probability requires:

$$E_1(T) = E_1(M) \rightarrow 2l + 5(1-l) = 3l + 1(1-l) \rightarrow l = 4/5$$

To play L and R, both with strictly positive probability requires:

$$E_2(L) = E_2(R) \rightarrow 3t + 2(1-t) = 4(1-t), \rightarrow t = 2/5$$

Nash Equilibrium:

$$((p_{1T}, p_{1M}, p_{1B}), (p_{2L}, p_{2R})) = ((2/5, 3/5, 0), (4/5, 1/5))$$

Existence of Nash equilibrium in a 2 x 2game

Consider a generic 2 x 2 game

If there is a dominant strategy then an equilibrium always exists

Consider a game with no strictly dominant strategy and no equilibria in pure strategies

		Player 2	
		Left	Right
Player 1	Up	X, -	Y, -
	Down	Z, -	W, -

Let be $X \geq Z$ and $W \geq Y$ with at least one strict inequality

Let q be the probability player 1 plays Up and p be the probability player 2 plays Left

$E_1(q, 1-q) = p\, q\, X + p\,(1-q)\, Z + (1-p)\, q\, Y + (1-p)(1-q)\, W$

$= q\,(p(X - Z) + (1 - p)(Y - W)) + pZ + (1 - p)\, W$

$E_1(q, 1\text{-}q) = \; q\,(p(X – Z) + (1 – p)(Y – W)\,) + pZ + (1 – p)\,W$

It is constant in q if $p(X - Z) + (1 - p)(Y - W) = 0$

i.e. for $p = \frac{W-Y}{X-Z+W-Y}$

The assumption "*$X \geq Z$ and $W \geq Y$ with at least one strict inequality*" ensures the existence of p

In such a case the best response of player 1 is to play any mixed strategy, i.e.

→ q(p)□[0, 1]

Repeating the same reasoning for player 2 we find that exists a value of q such that p(q)□[0, 1]

Then it is straightforward that an equilibrium exists.

This result is generalized for all finite game by the following theorem

Nash's Existence Theorem

In the *n – player normal form game* if *n* is finite and every player has a finite number of strategies then there exist at least one Nash equilibrium.

(proof in the book)