2. Set theory
(part one)

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Definition

A *set* is a collection of objects, called *elements* of the set.

We shall denote a set by capital letters (e.g. $A, B, C, \ldots$) and the elements of some set by small letters (e.g. $a, b, c, \ldots$).

Example

$$A = \{a, b, c\}\,.$$

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We have that a, b, c belong to A.

We use the symbol "$\in$" which means "*belongs to*", so that when we write

$$a \in A$$

we mean that a is an element of A.

Example

If we have $A = \{a, b, c\}$. Then, we can say that $a \in A$, but $f \notin A$.

We can characterize a set by a property. We write $\{x : P(x)\}$ as "the set of all the elements x which satisfy property $P(x)$".

Remark

There is a unique set which has no elements (i.e., void). Such a set, the empty set*, is denoted by* $\emptyset$*.*

Example

- $A = \{x :$
  x belongs to the first four letters of the Greek alphabet$\}$
  $= \{\alpha, \beta, \gamma, \delta\}$;
- $B = \{x : x$ is a desert in Italy$\} = \emptyset$;
- $C = \{x : x$ is a perfect square number$\}$
  $= \{1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$.

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Quantifiers

A very useful notation concerns the so called *quantifiers*. In particular, we introduce

i) the *existential quantifier* $\exists$ to be read “exists”;

ii) the *universal quantifier* $\forall$ to be read “for all” or equivalently “for every”.

There is another quantifier $\exists!$ to be read as “there exists and it is unique”.

Example

Let $S = \{1, 2, 3, 4\}$. Then

- $\exists x \in S : x < 3$. Indeed, the set S contains the elements 1 and 2, which are both strictly less than 3.
- $\exists! x \in S : x > 3$. Indeed, the set S contains only one element, 4, which is strictly greater than 3.
- $\forall x \in S : x < 5$. In fact, every element belonging to the set S is strictly less than 5. Thus, $\nexists x \in S : x \geq 5$.

Inclusion

Definition

If for two sets A, B it happens that all the elements of A are also elements of B, then we say that "A is a subset of B" and we write

$$A \subseteq B.$$

If it is not true that $A \subseteq B$ then we simply write

$$A \not\subseteq B.$$

In this case there exists an element of A which does not belong to B.

We say that $\subseteq$ is the *inclusion relation*. When $A \subseteq B$, we may also say that "A is contained in B".

Two sets A, B are equal if they have the same elements. So, the previous definition of inclusion naturally offers a satisfactory definition of "equality" referred to sets.

Definition

Two sets A, B are *equal*, $A = B$, if $A \subseteq B$ and $B \subseteq A$.

If for two sets A, B we have that $A \subseteq B$ and $B \not\subseteq A$ then we say that "A is a proper subset of B" and we write

$$A \subsetneq B.$$

More explicitly, $A \subsetneq B$ if every element of A is an element of B and there is some element of B not belonging to A. Or also,

$$\forall a \in A \Rightarrow a \in B \quad \text{and} \quad \exists b \in B : b \notin A.$$

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Example

Let

$$\begin{aligned}
&A = \{1, 2, 3, 4, 5, 6\}\\
&B = \{x : x \textit{ is an even number}\}\\
&C = \{x : x \textit{ is an even or an odd number}\}\\
&D = \{x : x \textit{ is a natural number}\}.
\end{aligned}$$

Then $A \subsetneqq D$, $A \not\subseteq B$, $B \not\subseteq A$, *and* $C = D$.

Union and intersection

Definition

Given two sets A, B, we define the *union* $A \cup B$ and the *intersection* $A \cap B$ as follows:

i) $A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}$;

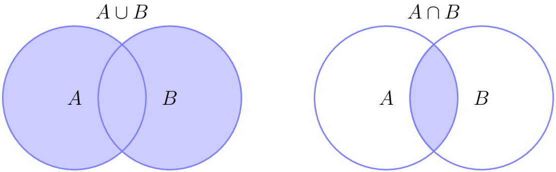
ii) $A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}$.

Example

Let $A = \{1,2,3,4\}$ and $B = \{1,2,5,7\}$ be two sets. Then

- $A \cup B = \{1,2,3,4,5,7\}$;
- $A \cap B = \{1,2\}$.

A set may be represented by means of a *Venn's diagram* (the colored zone denotes the set considered).



Definition

Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$ (i.e., they are empty intersecting).

Example

Let $A = \{1, 2, 3\}$ and $B = \{5, 7\}$ be two sets.
Then

$$A \cap B = \emptyset.$$

Difference and complement

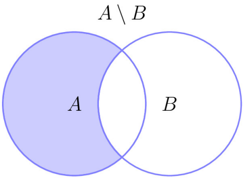
Definition

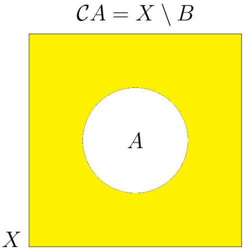
If A, B are any two sets, then we define the *difference* "A minus B" as follows:

$$A \setminus B = \{x \in A : x \notin B\}.$$

If we agree on the fact that all the sets referred to are subsets of a unique set X (called the *universe*), then we define the *complement* of a set A (w.r.t. X) as the set

$$\mathcal{C}A = X \setminus A = \{x \in X : x \notin A\}.$$





Example

Let $X = \mathbb{R}$ be the set of all real numbers,

$$A = \{x \in X : x \leq -8\}, \quad B = \{-8\}.$$

Thus,

$$\complement A = \{x \in X : x > -8\}, \quad A \setminus B = \{x \in X : x < -8\}.$$

The most relevant properties of the complement are perhaps represented by the *De Morgan's laws*.

Proposition

For any two sets A and B the following equalities hold:

P.1 $\mathcal{C}(A \cup B) = \mathcal{C}A \cap \mathcal{C}B$;

P.2 $\mathcal{C}(A \cap B) = \mathcal{C}A \cup \mathcal{C}B$.

Therefore the complement of the union (respectively, intersection) is the intersection (respectively, union) of the complements.

Definition

For any two sets A, B we may define the *Cartesian product* $A \times B$ as the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, i.e.

$$A \times B = \{(a, b) : (a \in A) \text{ and } (b \in B)\}.$$

It should be noted that if A and B are two sets consisting of m and n elements respectively, then the Cartesian product $A \times B$ consists of $m \cdot n$ elements.

Example

Let $A = \{1, 2\}$ and $B = \{4, 5\}$ be two sets. Then

- $A \times B = \{(1,4), (1,5), (2,4), (2,5)\}$;
- $B \times A = \{(4,1), (4,2), (5,1), (5,2)\}$.

Power set

Definition

If A is a set then the *power set* $\mathcal{P}(A)$ of A is the set whose elements are all the subsets of A.

For example, if we consider the set $A = \{a, b, c\}$ then we get

$$\mathcal{P}(A) = \{\emptyset, A, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

In this case we have that A is a set consisting of 3 elements and $\mathcal{P}(A)$ consists of $2^3 = 8$ elements.

Proposition

If A is a set of n elements, then $\mathcal{P}(A)$ is a set of 2^n elements.

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