3. Real numbers (part one)

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Natural numbers

We denote by $\mathbb{N}$ the set of the *natural numbers*, i.e.,

$$\mathbb{N} = \{0, 1, 2, \ldots\}.$$

The natural (total) order $\leq$ on $\mathbb{N}$ is defined as usual.
In addition, the operations of addition $+$ and multiplication $\cdot$ are defined as usual.

Integer numbers

The introduction of the set of the *integers* $\mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}$ has an algebraic motivation.

Indeed it is not true in general that an equation of the form

$$a + x = b \qquad (a, b \in \mathbb{N})$$

has a solution $x \in \mathbb{N}$.

For example, the equation $3 + x = 2$ has no solution in $\mathbb{N}$.

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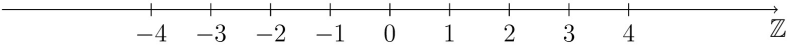
If we now consider the set $\mathbb{Z}$, we have that an equation of the form

$$a + x = b \qquad (a, b \in \mathbb{Z})$$

has always a (unique) solution $x = b - a \in \mathbb{Z}$.



Representation of the integers of the number line

Rational Numbers

Let us now consider the set $\mathbb{Q} = \left\{ \frac{h}{k} : h \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\} \right\}$ of the *rational numbers*.

Also the extension from $\mathbb{Z}$ to $\mathbb{Q}$ has an algebraic motivation, since it is not true in general that an equation of the form

$$a \cdot x = b \qquad (a, b \in \mathbb{Z}, a \neq 0)$$

has a solution $x \in \mathbb{Z}$.

For example, the equation $2x = 3$ has no solution in $\mathbb{Z}$.

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If we now consider the set $\mathbb{Q}$, an equation of the form

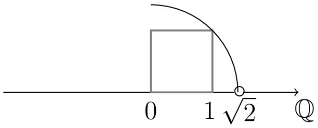
$$a \cdot x = b \qquad (a, b \in \mathbb{Q}, a \neq 0)$$

has a (unique) solution $x = \frac{b}{a} \in \mathbb{Q}$.

The inadequacy of the set $\mathbb{Q}$ is of *geometric nature*.

Indeed, the *rational number line* (i.e., the number line where only the rational numbers are represented) presents *holes*.

As an example, $\sqrt{2}$ is not a rational number, which means that for no rational number $r \in \mathbb{Q}$ it happens that $r^2 = 2$.



Irrationality of $\sqrt{2}$

Real Numbers

We have seen that the *rational (number) line* (i.e., the oriented straight line on which only points with rational are represented) has holes.

In order to guarantee that the number line is “without holes” according to the geometric intuition, the set $\mathbb{R}$ of the *real numbers* is introduced.

Real numbers

The set of *real numbers* $(\mathbb{R}, +, \cdot, \leq)$ has the following properties:

- $+$ and $\cdot$ are *associative*, and *commutative*;
- $\cdot$ *is distributive with respect to* $+$*:* $(a+b)\cdot c = a\cdot c + b\cdot c$ for all $a, b, c \in \mathbb{R}$;
- *Existence of the neutral element* 0 for the addition $+$ and the *neutral element* 1 for the multiplication $\cdot$;
- *Existence of the opposite:* for every $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$;
- *Existence of the reciprocal of every number different from zero:* for every $a \in \mathbb{R}$ such that $a \neq 0$ there exists an element $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$;

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- *Compatibility with the sum*: for all $a, b, c \in \mathbb{R}$, if $a \leq b \Rightarrow a + c \leq b + c$;
- *Compatibility with the product*: for all $a, b, c \in \mathbb{R}$, if $(a \leq b \text{ and } 0 \leq c) \Rightarrow a \cdot c \leq b \cdot c$;
- *Dedekind axiom*: given $A, B \subseteq \mathbb{R}$ with $A, B \neq \emptyset$, for all $a \in A$ and $b \in B$, if $a \leq b$ then $\exists c \in \mathbb{R}$ such that $a \leq c \leq b$.

Remark

Every real number corresponds to a unique point on the (real) line and vice versa.

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Intervals in $\mathbb{R}$

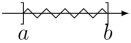
Definition

Let I be a non-empty subset of $\mathbb{R}$. Then I is said to be an *interval* if the following property holds:

$$\text{for every } x_1 < x_2 \; (x_1, x_2 \in I), \;\; \text{if} \;\; x_1 < x < x_2 \text{ then } x \in I.$$

So an interval in $\mathbb{R}$ is any subset of $\mathbb{R}$ which contains every point x between x_1 and x_2.

It can be shown that a subset *I* of $\mathbb{R}$ is an interval if and only if *I* belongs to one of the following types

	$a \leq x \leq b$	$[a, b]$
	$a < x < b$	$]a, b[$
	$a < x <\leq b$	$]a, b]$
	$a \leq x < b$	$[a, b[$
	$x \geq a$	$[a, +\infty[$
	$x > a$	$]a, +\infty[$
	$x \leq a$	$]-\infty, a]$
	$x < a$	$]-\infty, a[$
	$z \in \mathbb{R}$	$\mathbb{R}$