October 1

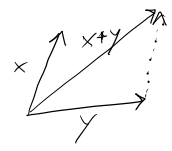
Def Let X be a Vector space on $K=\mathbb{R},\mathbb{C}$

A norm $\|\cdot\|$ on X is a function

$$X \ni x \longrightarrow \|x\| \in \mathbb{R} \qquad \|x\| \geq 0$$

s.t.

1) $\|x\| = 0 \iff x = 0$
2) $\|x+y\| \leq \|x\| + \|y\|$
3) $\|\lambda x\| = |\lambda| \, \|x\|$



$\lambda = u + iv \qquad |\lambda| = \sqrt{u^2+v^2}$

$(X, \|\cdot\|)$

$$d(x,y) = \|x-y\|$$

$$d : X \times X \longrightarrow [0, +\infty)$$

1) $d(x,y) = 0 \iff x = y$
2) $d(x,y) = d(y,x)$
3) $d(x,y) \leq d(x,z) + d(z,y) \qquad \forall x,y,z \in X.$

From (X,d) it is possible to define a Topology

Two metrics d_1 and d_2 are called equivalent if $\exists$

$C \geq 1$ s.t.

$$\frac{1}{C} d_1(x,y) \leq d_1(x,y) \leq C\, d_2(x,y) \qquad \forall x,y \in X$$

A metric space is complete if every Cauchy sequence $\{x_n\}$ in (X,d) is convergent.

When we have a normed space $(X, \|\cdot\|)$ the corresponding metric induces on X a topology which makes X into a TVS.

$$X \times X \longrightarrow X$$
$$(x,y) \longrightarrow x+y$$
$$K \times X \longrightarrow X$$
$$(\lambda, x) \longrightarrow \lambda x$$

A normed space is called a Banach space when it is complete.

$L^p(\Omega) = \{ f \in L^1_{loc}(\Omega) : \int_\Omega |f|^p \, dx < +\infty \}$

$\Omega \subseteq \mathbb{R}^d$

$1 \le p < +\infty$

$\|f\|_{L^p} = \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}}$

$p = 2$

$L^2(\Omega)$

$L^\infty(X, d\mu)$

$\|f\|_{L^\infty} = \sup \{ c \ge 0 : \mu(\{x : |f(x)| \ge c\}) > 0 \}$

There are all Banach spaces

$BC^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$

with the $|\cdot|_{L^\infty}$ $BC^0(\Omega)$ is a Banach space.

If $f \in BC^0(\Omega)$ then

$\|f\|_{L^\infty} = \sup_{x \in \Omega} |f(x)|$

Let $\{f_n\}$ be Cauchy sequence

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall n, m \ge N \Rightarrow |f_n - f_m|_{L^\infty} < \varepsilon$

and let us show that $f_n \to f$ in $BC^0(\Omega)$.

$\forall x \in \Omega$ and $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$

$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{L^\infty(\Omega)}$

$\Rightarrow \forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $n, m \ge N_\varepsilon \Rightarrow |f_n(x) - f_m(x)| \le \|f_n - f_m\|_{L^\infty} < \varepsilon$

$f(x) = \lim_{n \to +\infty} f_n(x)$

$f(x)$ is a measurable function

$|f_n(x) - f_m(x)| < \varepsilon$ taking limits for $n \to +\infty$

$|f(x) - f_m(x)| \le \varepsilon$

$\Rightarrow |f(x)| = |f(x) - f_m(x) + f_m(x)| \le$

$\le |f(x) - f_m(x)| + |f_m(x)|$

$|f(x)| \le \varepsilon + |f_m(x)| \le \varepsilon + \|f\|_{L^\infty}$

$f \in L^\infty(\Omega)$ $\quad m \ge N$

$f = \lim_n f_m$ in $L^\infty(\Omega)$

$|f(x) - f_m(x)| = \lim_{n \to +\infty} |f_n(x) - f_m(x)|$

$\le \limsup_{n \to +\infty} |f_n - f_m|_{L^\infty(\Omega)} \le \varepsilon$

$\forall x \in \Omega \quad |f - f_m|_{L^\infty(\Omega)} \le \varepsilon$

$\forall n > N_\varepsilon \cdot f = \lim_{n \to +\infty} f_n$ in $L^\infty(\Omega)$

Fix $n > N_\varepsilon$ and let $\delta > 0$ s.t

$|f_n(x) - f_n(x_0)| < \varepsilon \quad \forall x \in D_\Omega(x_0, \delta)$

$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$

$\le |f(x) - f_n(x)| + |f_n(x_0) - f(x_0)| + |f_n(x) - f_n(x_0)|$

$< 3\varepsilon$

$BC^0(\Omega)$ is Banach space

$C^{0,\vartheta}(\Omega)$ $\quad 0<\vartheta<1$

This is the set of functions f in $BC^0(\Omega)$ s.t. ~~$\exists C>0$~~ given f $\exists\ C>0$ s.t.

$$\frac{|f(x)-f(y)|}{|x-y|^\vartheta} \le C$$

$\forall\ x \neq y$.

$$[f]_{C^{0,\vartheta}(\Omega)} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\vartheta}$$

$C^{0,\vartheta}(\Omega)$

$$\|f\|_{C^{0,\vartheta}(\Omega)} = \|f\|_{L^\infty(\Omega)} + [f]_{C^{0,\vartheta}(\Omega)}$$

What I defined is a norm. Want to show that we have a Banach space

Let $\{f_n\}$ be a Cauchy sequence in $C^{0,\vartheta}(\Omega)$

then $\{f_n\}$ is also a $\|\ \|$ in $BC^0(\Omega)$

$$\|f_n - f_m\|_{L^\infty(\Omega)} \le \|f_n - f_m\|_{C^{0,\vartheta}}$$

Then $\exists\ f \in BC^0(\Omega)$ s.t. $f_n \to f$ in $BC^0(\Omega)$

We need to show that $f \in C^{0,\vartheta}(\Omega)$ and that

$f_n \to f$ in $C^{0,\vartheta}(\Omega)$

$$\frac{|f(x)-f(y)|}{|x-y|^\vartheta} = \frac{|(f(x)-f_n(x)) + (f_n(x)-f_n(y)) + (f_n(y)-f(y))|}{|x-y|^\vartheta}$$

$$\le 2\,\frac{\|f-f_n\|_{L^\infty(\Omega)}}{|x-y|^\vartheta} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\vartheta}$$

$$\le 2\,\frac{\|f-f_n\|_{L^\infty(\Omega)}}{|x-y|^\vartheta} + [f_n]_{C^{0,\vartheta}}$$

$$\frac{|f(x)-f(y)|}{|x-y|^\vartheta} \le 0 + \limsup_{n\to+\infty} [f_n]_{C^{0,\vartheta}} \le C \qquad \forall\ x\neq y$$

f_n is Cauchy in $C^{0,\vartheta}(\Omega)$ in particular ~~by~~ that $\forall\ \varepsilon>0\ \exists\ N_\varepsilon$ s.t. $n,m \ge N_\varepsilon$

$$\Rightarrow [f_n - f_m]_{C^{0,\vartheta}} < \varepsilon$$

$$[f_n]_{C^{0,\vartheta}} \le [f_n - f_m]_{C^{0,\vartheta}} + [f_m]_{C^{0,\vartheta}}$$
$$< \varepsilon + [f_m]_{C^{0,\vartheta}}$$

$$\Rightarrow \limsup_{n\to+\infty} [f_n]_{C^{0,\vartheta}} \le \varepsilon + [f_m]_{C^{0,\vartheta}}$$

We need to show that

$$\lim_{n \to +\infty} \|f - f_n\|_{C^{0,\vartheta}(\Omega)} = 0$$

$$\frac{\left|\left(f(x) - f_n(x)\right) - \left(f(y) - f_n(y)\right)\right|}{|x-y|^{\vartheta}}$$

We need to show that

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \text{ s.t. } n > N_\varepsilon \Rightarrow [f - f_n]_{C^{0,\vartheta}(\Omega)} < \varepsilon$$

$m, n > N_\varepsilon$

$$\frac{\left|\left(f(x) - f_n(x)\right) - \left(f(y) - f_n(y)\right)\right|}{|x-y|^{\vartheta}} =$$

$$= \lim_{m \to +\infty} \frac{\left|\left(f_m(x) - f_n(x)\right) - \left(f_m(y) - f_n(y)\right)\right|}{|x-y|^{\vartheta}}$$

$$\leq \limsup_{m \to +\infty} \underbrace{[f_m - f_n]_{C^{0,\vartheta}(\Omega)}}_{< \varepsilon} \leq \varepsilon$$

$$\forall \quad x \neq y$$

$$\frac{\left| \left(f(x) - f_n(x) \right) - \left(f(y) - f_n(y) \right) \right|}{|x-y|^\vartheta} \leq \varepsilon$$

if $n \geq N_\varepsilon$

$$[f - f_n]_{C^{0,\vartheta}(\Omega)} \leq \varepsilon$$

$$f_n \xrightarrow{n \to +\infty} f \text{ in } C^{0,\vartheta}(\Omega)$$

$1 \le p < +\infty$ $\quad \Omega \subseteq \mathbb{R}^d$

$0 < \vartheta < 1$ ~~Sobolev~~ Sobolev space $W^{\vartheta,p}(\Omega)$

$W^{\vartheta,p}(\Omega) = \{ f \in L^p(\Omega) : \quad s > 0$

$$[f]^p_{W^{\vartheta,p}(\Omega)} := \int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p + d}}\, dx\, dy < \infty \}$$

$$\|f\|_{W^{\vartheta,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{\vartheta,p}(\Omega)}$$

It is a norm. We will

$D^\vartheta f \in L^p(\Omega)$

$W^{n,p}(\Omega)$ $\quad f, \partial_x^\alpha f \in L^p(\Omega)$ $|\alpha| \le n$

D

Let $\{f_n\}$ be Cauchy in $W^{\vartheta,p}(\Omega)$

$\Rightarrow \{f_n\}$ is Cauchy in $L^p(\Omega)$.

$\forall \varepsilon > 0 \quad \exists N_\varepsilon$ s.t. $n, m > N_\varepsilon \Rightarrow [f_n - f_m]_{\vartheta,p} < \varepsilon$

$\exists f = \lim_{n\to\infty} f_n$ in $L^p(\Omega)$

Want $f \in W^{\vartheta,p}(\Omega)$ and

$$\frac{|f(x)-f(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \le \frac{|f_n(x)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} + \frac{|f(x)-f_n(x)|}{|x-y|^{\vartheta+\frac{d}{p}}} + \frac{|f(y)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}}$$

$$\left(\int_{\substack{|x-y|\ge\delta \\ x,y\in\Omega}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy \right)^{\frac{1}{p}} \le \underbrace{[f_n]_{\vartheta,p}}_{\le C} +$$

$$+ 2\left(\int_\Omega dy\, |f(y)-f_n(y)|^p \int_{|x-y|\ge\delta} \frac{1}{|x-y|^{\vartheta p+d}}\,dx \right)^{\frac{1}{p}}$$

$$C \int_\delta^{+\infty} \frac{1}{r^{\vartheta p+1}}\,dr \le \delta^{-p\vartheta}$$

$$\le \underbrace{[f_n]_{\vartheta,p}}_{\le C} + 2C^{\frac{1}{p}} \delta^{-\vartheta} \underbrace{|f-f_n|_{L^p(\Omega)}}_{n\to+\infty}$$

$n, m > N_\varepsilon$. $\qquad n \to +\infty$

$\to 0$

$\forall\, \delta > 0$

$$\left(\int_{\substack{|x-y|\geq\delta \\ x,y\in\Omega}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta+d}}\, dx\, dy \right)^{\frac{1}{p}} \leq C$$

$$\lim_{\delta\to 0^+} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta+d}}\, \chi_{\{|x-y|\geq\delta\}} = \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta+d}}$$

$$L^p(\Omega\times\Omega)$$

By Fathou's lemma

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta+d}}\, dx\, dy \leq \liminf_{\delta\to 0} \int_{\substack{|x-y|\geq\delta \\ \Omega\times\Omega}} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta+d}}\, dx\, dy \leq C$$

$$\Rightarrow f \in W^{\vartheta,p}(\Omega)$$

$$[f-f_n]_{\vartheta,p} \xrightarrow{n\to+\infty} 0$$