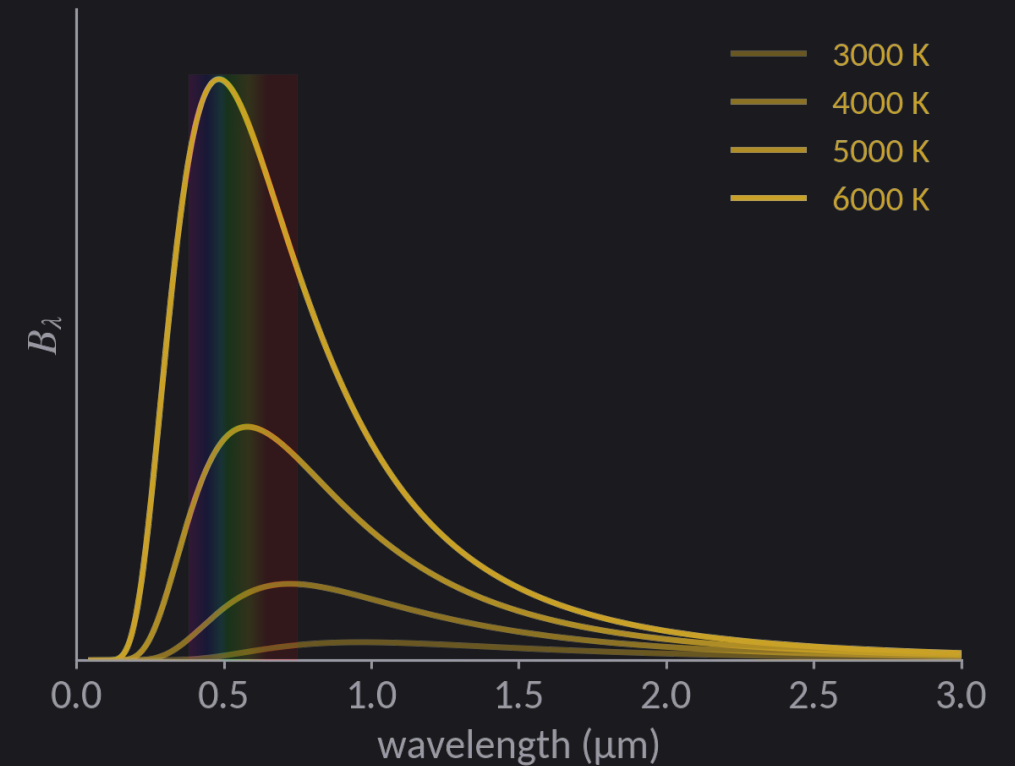


Blackbody radiation

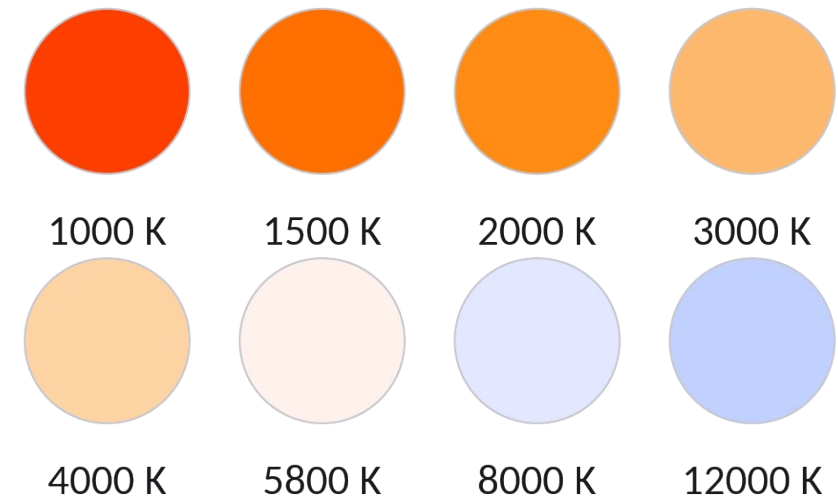
From a measurement to the quantum of action

Modern Physics · Lecture 1



Is there a universal thermal spectrum?

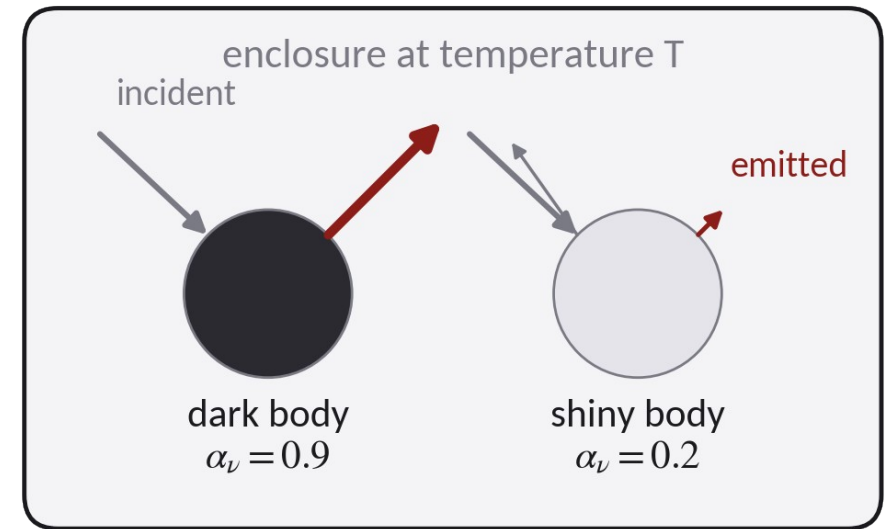
- Every body at a temperature $T > 0$ emits electromagnetic radiation.
- How much it emits, and at which wavelengths, depends on the material, on the surface and on the direction.
- Kirchhoff (1860) asked whether some part of the problem depends only on T .
- It does: the blackbody spectrum. Finding its shape took forty years and needed a new constant of nature, h .



Colour of a blackbody at each temperature, computed from Planck's law and the CIE 1931 colour-matching functions.

Black means a perfect absorber

- Absorptivity α_ν : the fraction of incident power at frequency ν that is absorbed. A blackbody has $\alpha_\nu = 1$ at every ν .
- Put two different bodies in an enclosure at temperature T . At equilibrium, at every frequency, each one must emit exactly what it absorbs.
- Otherwise energy would flow from one to the other at the same temperature.



A good absorber is a good emitter. The blackbody is the best possible thermal emitter, and its spectrum carries no memory of the material.

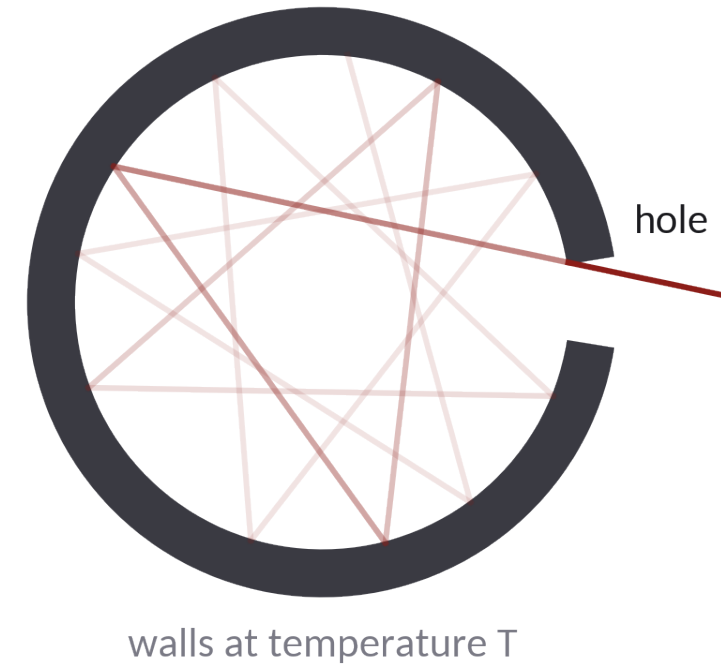
Kirchhoff's law

$$\varepsilon_\nu(T) = \alpha_\nu(T)$$

$$\varepsilon_\nu(T) = \frac{M_\nu(T)}{M_{\nu, \text{BB}}(T)}, \quad 0 \leq \varepsilon_\nu = \alpha_\nu \leq 1$$

A real blackbody: a cavity with a small hole

- Light entering a small hole is reflected many times inside the cavity and is almost surely absorbed before it can come out.
- So the hole has $\alpha \approx 1$, and by Kirchhoff it emits like a blackbody.
- The hole does not produce the radiation. It samples the equilibrium field that fills the cavity.
- Lummer and Wien (1895) proposed to use a heated cavity as the laboratory blackbody.
- What spoils it: temperature gradients along the walls, a hole that is too large, walls with low absorptivity.



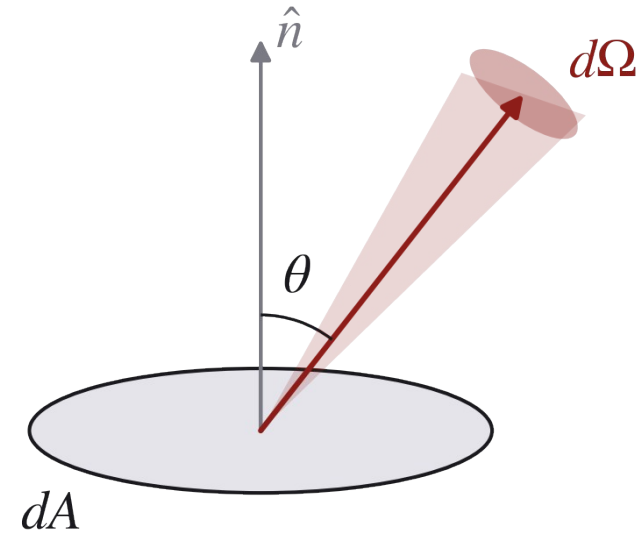
A ray traced through 12 reflections. With wall absorptivity 0.5 only $0.5^{12} \approx 2 \times 10^{-4}$ of the power would survive.

What we measure: three spectral quantities

Quantity	Meaning	Units
u_ν	energy per volume and per unit frequency inside the cavity	$\text{J m}^{-3} \text{Hz}^{-1}$
B_ν	power per projected area, per solid angle, per unit frequency (spectral radiance)	$\text{W m}^{-2} \text{sr}^{-1} \text{Hz}^{-1}$
M_ν	power per area and per unit frequency, into all directions of the half space	$\text{W m}^{-2} \text{Hz}^{-1}$

They are not synonyms. Geometry connects them. For an emitter whose radiance is the same in every direction (Lambertian):

$$M_\nu = \int_{\text{half space}} B_\nu \cos \theta d\Omega = \pi B_\nu$$



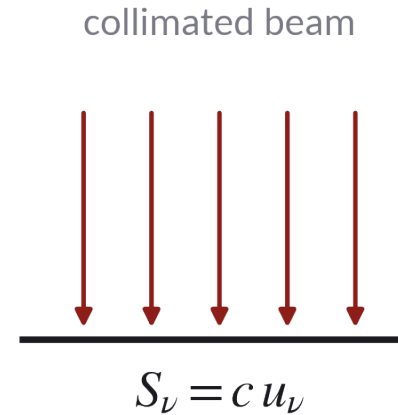
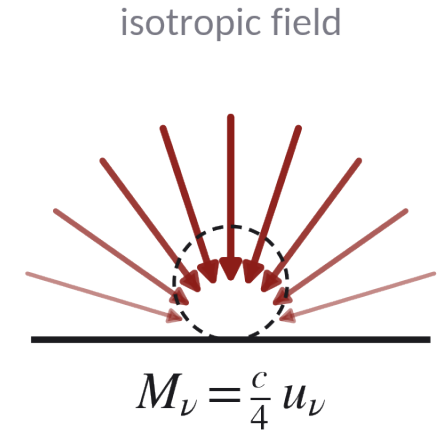
The factor $\cos \theta$ is the projected area seen from direction θ .

From the cavity to the hole: the factor $c/4$

- A collimated beam of energy density u_ν carries $S_\nu = c u_\nu$ through a surface normal to it.
- Inside the cavity the field is isotropic: the energy per unit solid angle is $u_\nu/4\pi$, and only the part moving towards the hole gets out, obliquely.
- Weighting by $\cos \theta$ and integrating over the half space:

$$M_\nu = \frac{c}{4} u_\nu \qquad B_\nu = \frac{c}{4\pi} u_\nu$$

Mixing up $S_\nu = c u_\nu$ and $M_\nu = c u_\nu / 4$ is one of the most common mistakes in this topic.

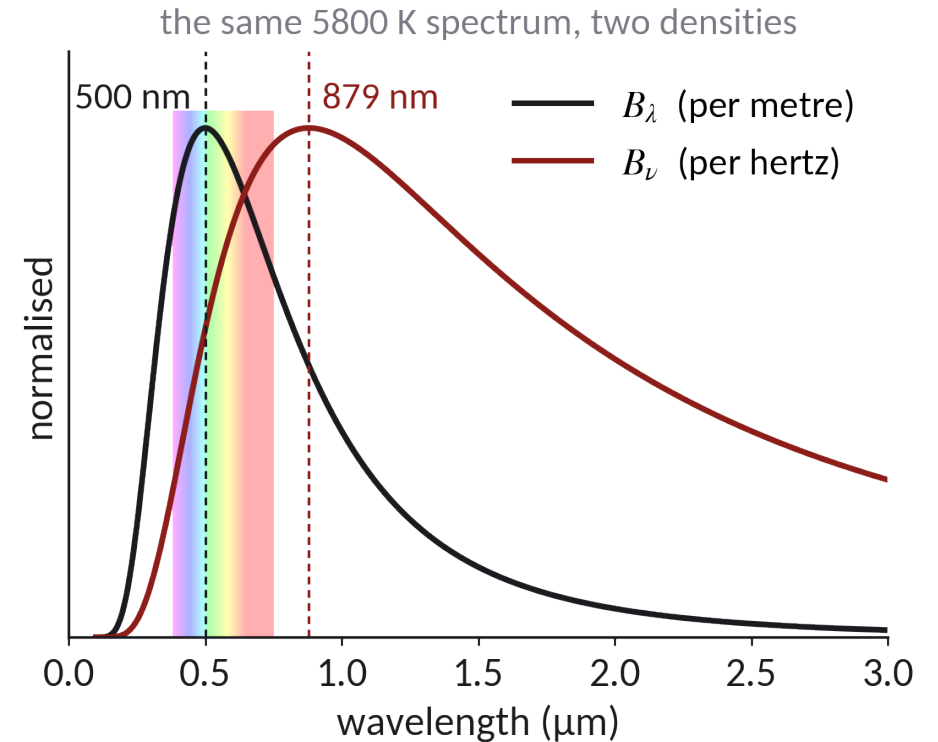


Per hertz or per metre? The peak moves

- A spectrum can be given per unit frequency or per unit wavelength. The energy in the same physical interval must be the same:

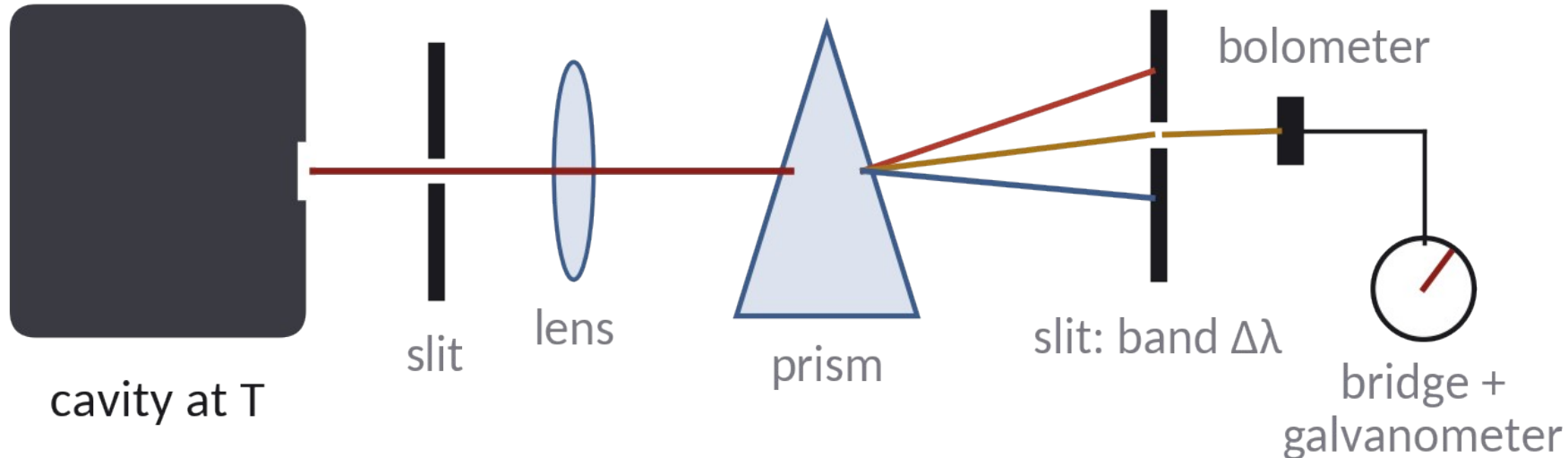
$$u_\nu |d\nu| = u_\lambda |d\lambda| \implies u_\lambda = u_\nu \frac{c}{\lambda^2}$$

- The two curves have different shapes and different maxima.
- For the Sun (5800 K): B_λ peaks at 500 nm; B_ν peaks at a frequency whose wavelength is 879 nm.
- So $\lambda_{\max} \neq c/\nu_{\max}$. Always say which spectral density you are plotting.



How the spectrum was measured (Berlin, 1890s)

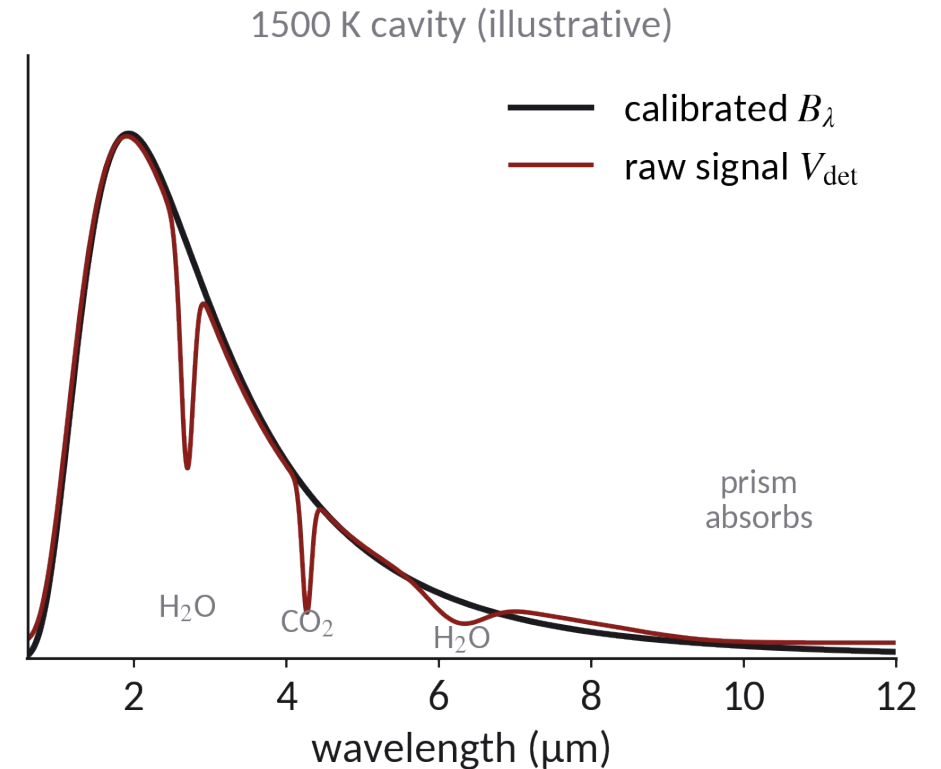
- An electrically heated cavity at uniform, measured temperature. A slit and a lens select a beam; a prism (fluorite, rock salt) or a grating disperses it.
- A second slit selects a narrow band $\Delta\lambda$. A bolometer (a blackened platinum strip) turns the absorbed power into a change of resistance, read with a Wheatstone bridge and a galvanometer.
- Turning the prism scans the spectrum, one wavelength at a time. Lummer, Pringsheim, Rubens and Kurlbaum worked at the Physikalisch-Technische Reichsanstalt and the Technische Hochschule.



What the detector gives is not yet B_λ

$$V_{\text{det}}(\lambda) = G R_{\text{det}}(\lambda) \tau_{\text{opt}}(\lambda) \mathcal{G} B_\lambda(T) \Delta\lambda + V_0$$

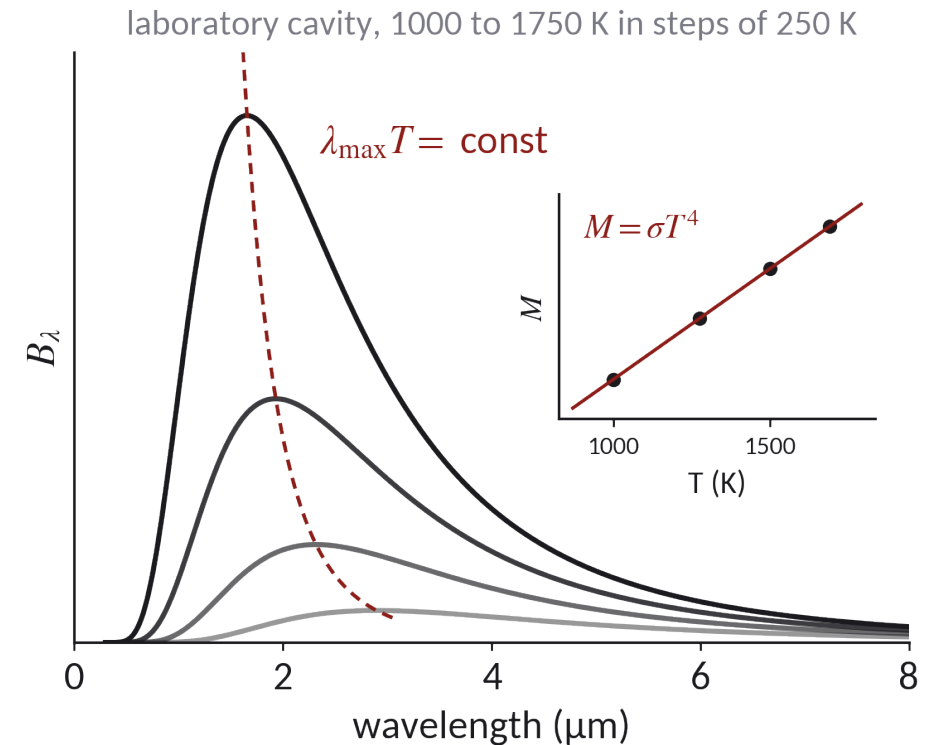
- G : electronic gain. R_{det} : responsivity of the detector. τ_{opt} : transmission of air and optics. The calligraphic G : area times solid angle. V_0 : background.
- To get B_λ : subtract dark signal and background, divide by R_{det} and τ_{opt} , know $\Delta\lambda$, measure T .
- Systematics: emissivity of the hole below 1, temperature gradients, stray light, overlapping grating orders, finite band, detector non-linearity.
- A good check: change T . The peak must move as $1/T$ and the area must grow as T^4 .



Illustrative: water and CO₂ in the air and the prism's absorption distort the raw signal.

What was known by 1900

- Stefan (1879, from data) and Boltzmann (1884, from thermodynamics): the total emitted power is $M = \sigma T^4$.
- Wien (1893): the peak moves as $\lambda_{\max} T = \text{constant}$.
- Wien (1896) proposed $u_\nu = A \nu^3 e^{-a\nu/T}$, with A and a constants. It fits the short-wavelength side well.
- Lummer and Pringsheim, then Rubens and Kurlbaum (1900): in the far infrared the data lie clearly above Wien's formula.
- Two laws are right. The shape of the curve is still unknown.



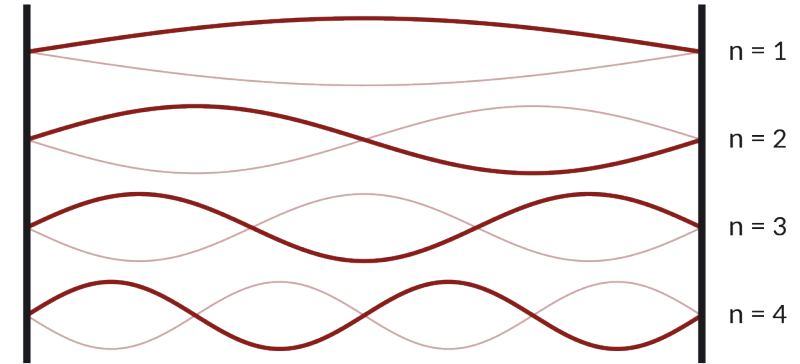
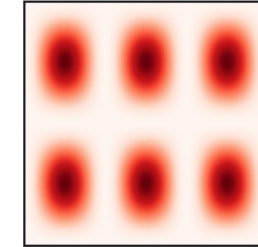
Computed from Planck's law at laboratory temperatures.

Classical step 1: the modes of the cavity

- Take a cubic cavity of side L with conducting walls. The tangential electric field must vanish at the walls, so the field forms standing waves.
- Allowed wave vectors, each with two independent polarizations:

$$k_i = \frac{n_i \pi}{L}, \quad n_i = 0, 1, 2, \dots \quad \nu = \frac{c |\mathbf{k}|}{2\pi}$$

- Each mode behaves as an independent harmonic oscillator of frequency ν .
- The question becomes: how many modes are there between ν and $\nu + d\nu$, and how much energy does each one hold?



a 2D mode:
 $(n_x, n_y) = (3, 2)$

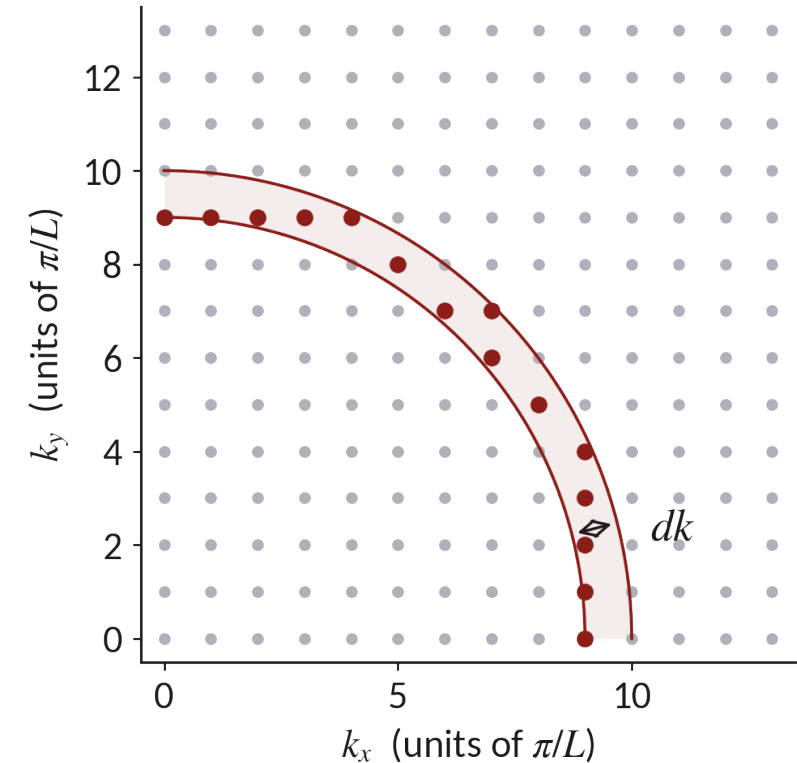
Counting the modes

- Each mode is a point of a lattice in k -space, with cell volume $(\pi/L)^3$.
- Count the points in a thin shell between k and $k + dk$, only in the positive octant ($n_i \geq 0$), and multiply by two polarizations:

$$dN = 2 \cdot \frac{1}{8} \cdot \frac{4\pi k^2 dk}{(\pi/L)^3} = \frac{V}{\pi^2} k^2 dk$$

$$g(\nu) = \frac{1}{V} \frac{dN}{d\nu} = \frac{8\pi\nu^2}{c^3}$$

This is pure electromagnetism and geometry, with nothing quantum in it. It stays valid in Planck's theory.



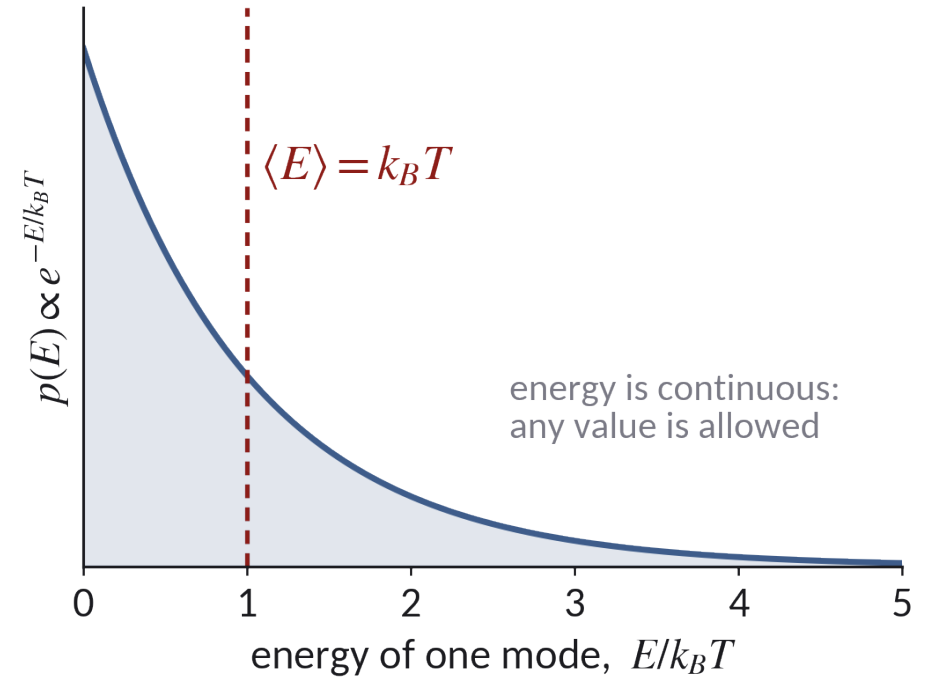
A 2D slice of the 3D count: the red points lie in the shell.

Classical step 2: the energy of each mode

- In thermal equilibrium the energy E of an oscillator follows the Boltzmann distribution $p(E) \propto e^{-E/k_B T}$.
- Classically E is continuous: any value from 0 to ∞ is allowed.

$$\langle E \rangle = \frac{\int_0^\infty E e^{-E/k_B T} dE}{\int_0^\infty e^{-E/k_B T} dE} = k_B T$$

- The average does not depend on the frequency: every mode gets $k_B T$, whatever ν is.
- This is equipartition: $k_B T/2$ for each of the two quadratic terms of the oscillator.

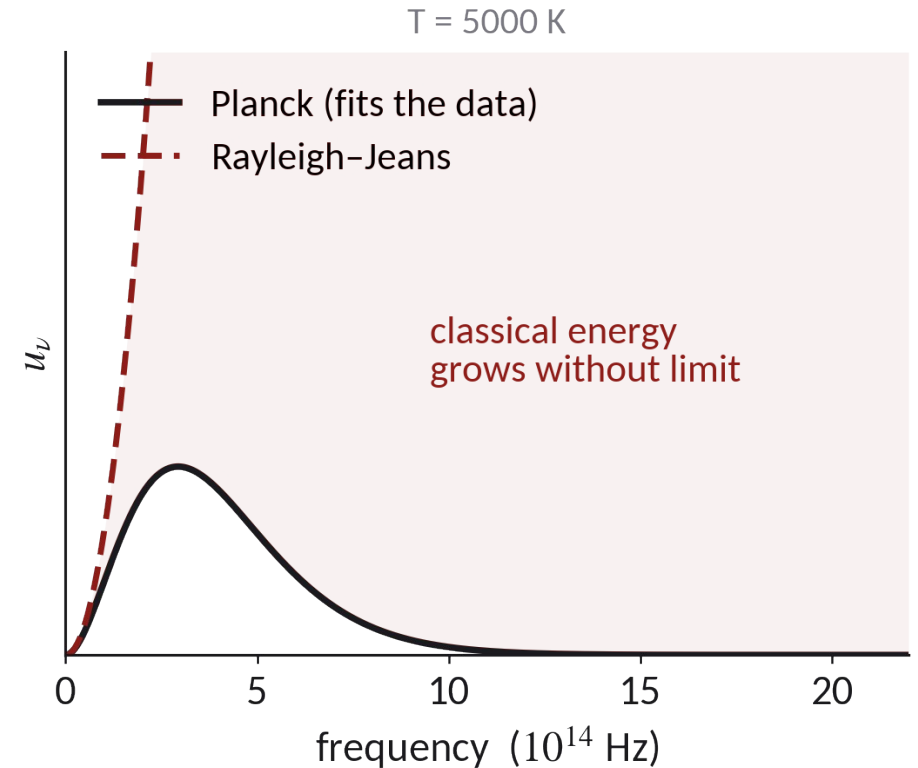


Rayleigh-Jeans and the ultraviolet catastrophe

Number of modes $\times$ energy per mode:

$$u_{\nu}^{\text{RJ}} = \frac{8\pi\nu^2}{c^3} k_B T \quad \int_0^{\infty} u_{\nu}^{\text{RJ}} d\nu = \infty$$

- Correct at low frequency, where $h\nu \ll k_B T$.
- But the number of modes grows as ν^2 , and each one gets $k_B T$. The total energy diverges.
- The mode count is right. The energy per mode is wrong.
- Rayleigh (1900) and Jeans (1905). The name 'ultraviolet catastrophe' is Ehrenfest's (1911).

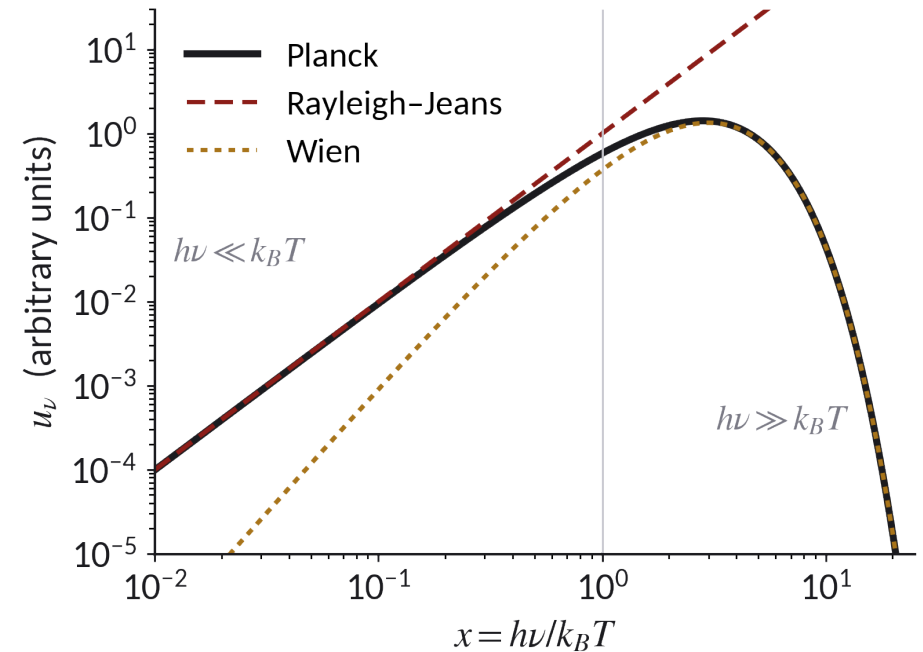


Two half-right laws

- Rayleigh–Jeans fits the low-frequency side and diverges at high frequency.
- Wien fits the high-frequency side and falls below the data at low frequency.

$$u_{\nu}^{\text{Wien}} \propto \nu^3 e^{-a\nu/T}$$

- The correct law must reduce to Rayleigh–Jeans for $h\nu \ll k_B T$ and to Wien for $h\nu \gg k_B T$.
- In October 1900 Planck found a formula that interpolates between the two. In December he derived it.



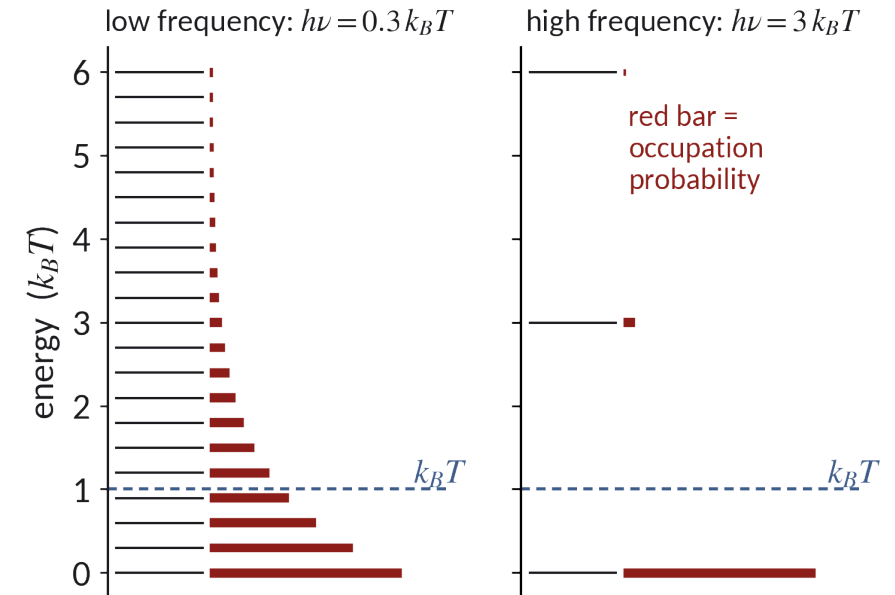
Planck's step: energy in multiples of $h\nu$

Assume that an oscillator of frequency ν can only have the energies $E_n = nh\nu$. The integral becomes a geometric series:

$$E_n = n h \nu \quad Z = \sum_{n=0}^{\infty} e^{-n h \nu / k_B T} = \frac{1}{1 - e^{-h \nu / k_B T}}$$

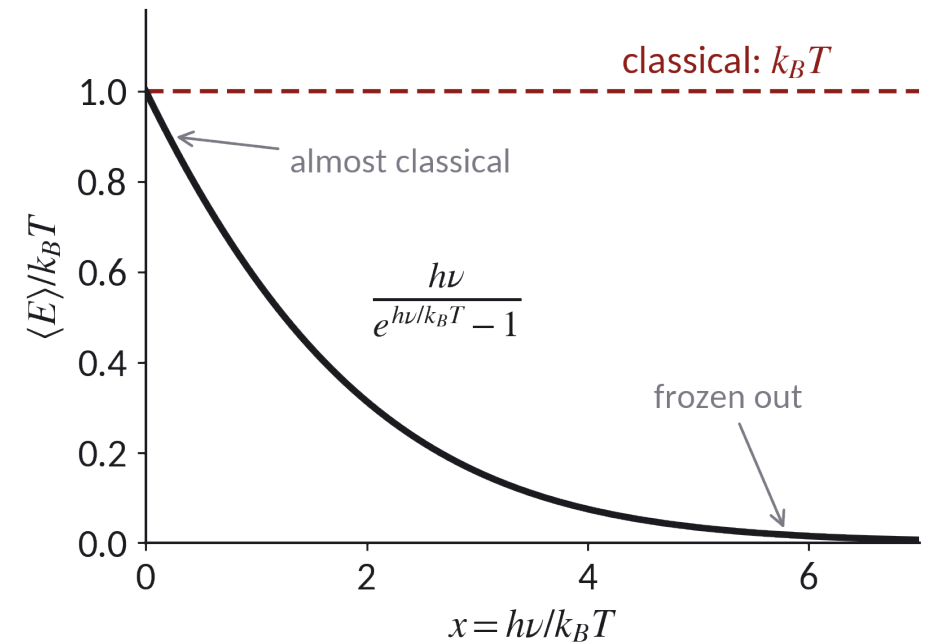
$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = \frac{h \nu}{e^{h \nu / k_B T} - 1}, \quad \beta = \frac{1}{k_B T}$$

- From the data Planck found $h = 6.55 \times 10^{-34} \text{ J s}$ and $k_B = 1.346 \times 10^{-23} \text{ J/K}$ (1901). Since 2019 the SI fixes $h = 6.62607015 \times 10^{-34} \text{ J s}$ exactly.
- The ladder on the right is to scale: when $h\nu \ll k_B T$ many levels are occupied; when $h\nu \gg k_B T$ almost only $n = 0$.



What quantization does to a single mode

- One number decides everything: $x = h\nu/k_B T$, the size of the step compared with the thermal energy.
- $x \ll 1$: the steps are small compared with $k_B T$, many levels are populated, and $\langle E \rangle \rightarrow k_B T$. Classical physics survives.
- $x \gg 1$: even one quantum costs too much, and $\langle E \rangle \approx h\nu e^{-x}$. The mode is frozen out.
- The high-frequency modes are switched off. That removes the catastrophe.



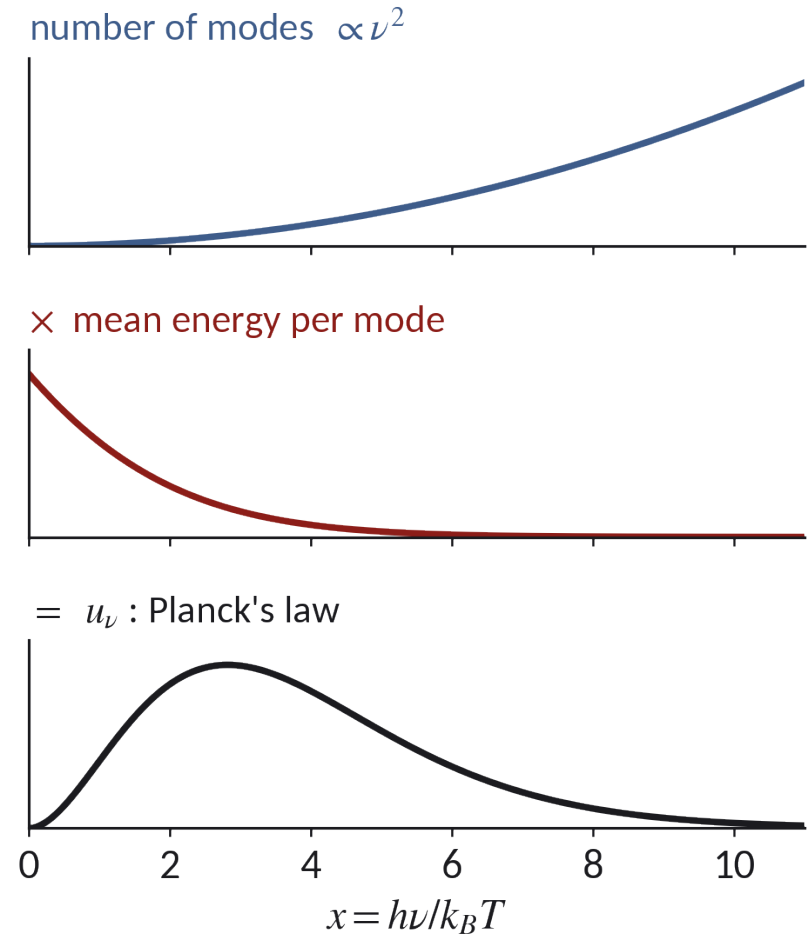
Planck's law = number of modes $\times$ energy per mode

$$u_\nu(T) = \frac{8\pi\nu^2}{c^3} \cdot \frac{h\nu}{e^{h\nu/k_B T} - 1} = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}$$

With $B_\nu = (c/4\pi) u_\nu$ and the Jacobian c/λ^2 :

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \quad B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}$$

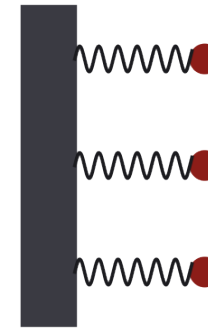
- Always pair each formula with its quantity and its units: u_ν is energy per volume in the cavity; B_ν is what an ideal radiometer sees.
- Checks: for $x \ll 1$ it becomes Rayleigh–Jeans, for $x \gg 1$ it takes Wien's form.



What exactly was quantized?

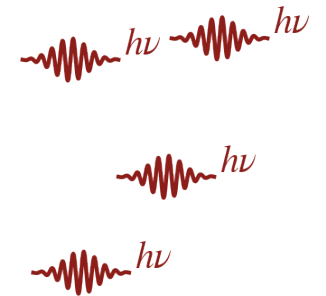
- Planck, 1900: to count the ways of sharing energy among the material oscillators in the walls, he divided it into elements $h\nu$. He saw it as a formal step.
- Einstein, 1905: radiation itself, in the Wien regime, behaves as if made of independent quanta of energy $h\nu$.
- Debye, 1910: quantize each mode of the field directly. That is the derivation we just did.
- Same formula, different physical pictures. The blackbody spectrum alone cannot decide between them.

Planck, 1900



energy of the wall oscillators
counted in elements $h\nu$

Einstein, 1905



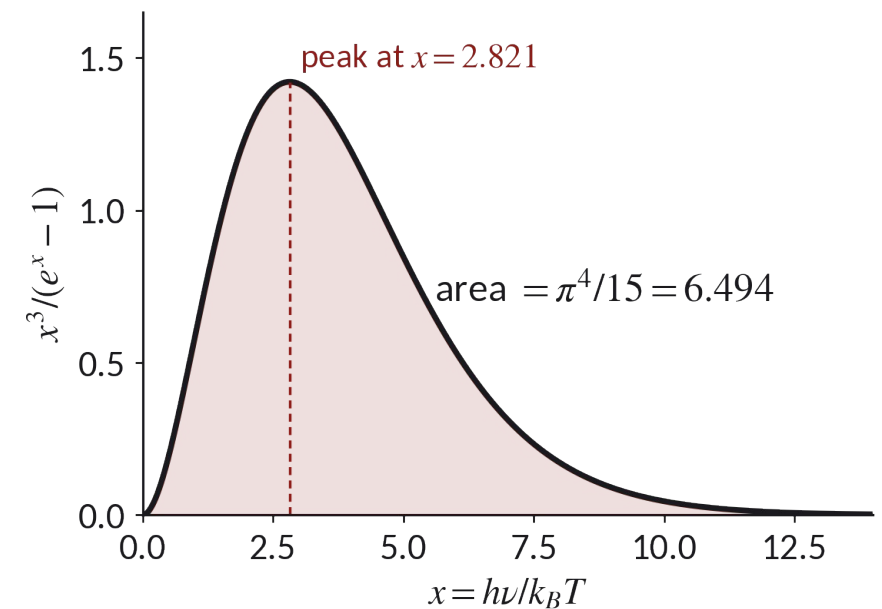
the radiation itself behaves
as quanta of energy $h\nu$

Consequences: Stefan–Boltzmann and Wien's displacement

$$u = aT^4, \quad a = \frac{8\pi^5 k_B^4}{15 h^3 c^3} \quad M = \frac{c}{4} u = \sigma T^4, \quad \sigma = \frac{2\pi^5 k_B^4}{15 h^3 c^2}$$

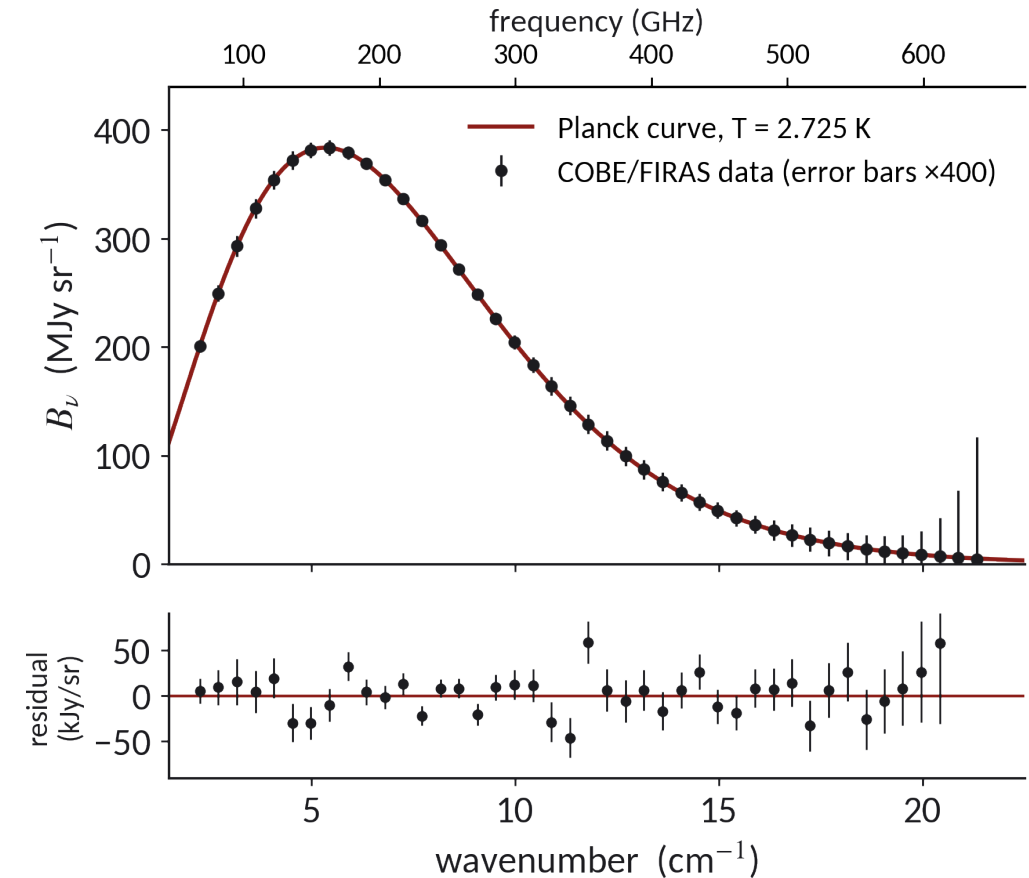
$$5(1 - e^{-x}) = x \implies x = 4.965 \implies \lambda_{\max} T = 2.898 \text{ mm K}$$

- Planck's law gives σ from h , k_B and c . The integral is $\int x^3 / (e^x - 1) dx = \pi^4/15$.
- $\sigma = 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, $a = 7.566 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$.
- A 3000 K filament peaks at 966 nm, in the infrared; the Sun (5800 K) at 500 nm; the cosmic background (2.725 K) at 1.06 mm.
- From 3000 K to 5800 K the emitted power per area grows by $(5800/3000)^4 \approx 14$.



The best blackbody ever measured

- COBE/FIRAS (1989–1990) measured the cosmic microwave background between 68 and 640 GHz.
- A single Planck curve, with one parameter $T = 2.725$ K, goes through all 43 points.
- The residuals scatter by about 50 parts per million of the peak.
- So matter and radiation were in thermal equilibrium in the early universe. Mather and Smoot shared the 2006 Nobel Prize for COBE.

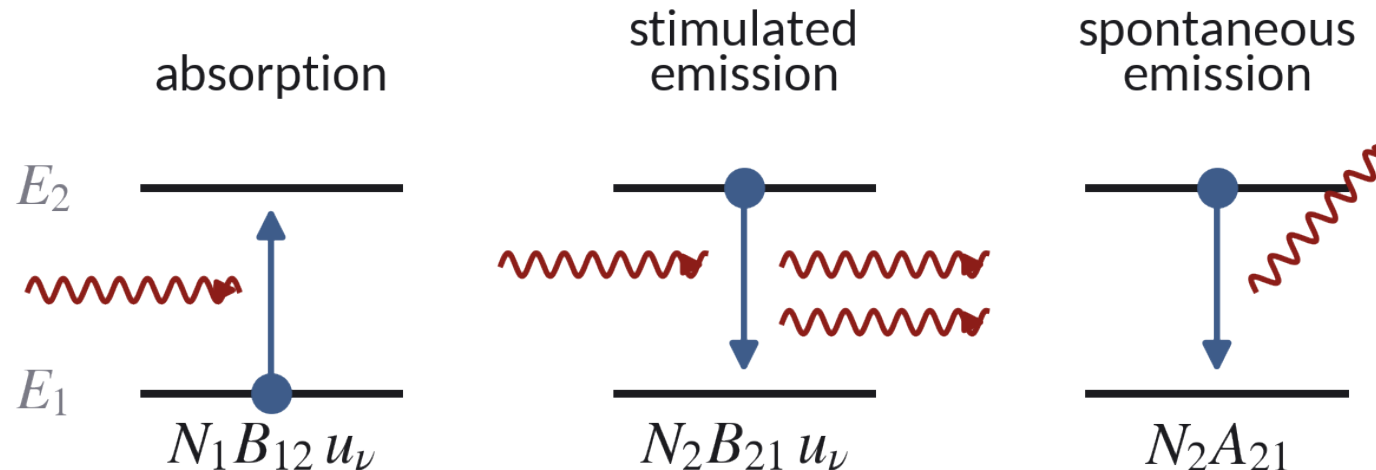


Data: COBE/FIRAS, Fixsen et al., *ApJ* 473, 576 (1996), from NASA LAMBDA.

Einstein 1917: three elementary processes

- Two atomic levels $E_1 < E_2$, with $E_2 - E_1 = h\nu$, in a radiation field of spectral density u_ν .
- Absorption and stimulated emission are driven by the field. Spontaneous emission is not.
- In equilibrium the upward and downward rates balance, and the populations follow Boltzmann:

$$N_1 B_{12} u_\nu = N_2 (B_{21} u_\nu + A_{21}), \quad \frac{N_2}{N_1} = e^{-h\nu/k_B T}$$

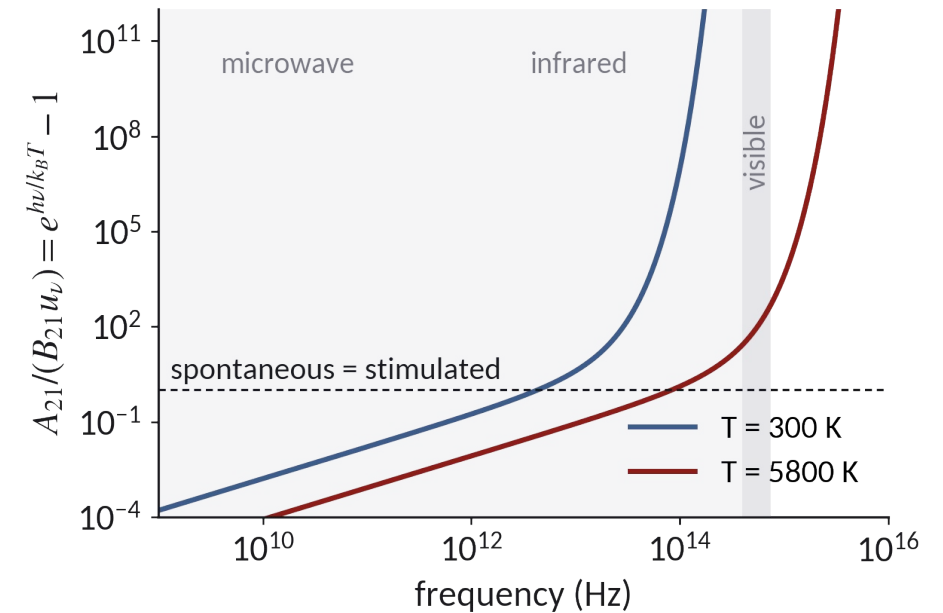


What Einstein's balance requires

At high temperature u_ν must grow without saturating, and at low frequency it must become Rayleigh–Jeans. This fixes:

$$B_{12} = B_{21}, \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3}$$

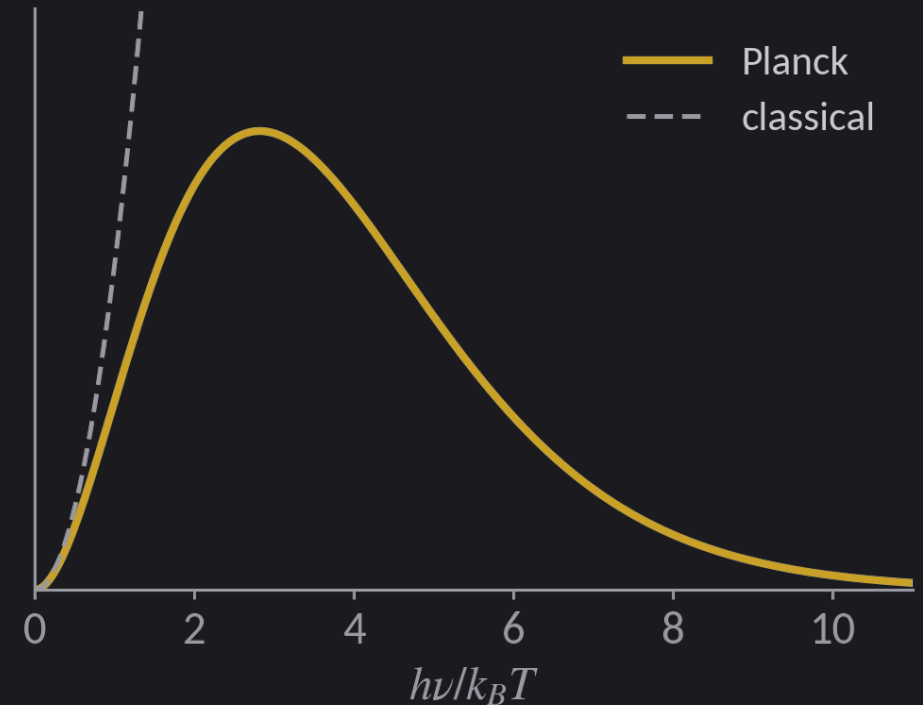
- With these relations u_ν is exactly Planck's law.
- Without stimulated emission it cannot work. That term is the physics of the laser.
- The ratio of spontaneous to stimulated emission is $e^{h\nu/kBT} - 1$: stimulated emission wins at radio and microwave frequencies, spontaneous emission wins in the visible.
- Einstein also gave each event a momentum $h\nu/c$. That leads to the photoelectric effect and Compton scattering.



Summary

measurement $\rightarrow u_\nu, B_\nu, M_\nu \rightarrow g(\nu) \rightarrow \langle E \rangle \rightarrow u_\nu(T)$

- A blackbody is defined by an absorptivity equal to 1, and realised by a cavity with a small hole.
- Radiance, emittance and energy density are linked by geometry:
 $M_\nu = \pi B_\nu = (c/4) u_\nu$.
- The mode count $g(\nu) = 8\pi\nu^2/c^3$ is classical, and it stays.
- What changes is the energy per mode: $E_n = nh\nu$ turns $k_B T$ into $h\nu/(e^{h\nu/k_B T} - 1)$.
- One new constant, h , fixes the spectrum at every temperature, from a filament to the cosmic background.



Questions before you leave

1. Why does $c/4$ appear for the field in a cavity, and c for a collimated beam?
2. Which part of the Rayleigh–Jeans derivation survives in Planck's theory?
3. Why is the peak of B_λ not at $c/\nu_{\max}$?
4. What distinguishes a calibrated spectrum from a raw detector signal?
5. Why does Einstein need stimulated emission to recover Planck's law?

