1.B.1 Since $y \succ z$ implies $y \succsim z$, the transitivity implies that $x \succsim z$. Suppose that $z \succsim x$. Since $y \succsim z$, the transitivity then implies that $y \succsim x$. But this contradicts $x \succ y$. Thus we cannot have $z \succsim x$. Hence $x \succ z$.

1.B.2 By the completeness, $x \succsim x$ for every $x \in X$. Hence there is no $x \in X$ such that $x \succ x$. Suppose that $x \succ y$ and $y \succ z$, then $x \succ y \succsim z$. By (iii) of Proposition 1.B.1, which was proved in Exercise 1.B.1, we have $x \succ z$. Hence $\succ$ is transitive. Property (i) is now proved.

As for (ii), since $x \succsim x$ for every $x \in X$, $x \sim x$ for every $x \in X$ as well. Thus $\sim$ is reflexive. Suppose that $x \sim y$ and $y \sim z$. Then $x \succsim y$, $y \succsim z$, $y \succsim x$, and $z \succsim y$. By the transitivity, this implies that $x \succsim z$ and $z \succsim x$. Thus $x \sim z$. Hence $\sim$ is transitive. Suppose x that $\sim y$. Then $x \succsim y$ and $y \succsim x$. Thus $y \succsim x$ and $x \succsim y$. Hence $y \sim x$. Thus $\sim$ is symmetric Property (ii) is now proved.

1.B.3 Let $x \in X$ and $y \in X$. Since $u(\cdot)$ represents $\succsim$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Since $f(\cdot)$ is strictly increasing, $u(x) \geq u(y)$ if and only if $v(x) \geq v(y)$. Hence $x \succsim y$ if and only if $v(x) \geq v(y)$. Therefore $v(\cdot)$ represents $\succsim$.

1.B.4 Suppose first that $x \succsim y$. If, furthermore, $y \succsim x$, then $x \sim y$ and hence $u(x) = u(y)$. If, on the contrary, we do not have $y \succsim x$, then $x \succ y$. Hence $u(x) > u(y)$. Thus, if $x \succsim y$, then $u(x) \geq u(y)$.

Suppose conversely that $u(x) \geq u(y)$. If, furthermore, $u(x) = u(y)$, then $x \sim y$ and hence $x \succsim y$. If, on the contrary, $u(x) > u(y)$, then $x \succ y$, and hence $x \succsim y$. Thus, if $u(x) \geq u(y)$, then $x \succsim y$. So $u(\cdot)$ represents $\succsim$.

1.B.5 First, we shall prove by induction on the number N of the elements of X that, if there is no indifference between any two different elements of X, then there exists a utility function. If $N = 1$, there is nothing to prove: Just assign any number to the unique element. So let $N > 1$ and suppose that the above assertion is true for $N - 1$. We will show that it is still true for N. Write $X = \{x_1,\ldots,x_{N-1},x_N\}$. By the induction hypothesis, $\succsim$ can be represented by a utility function $u(\cdot)$ on the subset $\{x_1,\ldots,x_{N-1}\}$. Without loss of generality we can assume that $u(x_1) > u(x_2) > \ldots > u(x_{N-1})$.

Consider the following three cases:

Case 1: For every $i < N$, $x_N \succ x_i$.

Case 2: For every $i < N$, $x_i \succ x_N$.

Case 3: There exist $i < N$ and $j < N$ such that $x_i \succ x_N \succ x_j$.

Since there is no indifference between two different elements, these three cases are are exhaustive and mutually exclusive. We shall now show how the value of $u(x_N)$ should be determined, in each of the three cases, for $u(\cdot)$ to represent $\succsim$ on the whole X.

If Case 1 applies, then take $u(x_N)$ to be larger than $u(x_1)$. If Case 2 applies, take $u(x_N)$ to be smaller than $u(x_{N-1})$. Suppose now that Case 3 applies. Let $I = \{i \in \{1,\ldots, N - 1\}: x_i \succ x_{N+1}\}$ and $J = \{j \in \{1,\ldots, N - 1\}: x_{N+1} \succ x_j\}$. Completeness and the assumption that there is no indifference implies that $I \cup J = \{1,\ldots, N - 1\}$. The transitivity implies that both I and J are "intervals," in the sense that if $i \in I$ and $i' < i$, then $i' \in I$; and if $j \in J$ and $j' > j$, then $j' \in J$. Let $i^* = \max I$, then $i^* + 1 = \min J$. Take

$u(x_N)$ to lie in the open interval $(u(x_{i^*+1}), u(x_{i^*}))$. Then it is easy to see that $u(\cdot)$ represents $\succsim$ on the whole X.

Suppose next that there may be indifference between some two elements of $X = \{x_1, \ldots, x_N\}$. For each n = 1,...,N, define $X_n = \{x_m \in X: x_m \sim x_n\}$. Then, by the reflexivity of ~ (Proposition 1.B.1(ii)), $\cup_{n=1}^{N} X_n = X$. Also, by the transitivity of ~ (Proposition 1.B.1(ii)), if $X_n \neq X_m$, then $X_n \cap X_m = \emptyset$. So let M be a subset of {1,...,N} such that $X = \cup_{m \in M} X_m$ and $X_m \neq X_n$ for any $m \in M$ and any $n \in M$ with $m \neq n$. Define an relation $\succsim^*$ on $\{X_m: m \in M\}$ by letting $X_m \succsim^* X_n$ if and only if $x_m \succsim x_n$. In fact, by the definition of M, there is no indifference between two different elements of $\{X_m: m \in M\}$. Thus, by the preceding result, there exists a utility function $u^*(\cdot)$ that represents $\succsim^*$. Then define $u: X \to \mathbb{R}$ by $u(x_n) = u^*(X_m)$ if $m \in M$ and $x_n \in X_m$. It is easy to show that, by the transitivity, $u(\cdot)$ represents $\succsim$.

1. Suppose the choice set in café jules is always the same, denote it C={coffee, tea, baguette, etc…}. A choice function c(.) picks one element out of C, for instance c(C) = coffee. So, in a standard model where the choice set is simply the set of options, you would always pick the same item. Clearly, we don't always pick the same item, depending on the time of the day, how hungry we are, etc. To capture this, we can consider a more general choice sets. For instance, consider the option of choosing coffee. This represents a more general option such as {coffee at 11a} or {coffee after a donut}. By suitably enriching the choice set you can account for the fluctuations in your preferences.

2. You prefer the train to the bus because in this comparison you focus on the comfort dimension. This hurts the bus relative to the train. You prefer the bus to the car ride because you focus on the frequency dimension. This helps the bus relative to the car. This leads to non-transitive choice, which violates rationality. The intuition is that your valuation of the bus depends on what you compare it with, therefore you do not assign a well defined utility value to the bus.

3.1 This is not a preference relation because it is not complete. (Half of the) teams who play in different groups never play each other. It could also happen that the relation is not transitive: team A may beat team B, which beats team C but then team C may beat team A.

3.2 Since completeness and transitivity are violated, this tournament organization does not provide a rational choice algorithm. In particular it is not possible to assign a clear “quality” value that ranks all the teams.