1. An individual is characterized by βδ-preferences where $\beta = 0.7$ and $\delta = 0.9$ and his instantaneous utility function is $u(x) = \ln(10 + x)$ where x is the spending. At $t = 1$ Paul receives an endowment $W = 10$ to spend in $t = 2$, $t = 3$ and $t = 4$. (Assume R>1)

   a) Compute the optimal plan of spending from the perspective of $t = 1$.
   b) Compute the optimal plan of spending from the perspective of $t = 2$.
   c) Assume the agent forecasts correctly his future behaviour (i.e. he is sophisticated). Compute the optimal share of W invested in illiquid asset in $t = 1$

Solution

By c_i we denote the spending in period $i = 2, 3, 4$.
By w_i we denote the endowment in period $i = 1, 2, 3, 4$; then $w_1 = 10$

a) The problem is

$$\max_{\{c_2,c_3,c_4\}} \beta\delta u(c_2) + \beta\delta^2 u(c_3) + \beta\delta^3 u(c_4)$$
$$s.t.\, c_2 \leq Rw_1, c_3 \leq Rw_2, c_4 \leq Rw_3 \text{ and } c_2, c_3, c_4 \geq 0$$

Note that $w_2 = Rw_1 - c_2$, $w_3 = Rw_2 - c_3$ and that in the solution must be $w_4 = 0$ (no resources left)

Then we can rewrite the problem as

$$\max_{\{w_2,w_3\}} \beta\delta u(Rw_1 - w_2) + \beta\delta^2 u(Rw_2 - w_3) + \beta\delta^3 u(Rw_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

Using the assumption given in the text we have:

$$\max_{\{w_2,w_3\}} 0.7 \cdot 0.9 ln(10 + Rw_1 - w_2) + 0.7 \cdot 0.9^2 ln(10 + Rw_2 - w_3) + 0.7 \cdot 0.9^3 ln(10 + Rw_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

FOCs are:

$$-\frac{0.7 \cdot 0.9}{10 + Rw_1 - w_2} + \frac{R \cdot 0.7 \cdot 0.9^2}{10 + R \cdot w_2 - w_3} = 0$$
$$-\frac{0.7 \cdot 0.9^2}{10 + R \cdot w_2 - w_3} + \frac{R \cdot 0.7 \cdot 0.9^3}{10 + R \cdot w_3} = 0$$

Replacing $c_2 = w_1 - w_2$, $c_3 = w_2 - w_3$, $c_4 = w_3$ and simplifying

$$\frac{1}{10 + c_2} = \frac{R \cdot 0.9}{10 + c_3}$$
$$\frac{1}{10 + c_3} = \frac{R \cdot 0.9}{10 + c_4}$$