Lecture 6

Dynamic games with imperfect information

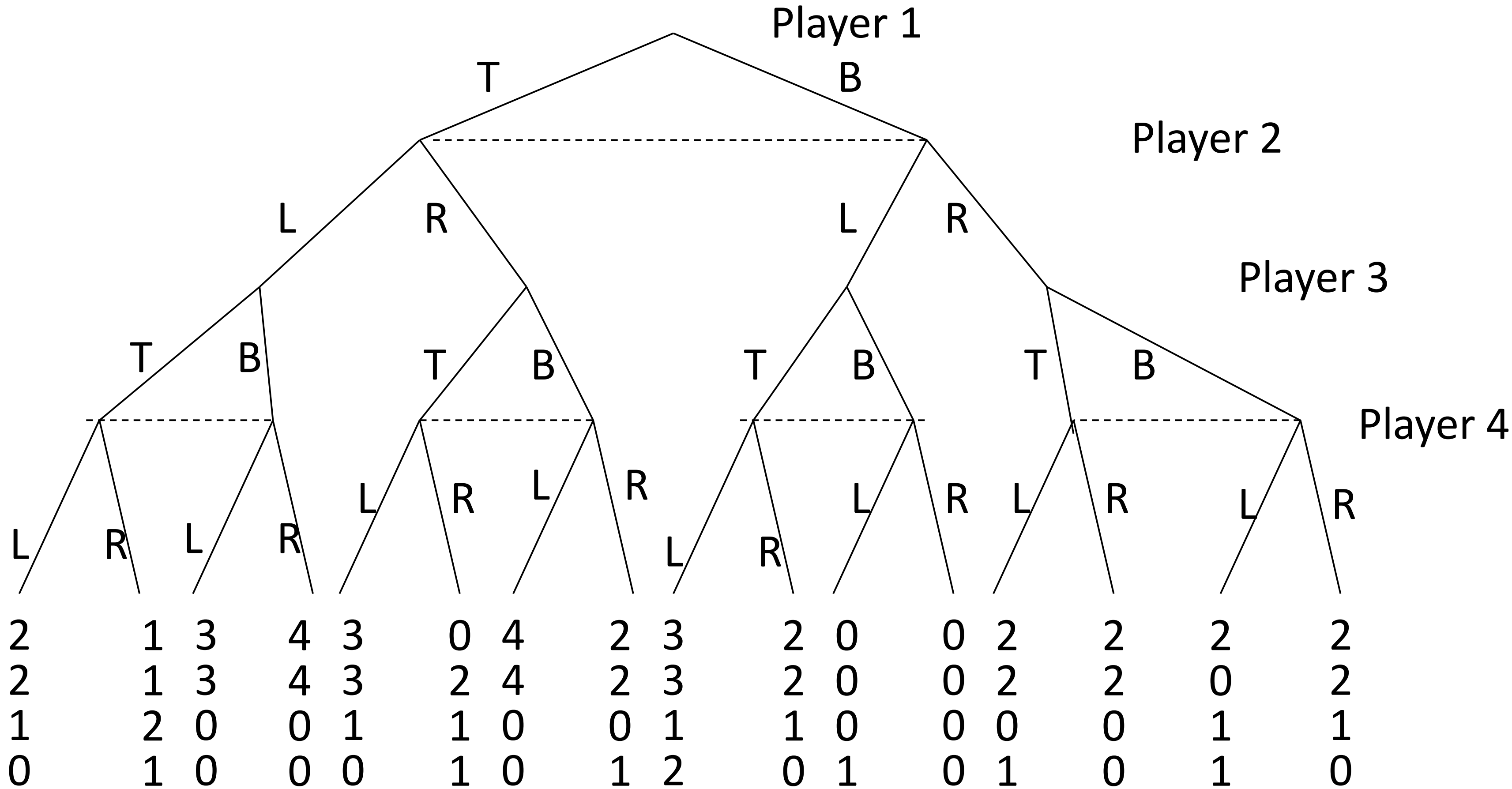
Backward Induction in dynamic games of imperfect information

- We start at the end of the trees
- first find the Nash equilibrium (NE) of the last subgame
- then taking this NE as given, find the NE in the second last subgame
- continue working backwards

If in each subgame there is only one NE, this procedure leads to a **Unique Subgame Perfect Nash equilibrium**

Example: two stage game of imperfect information

- **Stage 1**: Players 1 and 2 move simultaneously taking, respectively, actions $a_1 \in A_1$ and $a_2 \in A_2$
- **Stage 2:** Players 3 and 4 observe (a_1, a_2), then move simultaneously taking, respectively, actions $a_3 \in A_3$ and $a_4 \in A_4$
- **Payoffs:** $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$
- **Solution:**
  - We solve the simultaneous - move game between players 3 and 4 in the second stage:
  - Players 1 and 2 anticipate the behaviour of players 3 and 4


Player 1
T
B
Player 2
L
R
L
R
Player 3
T
B
T
B
T
B
T
B
Player 4
L
R
L
R
L
R
L
R
L
R
L
R
L
R
L
R
2 1 3 4 3 0 4 2 3 2 0 0 2 2 2 2
2 1 3 4 3 2 4 2 3 2 0 0 2 2 0 2
1 2 0 0 1 1 0 0 1 1 0 0 0 0 1 1
0 1 0 0 0 1 0 1 2 0 1 0 1 0 1 0

		Player 2	
		L	R
Player 1	T	G1	G2
	B	G3	G4

G1

		Player 4	
		L	R
Player 3	T	$2, 2,\underline{1},0$	$1,1 , \underline{2}, \underline{1}$
	B	$3,3, 0,\underline{0}$	$4,4, 0, \underline{0}$

G2

		Player 4	
		L	R
Player 3	T	$3,3,\underline{1},0$	$0,2,\underline{1}, \underline{1}$
	B	$4,4,0,0$	$2,2,0, \underline{1}$

G3

		Player 4	
		L	R
Player 3	T	$3,3,\underline{1},\underline{2}$	$2,2,\underline{1}, 0$
	B	$0,0,0,\underline{1}$	$0,0,0, 0$

G4

		Player 4	
		L	R
Player 3	T	$2,2,0,\underline{1}$	$2,2,0,0$
	B	$2,0,\underline{1},\underline{1}$	$2,2,\underline{1}, 0$

		Player 2	
		L	R
Player 1	T	1,1 , 2, 1	0,$\underline{2}$,1, 1
	B	$\underline{3},\underline{3}$,1,2	$\underline{2}$,0,1,1

G1

		Player 4	
		L	R
Player 3	T	2, 2,$\underline{1}$,0	1,1 , $\underline{2}$, $\underline{1}$
	B	3,3, 0,$\underline{0}$	4,4, 0, $\underline{0}$

G2

		Player 4	
		L	R
Player 3	T	3,3,$\underline{1}$,0	0,2,$\underline{1}$, $\underline{1}$
	B	4,4,0,0	2,2,0, $\underline{1}$

G3

		Player 4	
		L	R
Player 3	T	3,3,$\underline{1}$,$\underline{2}$	2,2,$\underline{1}$, 0
	B	0,0,0,$\underline{1}$	0,0,0, 0

G4

		Player 4	
		L	R
Player 3	T	2,2,0,$\underline{1}$	2,2,0,0
	B	2,0,$\underline{1}$,$\underline{1}$	2,2,$\underline{1}$, 0

Backward induction outcome:

(B, L, T, L)

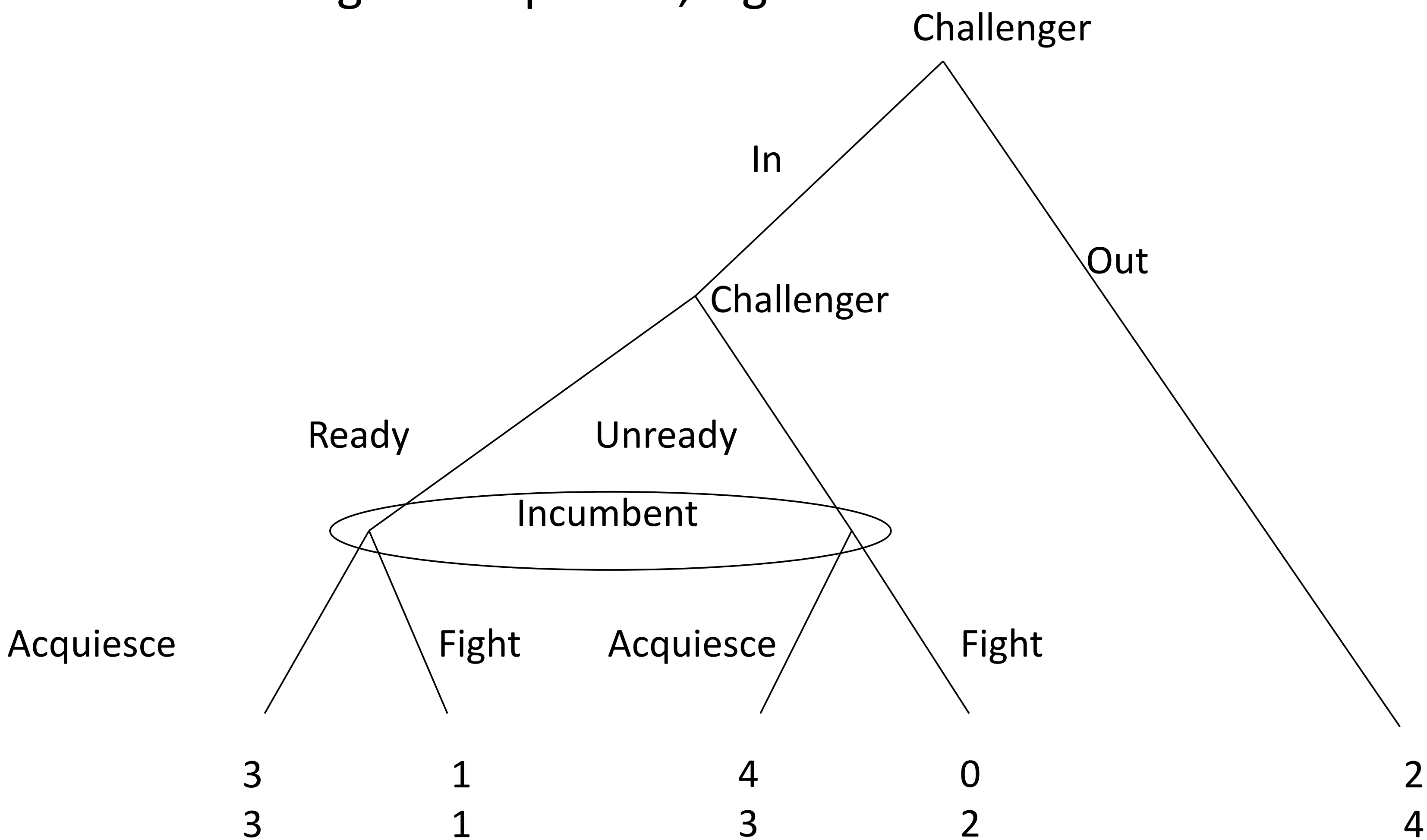
Subgame perfect Nash equilibrium

(B, L, (T, T, T, B), (R, R, L, L))

Example 2

Challenger's strategies: {(Out Ready), (Out Unready) (In ready), (In Unready)}

Incumbent' strategies: Acquiesce, Fight



		Incumbent	
		Acquiesce	Fight
	Out Ready	2, $\underline{4}$	$\underline{2}$, $\underline{4}$
	Out Unready	2, $\underline{4}$	$\underline{2}$, $\underline{4}$
Challenger	In Ready	3, $\underline{3}$	1, 1
	In Unready	$\underline{4}$, $\underline{3}$	0, 2

Three Nash equilibria:

1. (Out Ready, Fight);
2. (Out Unready, Fight)
3. (In unready, Acquiesce)

Consider the subgame starting in the decision node after Challenger's choice ***In***

		Incumbent	
		Acquiesce	Fight
Challenger	Ready	3, <u>3</u>	<u>1</u>, 1
	Unready	<u>4</u>, <u>3</u>	0, 2

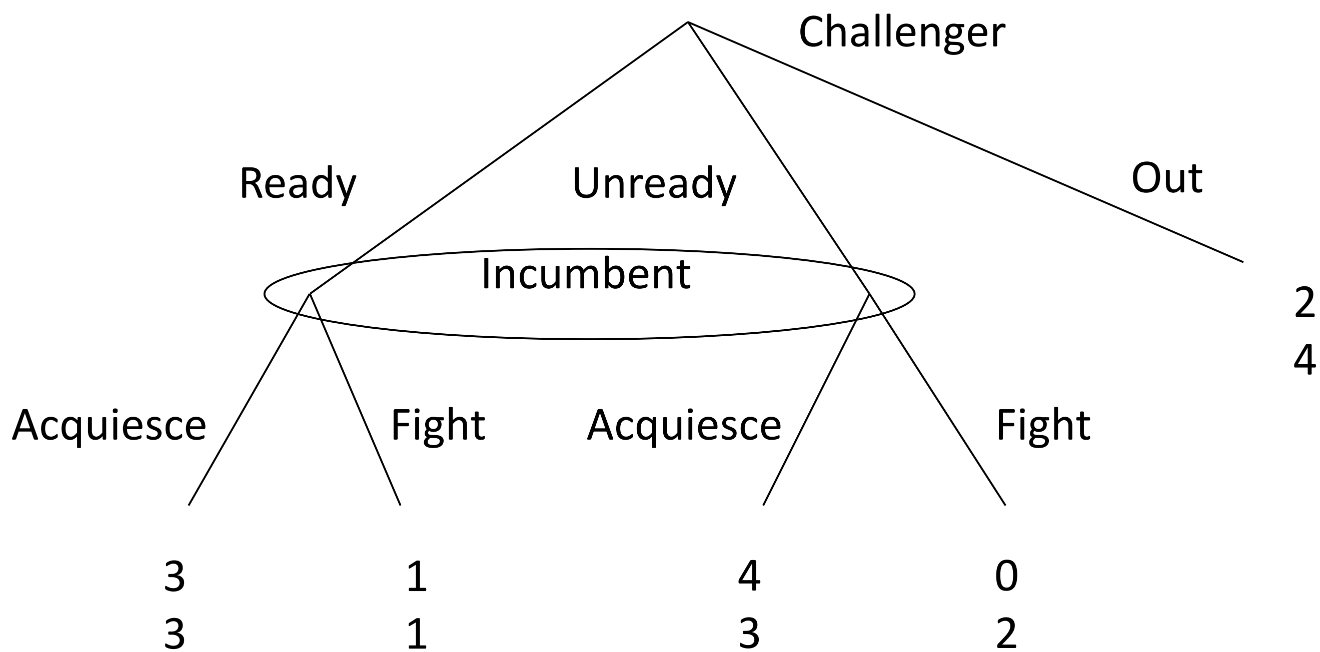
An unique Nash equilibrium: Unready, Acquiesce
Then, only (In unready, Acquiesce) is **subgame perfect NE**

Example 3

		Incumbent	
		Acquiesce	Fight
Challenger	Ready	3, $\underline{3}$	1,1
	Unready	$\underline{4}$, $\underline{3}$	0, 2
	Out	2, $\underline{4}$	$\underline{2}$, $\underline{4}$

Two Nash equilibria:
(Out, Fight)
(Unready, Acquiesce)

Both are SPNE



Applications with imperfect information

Bank Runs

- Two investors, one bank
- Each investor has deposited D with a bank
- The bank has invested $2D$ in a long term project
- If the bank liquidates the investment before the end, it will get back $2r$, where $D/2 < r < D$
- Otherwise the bank will get $2R$, where $R > D$

- Investors can make withdrawals from the bank at:
  - Date 1, before the end of the investment
  - Date 2, after the end of the investment
- It is enough that one investor makes withdrawal at date 1 to force the bank to liquidate the investment

Payoffs:

- Both investors make withdrawals at date 1:
  - each one receives r.
- Only one investor makes withdrawal at date 1:
  - he receives D, the other receives $2r – D$.
- Neither investor makes withdrawal at date 1:
  - Both investors will take a withdrawal decision at date 2
- Both investors make withdrawals at date 2:
  - each receive R
- Only one investor makes withdrawal at date 2:
  - he receives $2R - D$, the other receives D.
- Neither investor makes withdrawal at date 2:
  - The Bank returns R to each investor

Date 1

		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	r, r	D, 2r - D
	No withdraw	2r – D, D	Next stage

Date 2

		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	R, R	2R – D, D
	No withdraw	D, 2R – D	R, R

We solve the game in date 2

Date 2

		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	$\underline{R}, \underline{R}$	$\underline{2R - D}, D$
	No withdraw	$D, \underline{2R - D}$	R, R

In date 2's game there is only one Nash equilibrium:

{(Withdraw), (Withdraw)}

where each Investor gets *R*

In date 1 the two investors anticipate that in the case neither investor makes withdrawal at date 1, the game goes in the second stage (date 2) and that in date two the outcome will be (the NE):

{(Withdraw), (Withdraw)}

where each Investor gets *R.*

Then the game in date 1 can be written as:

Date 1			
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	r, r	$D, 2r - D$
	No withdraw	$2r – D, D$	R, R

We solve the game in date 1:

		Date 1	
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	$\underline{r}, \underline{r}$	$D, 2r - D$
	No withdraw	$2r - D, D$	$\underline{R}, \underline{R}$

There are two Nash equilibria in date 1 game:

{(Withdraw), (Withdraw)}

{(No withdraw), (No Withdraw)}

Game in Date 2, one NE: {(Withdraw), (Withdraw)}

Game in Date 1 (reduced), two NE:

1. {(Withdraw), (Withdraw)}
2. {(No withdraw), (No Withdraw)}

Whole game:

Two Backward Induction Outcomes (BIO):

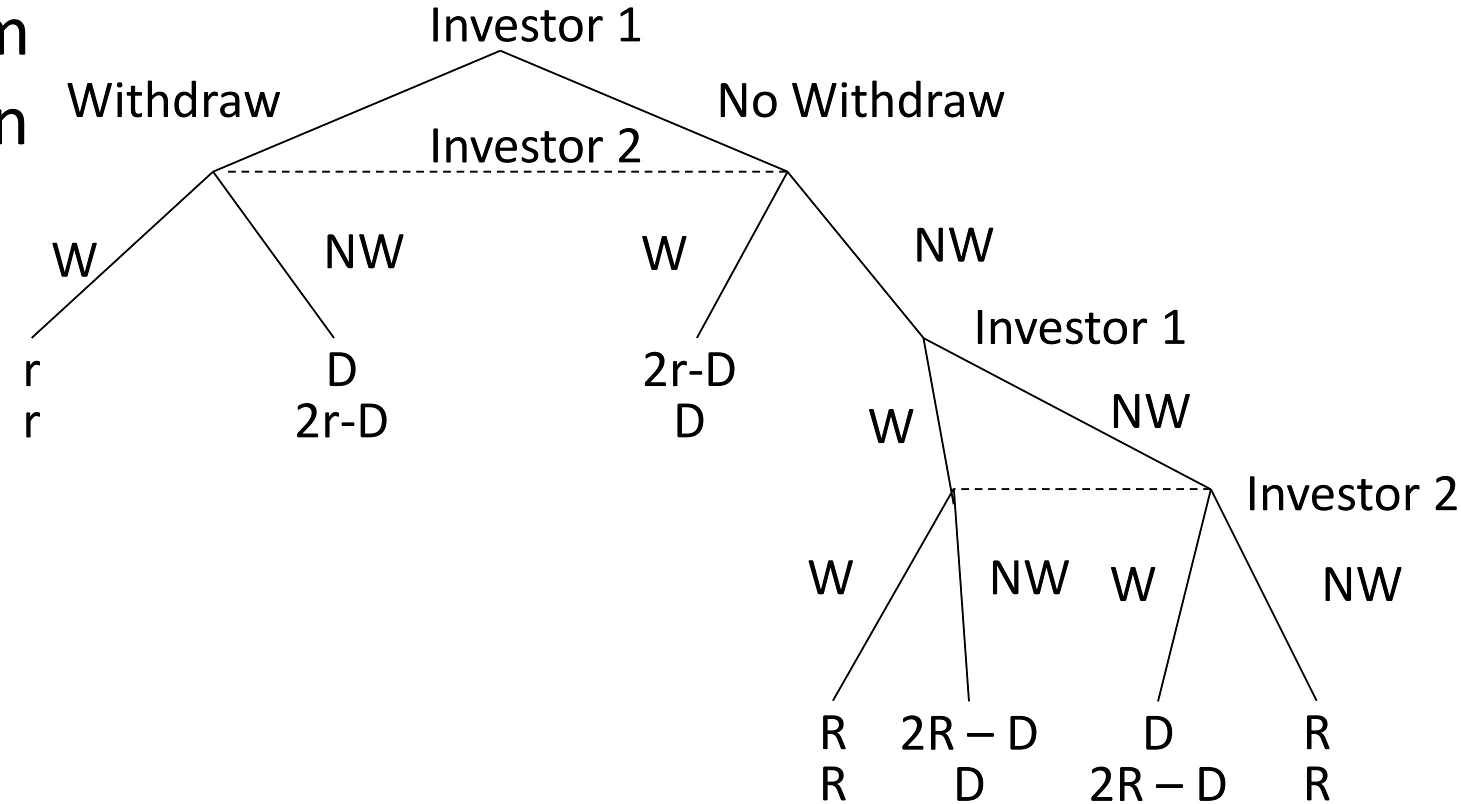
1) {(Withdraw), (Withdraw)} in date 1
2) {(No withdraw), (No Withdraw)} in date 1,
   {(Withdraw), (Withdraw)} in date 2

Two subgame perfect NE (SPNE):

1) {(Withdraw, Withdraw), (Withdraw, Withdraw)}
2) {(No withdraw, Withdraw), (No Withdraw, Withdraw)}

Note SPNE 1) supports BIO 1), SPNE 2) supports BIO 2)

Extensive form representation



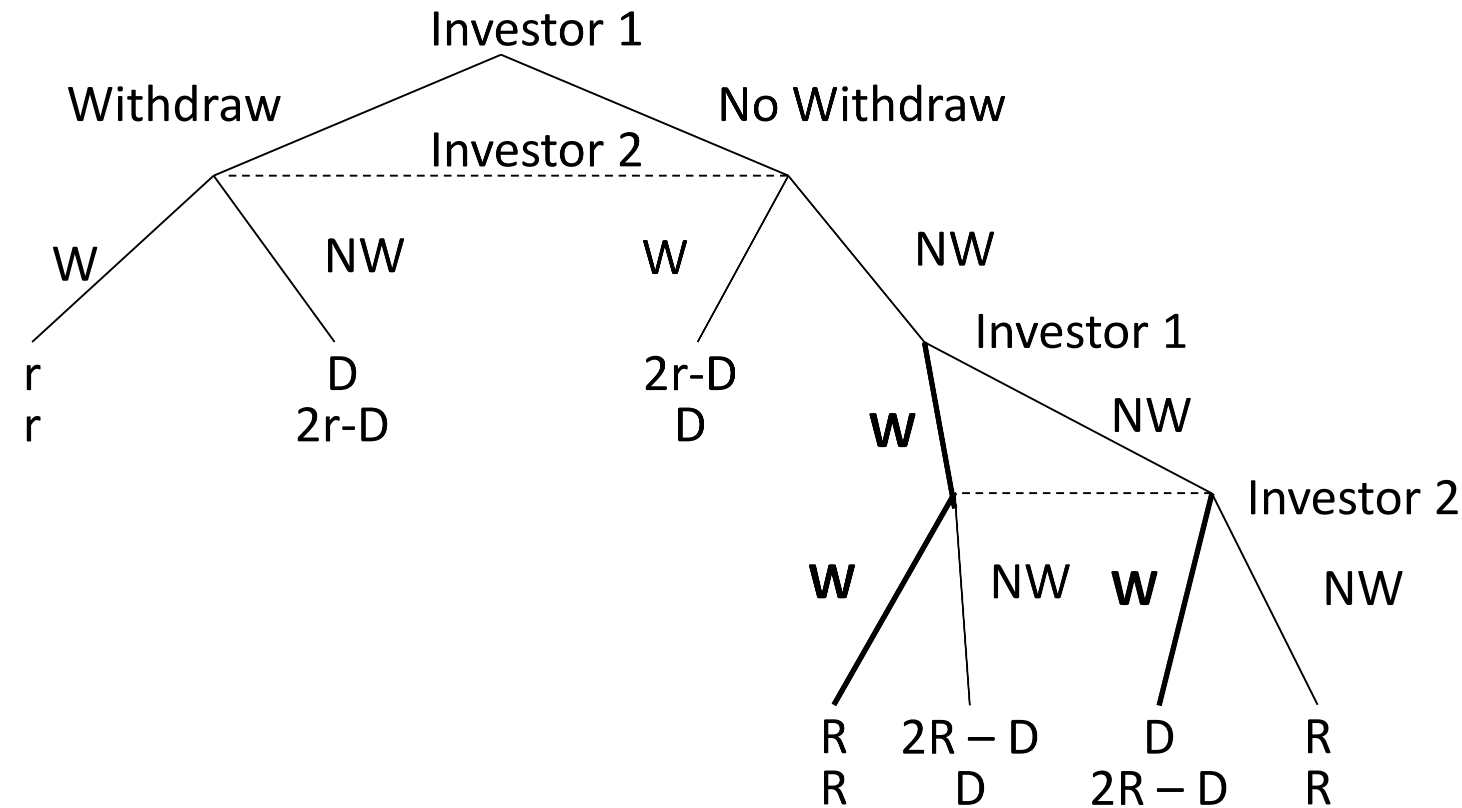
Investor 1: 2 information sets

Investor 2: 2 information sets

Investor 1’s strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Investor 2’s strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Backward Induction



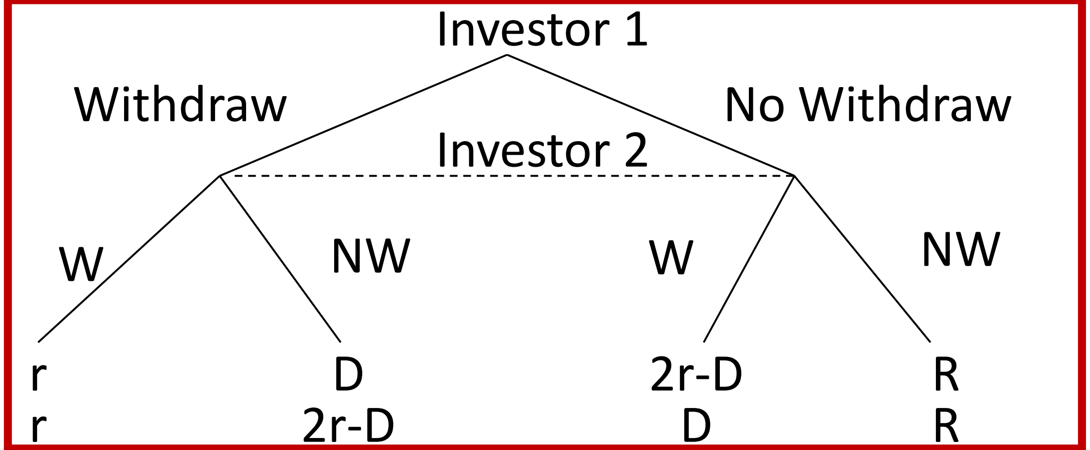
Investor 1: 2 information sets

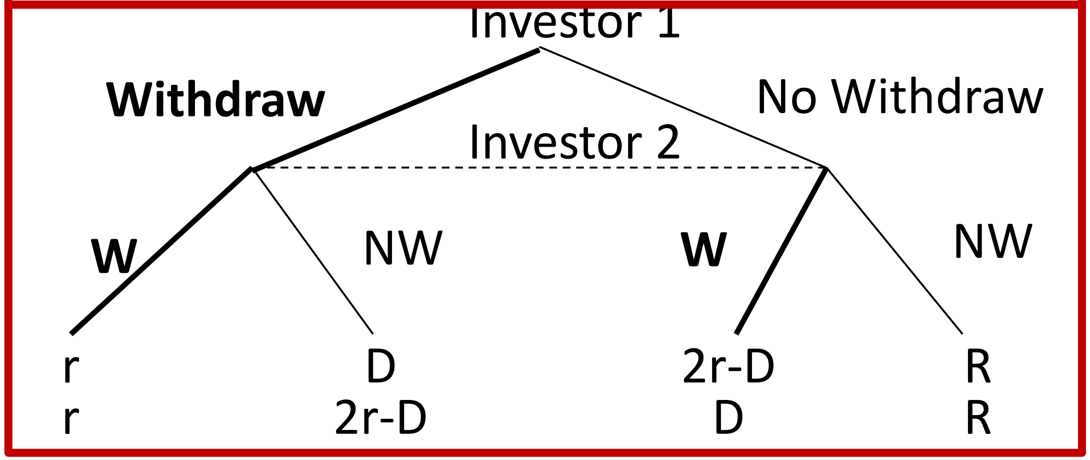
Investor 2: 2 information sets

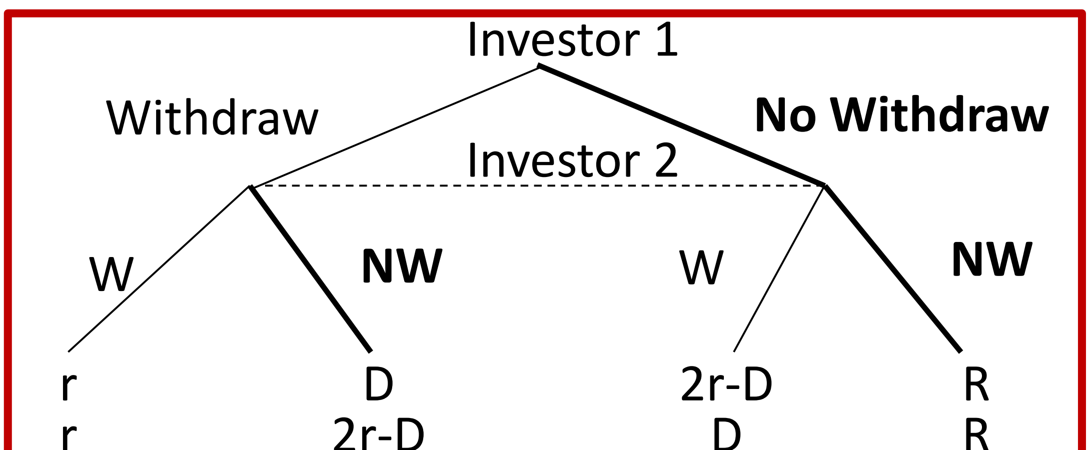
Investor 1's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Investor 2's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Reduced game







Tariffs and Imperfect international competition

- Two identical countries denoted by $i = 1, 2$.
- One homogeneous good is produced in each country by a firm, firm i in country i
- A share h_i of this product is sold in the home market and a share e_i is exported in the other country
- Governments choose tariffs, i.e. a tax on the import. Government of country i chooses tariff t_i.

In country i the market clearing price is:

$$P_i(Q_i) = a - Q_i$$

where $Q_i = hi + ej$

Firms have constant marginal cost, c, and no fixed cost

<u>Firm's payoff</u> (profits):

$$\pi_i = [a - hi - e_j]hi + [a - h_j - e_i]e_i - c[hi + e_i] - t_j\, e_i$$

<u>Government's payoff</u>

(consumer welfare + home firm's profit + tariff revenue)

$$W_i = 0.5\, Qi^2 + \pi_i + t_i e_j$$

- Timing
    1. Governments simultaneously choose tariffs (t_1, t_2)
    2. Firms observe (t_1, t_2) and simultaneously choose quantities $(h_1, e_1), (h_2, e_2)$.

- Backward induction solution
    1. We suppose that governments have chosen tariffs (t_1, t_2) and we find the optimal behaviour of firms as function of (t_1, t_2).
    2. We assume that governments correctly predict the optimal behaviour of firms for each possible combination of (t_1, t_2) and we find the optimal tariff rates.

We suppose that governments have chosen tariffs (t_1, t_2) and we find the optimal behaviour of firms as function of (t_1, t_2).

$$\max_{(h_1, e_1)} \pi_1$$

where

$$\pi_1 = [a - h_1 - e_2]h_1 + [a - h_2 - e_1]e_1 - c[h_1 + e_1] - t_2 e_1$$

Firm 1's FOCs:

$$[a - 2h_1 - e_2] - c = 0$$

$$[a - h_2 - 2e_1] - c - t_2 = 0$$

$$\rightarrow$$

$$h_1 = (a - e_2 - c) / 2$$

$$e_1 = (a - h_2 - c - t_2) / 2$$

For Firm 2:

$$\max_{(h_2,e_2)} \pi_2$$

where

$$\pi_2 =$$

$$[a - h_2 - e_1]h_2 + [a - h_1 - e_2]e_2 - c[h_2 + e_2] - t_1 e_2$$

Firm 2's FOCs:

$$[a - 2h_2 - e_1] - c = 0$$

$$[a - h_1 - 2e_2] - c - t_1 = 0$$

$$\rightarrow$$

$$h_2 = (a - e_1 - c)/2$$

$$e_2 = (a - h_1 - c - t_1)/2$$

We have to solve a system of 4 equations in 4 unknowns:

1. $h_1 = (a - e_2 - c) / 2$
2. $e_1 = (a - h_2 - c - t_2) / 2$
3. $h_2 = (a - e_1 - c) / 2$
4. $e_2 = (a - h_1 - c - t_1) / 2$

Solutions:

1. $h_1^* = (a - c + t_1) / 3$
2. $e_1^* = (a - c - 2t_2) / 3$
3. $h_2^* = (a - c + t_2) / 3$
4. $e_2^* = (a - c - 2t_1) / 3$

We assume that governments correctly predict the optimal behaviour of firms for each possible combination of (t_1 , t_2) and we find the optimal tariff rates.

The problem of country 1's government is:

$$\max_{\{t_1\}} W_1 = 0.5\,(Q_1^*)^2 + \pi_1^* + t_1 e_1^*$$

where

$$Q_1^* = h_1^* + e_2^* = \frac{(a - c + t_1)}{3} + \frac{(a - c - 2t_1)}{3}$$

$$= \frac{(2a - 2c - t_1)}{3}$$

$$\pi_1^* = [a - h_1^* - e_2^*]h_1^* + [a - h_2^* - e_1^*]e_1^* - c[h_1^* + e_1^*] - t_2 e_1^*$$

Using algebra

$$W_1 = \frac{(2(a-c) - t_1)^2}{18} + \frac{(a-c+t_1)^2}{9} + \frac{(a-c-2t_2)^2}{9} + \frac{t_1(a-c-2t_1)}{3}$$

Similarly we can write the problem of country 2's government

We compute the governments' FOCs and we find:

$t_1^* = (a - c)/\ 3$ $t_2^* = (a - c)/\ 3$

Then

Firm 1 will produce:

$h_1^* = 4(a - c)/\ 9$ $e_1^* = (a - c)/\ 9$

Firm 2 will produce:

$h_2^* = 4(a - c)/\ 9$ $e_2^* = (a - c)/\ 9$

Backward Induction outcome

$t_1^* = (a - c)/\ 3$ $\qquad$ $t_2^* = (a - c)/\ 3$

$h_1^* = 4(a - c)/\ 9$ $\qquad$ $e_1^* = (a - c)/\ 9$

$h_2^* = 4(a - c)/\ 9$ $\qquad$ $e_2^* = (a - c)/\ 9$

Subgame Perfect Nash Equilibrium (SPNE):

Note:

One info set for governments

infinite number of info set for firms, i.e. each possible combination of t_1 t_2

$t_1^* = (a - c)/ 3$ $t_2^* = (a - c)/ 3$

$h_1^* = (a - c + t_1) / 3$

$e_1^* = (a - c - 2t_2) / 3$

$h_2^* = (a - c + t_2) / 3$

$e_2^* = (a - c - 2t_1) / 3$