Lecture 7

Repeated Games

A repeated game: it is a game that is repeated for several periods.

The game that is played in a single period is called "***Stage game***"

In each period, players take their decisions and payoffs are consequently paid

1) **Finitely repeated game**
2) **Infinitely repeated games**

Note

By **BIO** we means ***b****ackward* ***i****nduction* ***o****utcome*. Sometime we call it ***s****ubgame* ***p****erfect outcome* (**SP**)

Finitely Repeated Games

Consider the following example

Prisoners' dilemma

		Player 2	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

The two players play this simultaneous game for two periods

Before the second play begins, players observe the outcome of the first play

No discount: payoff of entire game is the sum of the payoffs from the two periods

This is a game of complete and imperfect information

Note that

a) The players of the second period are the same of the first period

b) The NE equilibrium in the second period does not depend on the first period outcome.

We solve it using backward induction

Consider the second period

There is only one NE: {L1, L2}

		Player 2	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

Note, you can solve the game by iterated elimination of dominated strategies:

R1 is dominated by L1 and

R2 is dominated by L2

Players correctly predict the second period outcome, then, to each outcome of the first period, we add the payoff of the NE in the second period

		Player 2	
		L2	**R2**
Player 1	**L1**	2, 2	6, 1
	R1	1, 6	5, 5

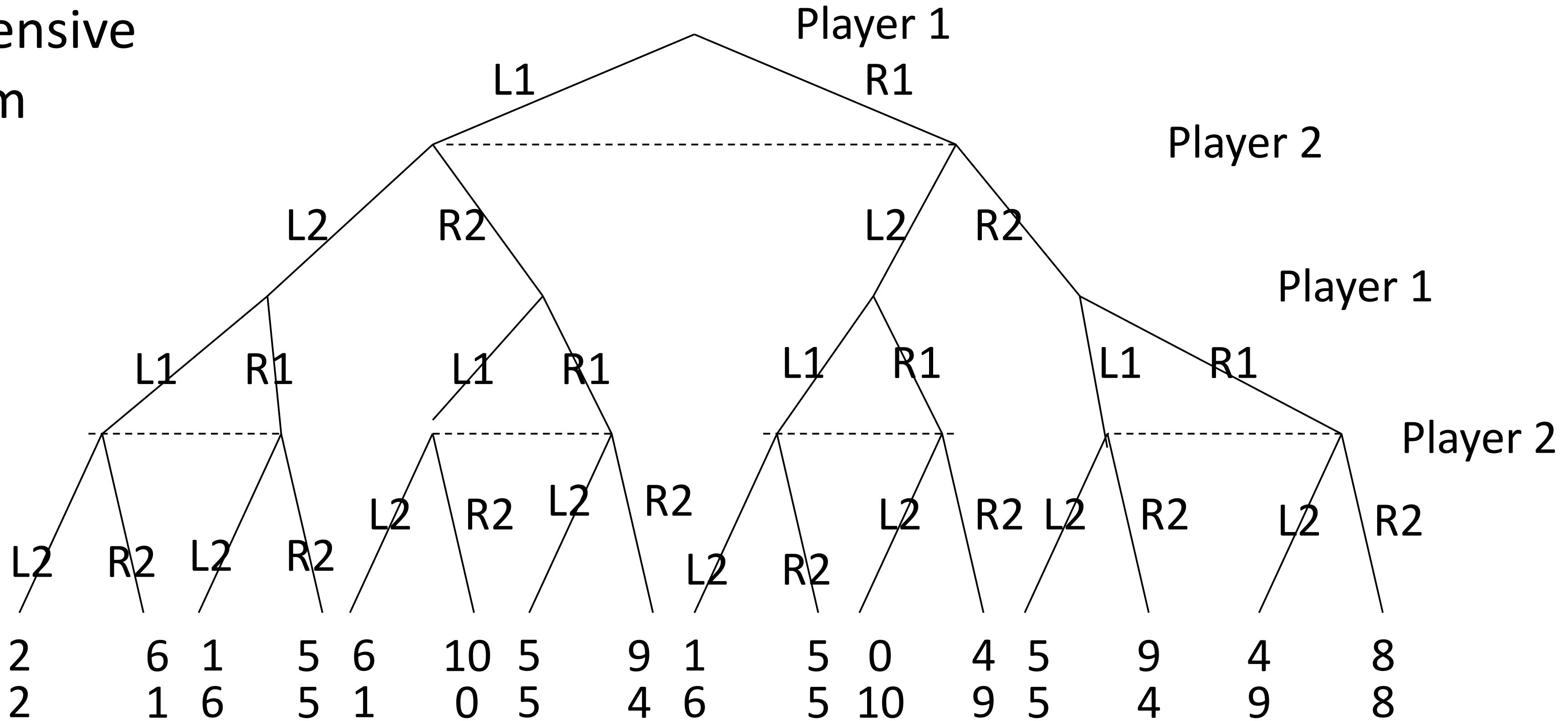
In the SPNE: Player 1 plays L1 in both periods, Player 2 plays L2 in both periods

Backward Induction outcome:

(L1, L2) in the first period and

(L1, L2) in the second period.

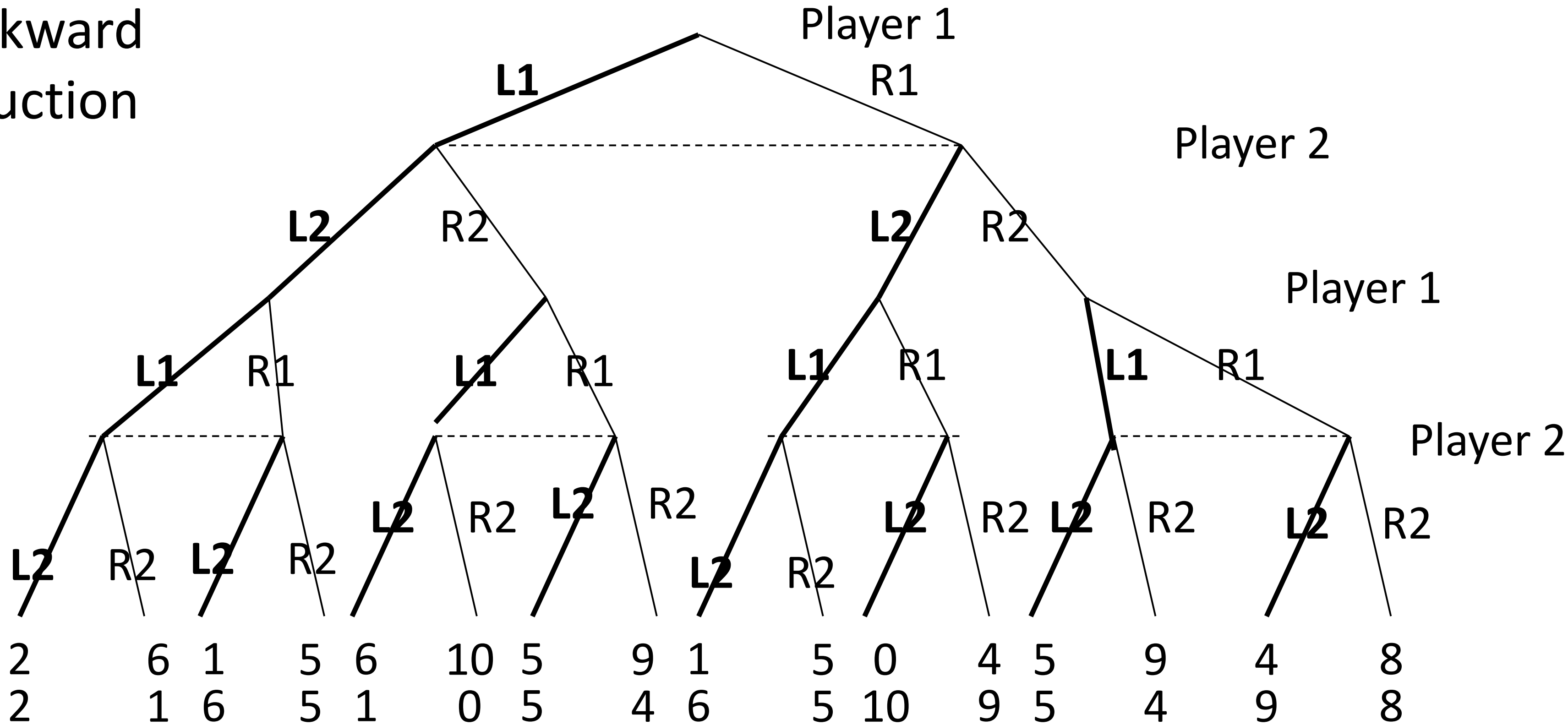
Extensive form



Note: 5 subgames, each player has 1 information set in period 1 and 4 information sets in period 2

The strategy of a player states one action for each information set, then 1 action for the first period and 4 actions for the second period.

Backward Induction



SPNE is: {(L1, L1, L1, L1, L1), (L2, L2, L2, L2, L2)}

BI outcome: (L1, L2) in the first period

(L1, L2) in the second period

First General Result

$G = (A_1,...A_N; u_1,...u_n)$ is the stage game

$G(T)$ denotes the finitely repeated game where G is played for T periods.

At the start of each period, all previous outcomes are observed.

The payoffs of $G(T)$ are the sum of the payoffs from the T stage games.

Proposition: if the stage game G has an unique NE, then for any finite T, the repeated game $G(T)$ has a unique SP outcome: The NE is played in every period of the game.

Consider the following example

Stage game		Player 2		
		L2	**M2**	**R2**
Player 1	**L1**	**1,1**	5, 0	0, 0
	M1	0, 5	4, 4	0, 0
	R1	0, 0	0, 0	**3, 3**

Stage game is played two periods

First period outcomes are observed before the second period starts

Two NE: (L1, L2) and (R1, R2)

For both players the preferred outcome is (M1, M2)

In this example, where the stage game has two NE, I will show that exists a SPNE where (M1, M2) is played in the first period.

Suppose that Players 1 and 2 have the following strategies:

Player 1: R1 if the first period outcome is (M1, M2) , otherwise L1

Player 2: R2 if the first period outcome is (M1, M2) , otherwise L2

Note, in each of the 9 subgames in the second period this strategy profile is a NE.

Stage game		**Player 2**		
		L2	**M2**	**R2**
Player 1	**L1**	**1,1**	5, 0	0, 0
	M1	0, 5	4, 4	0, 0
	R1	0, 0	0, 0	**3, 3**

Period 1		Player 2		
		L2	**M2**	**R2**
Player 1	**L1**	1,1	5, 0	0, 0
	M1	0, 5	**4, 4**	0, 0
	R1	0, 0	0, 0	3, 3

Period 2		Player 2		
		L2	**M2**	**R2**
Player 1	**L1**	**1,1**	5, 0	0, 0
	M1	0, 5	4, 4	0, 0
	R1	0, 0	0, 0	**3, 3**

The two players anticipate this outcome in period two, then the choices in first period can be represented as the following reduced game:

<table>
<tr><th colspan="2" rowspan="2">Stage 1</th><th colspan="3">Player 2</th></tr>
<tr><th>L2</th><th>M2</th><th>R2</th></tr>
<tr><td rowspan="3">Player 1</td><td>L1</td><td>2,2</td><td>6, 1</td><td>1, 1</td></tr>
<tr><td>M1</td><td>1, 6</td><td>7, 7</td><td>1, 1</td></tr>
<tr><td>R1</td><td>1, 1</td><td>1, 1</td><td>4, 4</td></tr>
</table>

This reduced game has **three NE: (L1, L2), (M1, M2) and (R1, R2)**

a) NE **(L1, L2)** corresponds to the BIO {(L1, L2), (L1, L2)} of the whole game

b) NE **(M1, M2)** corresponds to the BIO {(M1, M2), (R1, R2)} of the whole game

c) NE **(R1, R2)** corresponds to the BIO {(R1, R2), (L1, L2)} of the whole game

Second General result.

If the stage game G has multiple NE, then it may exist a BIO of the repeated game G(T) in which the outcome of some period t < T is not a NE of the stage game G.

Intuition: if players cooperate in period 1 (M1, M2) they play the high payoff equilibrium in period 2 (R1, R2) ,otherwise (in period 2) they play the low payoff equilibrium (L1, L2)

Infinitely Repeated Games

- The stage game G is played an infinite number of times
- At the start of each period, all previous outcomes are observed.

- Even if the stage game has an unique NE then may be a SP outcome of the infinitely repeated game in which no period outcome is a NE of the stage game G.

Stage Game		Player 2	
		L2	R2
Player 1	L1	1, 1	5, 0
	R1	0, 5	4, 4

- Differently from finitely repeated games, here it is not possible to sum up the payoffs from all stage games (infinite sequence):
  - To receive 4 every period is better than to receive 1 every period. But the sum of payoffs from all periods is infinite in both cases.
- We assume that players discount future payoffs by a discount factor $0 < d < 1$.
- d is the today's present value of £1 to be received one stage later.

Consider the infinite sequence of payoffs $p_1, \ p_2, \ p_3, \ldots\ldots.$ where p_i is the payoff from the stage game played in period i. At time 1 the present value of this sequence is:

$$p_1 + d\,p_2 + d^2\,p_3 + d^3\,p_4 \ldots\ldots\ldots\ldots.$$

Note (useful results)

The present value **at time 1** of an infinite sequence of payoffs $p_1, \ p_2, \ p_3 \ldots\ldots\ldots$ where $p_1 = p_2 = p_3 = \cdots.. = x$ is:

$$\frac{x}{1-d}$$

The present value **at time 1** of an infinite sequence of payoffs $p_2, p_3, p_4 \ldots\ldots..$ where $p_2 = p_3 = p_4 = \cdots.. = x$ is:

$$\frac{d\,x}{1-d}$$

By the infinitely repeated Prisoners' dilemma we show that is possible to get cooperation in each period (R1, R2) even if the unique NE of the stage game is no cooperation (L1, L2).

Suppose the following ***trigger strategy***:

player 1 strategy *is to play:*

i) R1 in period 1,

ii) in each period $t > 1$, R1 only if in all previous periods the outcome was (R1, R2); otherwise L1.

player 2 strategy *is to play*

i) R2 in period 1,

ii) in each period t>1, R2 only if in all previous periods the outcome was (R1, R2); otherwise L2.

Stage Game		**Player 2**	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

Trigger strategy:

player *i* strategy is to play *Ri* in period *1*, then *Ri* in period *t* only if in all previous periods the outcome was *(R1, R2)*, *Li* otherwise, where *i = 1, 2*

Stage Game		**Player 2**	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

To play *Ri* in every period has a present value of (at time 1):

$$\frac{4}{1-d}$$

In period 1 to deviate to L1 produces a immediate payoff of 5 but in all other periods (L1, L2) will be played with a payoff of 1 for each period.

Then the present value of the deviation is:

$$5 + \frac{d}{1-d}$$

To play *Ri* in stage 1 we need:

$$5 + \frac{d}{1-d} < \frac{4}{1-d}$$

We can repeat this reasoning at every period

Then to play *Ri* at every period is a **BI outcome**.

This BI outcome is based on the threat:

"if you don't cooperate I will play L1 for ever"

Solving the inequality we find the range of the discount's value that support this SPNE.

d > 0.25

Definition of Infinitely Repeated Game

$G \;=\; (A_1, \dots A_n;\; u_1, \dots u_n)$ is the stage game

$G(\infty, d)$ denotes the infinitely repeated game where G is played forever and d is the discount factor

At the start of each period, all previous outcomes are observed.

Each player's payoffs in $G(\infty, d)$ is the present value from the infinite sequence of stage games.

Definition of strategy in a repeated game

In the finitely repeated game $G(T)$ and in the infinitely repeated game $G(\infty, d)$ a player strategy specifies the action the player will take in each period, for each possible history of play through the previous periods

Note

"history of play through period t" means all players' choices from period 1 to period t

Definition of subgame in a repeated game

In the finitely repeated game $G(T)$ a subgame beginning at period $t+1$ is the repeated game $G(T-\ t)$

In the infinitely repeated game $G(\infty, d)$ a subgame beginning at period $t+1$ is identical to $G(\infty, d)$

At stage $t+1$, for each possible history, one subgame begins.

Subgame Perfect Nash Equilibrium (SNE)

A NE is SPNE if the players' strategies are a NE in every subgame.

(Selten 1965)

Theorem (Friedman 1971)

Let G be a finite, static game of complete information.

Let $(e_1, \ldots e_n)$ denote the payoff from a NE of G

Let $(x_1, \ldots x_n)$ denote any other feasible payoff from G

If $x_i > e_i$ for all i, and d is sufficiently close to 1, there exists a SPNE of $G(\infty, d)$ that achieves $(x_1, \ldots x_n)$ as average payoffs.

Note:

Average payoff is

$$(p_1 + d\, p_2 + d^2\, p_3 + d^3\, p_4 \ldots \ldots \ldots \ldots.) * (1 - d)$$

Another strategy to sustain a collusive outcome

Play ***Ri*** in the first period

In the t^{th} period , play ***Ri*** if both players played ***Ri*** in period *t-1,* play ***Ri*** if both players played ***Li*** in period *t-1*, and otherwise play ***Li***.

Stage Game		**Player 2**	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

To play ***Ri*** in every period has a present value of (at time 1):

$$\frac{4}{1-d}$$

In period 1 to deviate to L1 produces a immediate payoff of 5 but in the period 2 (**L1, L2**) will be played with a payoff of 1 for each period. Then again ***Ri*** will be played in periods 3 and after

Then the present value of the deviation is:

$$5\ + d + \frac{4d^2}{1-d}$$

To play *Ri* in stage 1 we need:

$$5\ + d + \frac{4d^2}{1-d} < \frac{4}{1-d} \quad \rightarrow\ d > 0.33$$

Stage Game		**Player 2**	
		L2	**R2**
Player 1	**L1**	1, 1	5, 0
	R1	0, 5	4, 4

Punishment is not an equilibrium

Stage Game		Player 2		
		L2	**R2**	**P2**
	L1	2, 2	6, 1	4.5, 1
Player 1	**R1**	1, 6	5, 5	4.5, 1
	P1	1, 4.5	1, 4.5	3, 3

Play ***Ri*** in the first period

In the t^{th} period , play ***Ri*** if

- either both players played ***Ri*** in period *t-1 or*
- both players played ***Pi*** in period *t-1,*

otherwise play ***Pi***.

Stage Game		Player 2		
		L2	**R2**	**P2**
	L1	2, 2	6, 1	4.5, 1
Player 1	**R1**	1, 6	5, 5	4.5, 1
	P1	1, 4.5	1, 4.5	3, 3

Two class of subgames:

- Collusive subgames → the outcome in t-1 was (R1, R2) or (P1, P2) (Collusive phase)
- Punishment subgames → the outcome in t-1 was neither (R1, R2) nor (P1, P2) (Punishment phase)

<u>Deviation in the Collusive phase</u>

$$\frac{5}{1-d} > 6 + d \cdot 3 + \frac{5 \cdot d^2}{1-d}$$
$$d > 0.5$$

<u>Deviation in the Punishment phase</u>

$$3 + \frac{d \cdot 5}{1-d} > 4.5 + d \cdot 3 + \frac{5 \cdot d^2}{1-d}$$
$$d > 0.75$$