UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dipartimento di Ingegneria e Architettura

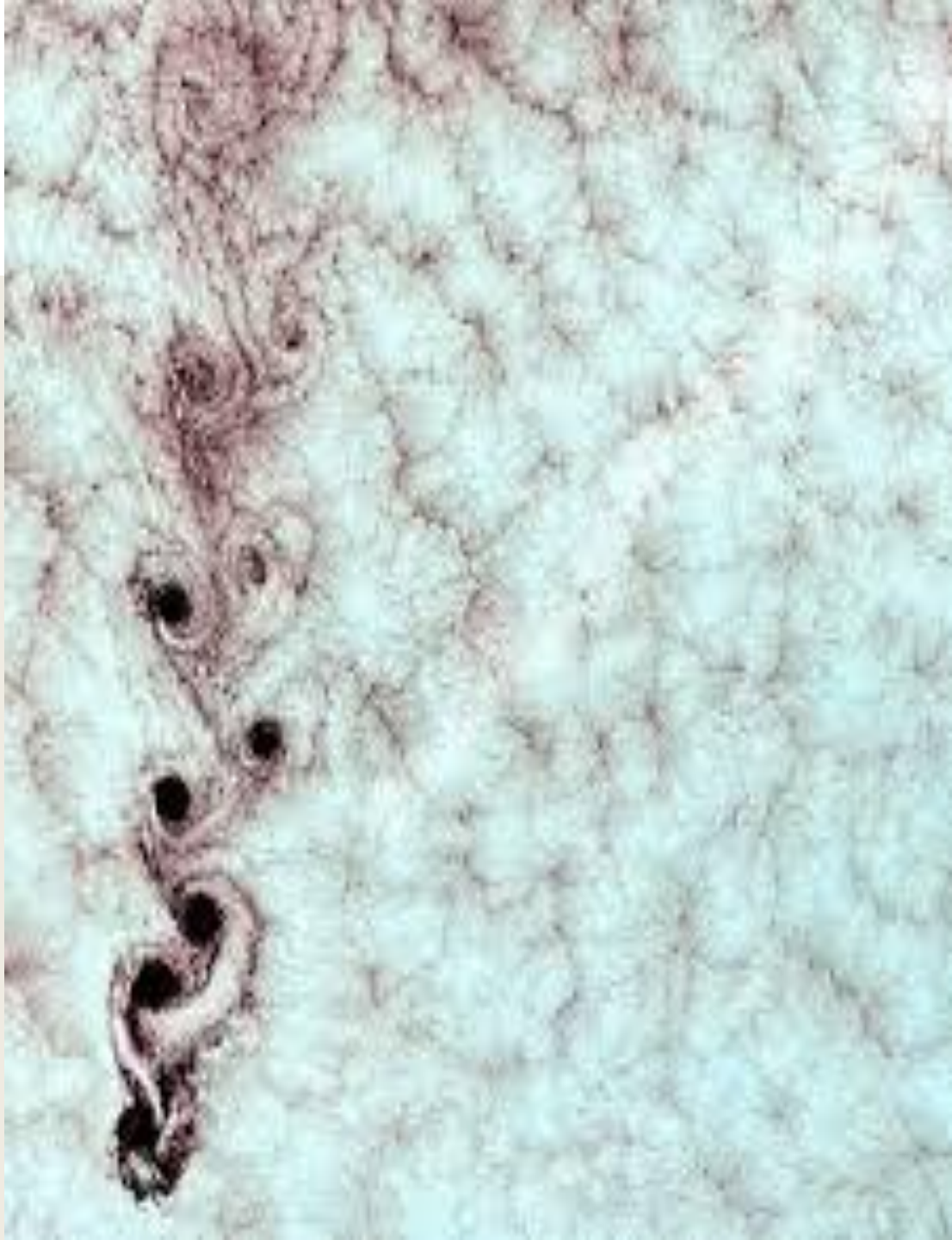
# Civil Engineering

## Hydraulics

## Unsteady flow in pressure pipes


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# Introduction – Unsteady flow

The fundamental quantities (pressure, velocity and discharge) vary instant by instant and section by section

UNSTEADY FLOW

$$H(x,t) = z(x,t) + \frac{p(x,t)}{\gamma} + \frac{V(x,t)^2}{2g}$$

Piezometric line and total head line vary in time

Specific energy does not necessarily decrease from upstream to downstream: along a trajectory a fluid particle can increase its mechanical energy.

STEADY FLOW

$$H(x) = z(x) + \frac{p(x)}{\gamma} + \frac{V(x)^2}{2g}$$

Piezometric line and total head line are constant in time

Specific energy decreases progressively from upstream to downstream, only because of resistance effects

In unsteady flow, energy is conserved globally → elastic energy of the fluid and of the pipe

Unsteady flow can be characterized by zones with large increases/decreases of pressure, velocity and discharge.

# Introduction – Unsteady flow

A change of discharge in a generic section propagates along the stream as a wave.

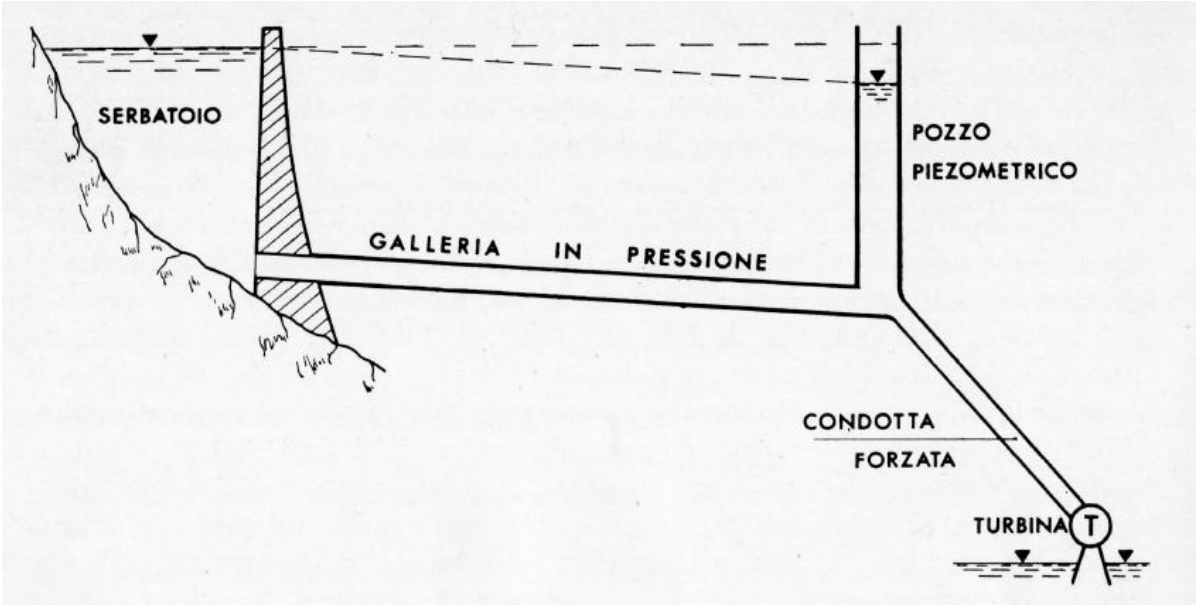
The passage of the wave causes pressure variations → compression or expansion of the liquid mass.

Celerity: propagation speed of the disturbance.

Along its path the disturbance can change and be reflected, depending on the change of boundary conditions. Of particular interest is the case in which the disturbance meets a wall or a free surface.

# Introduction – Hydropower plant

Examples of unsteady flow in a hydropower plant



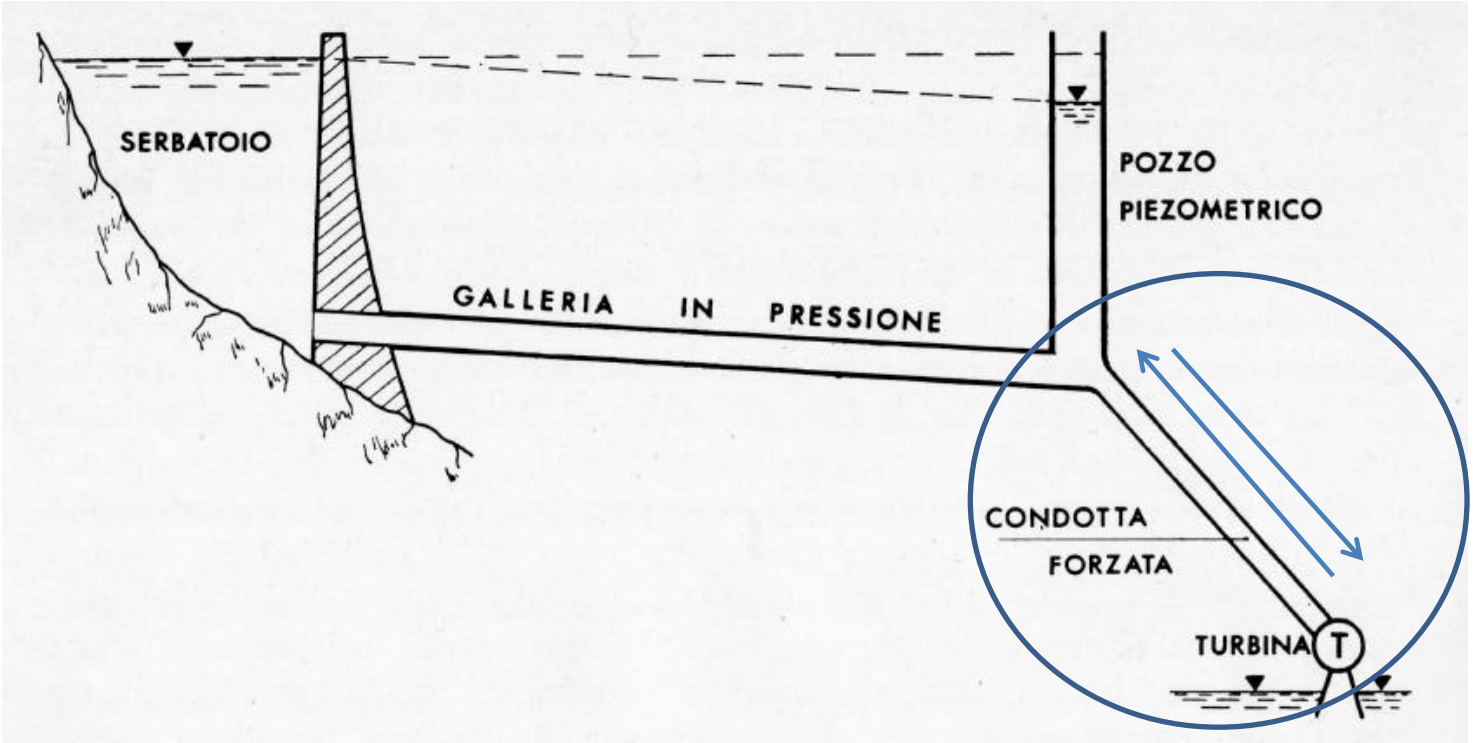

*SERBATOIO = reservoir;*
*GALLERIA IN PRESSIONE = pressure tunnel;*
*POZZO PIEZOMETRICO = surge tank;*
*CONDOTTA FORZATA = penstock;*
*TURBINA = turbine*

Components:

- Large-capacity reservoir
- Pressure tunnel (long, low pressures)
- Surge tank (piezometric shaft)
- Penstock (short, high head drop)
- Flow regulator (valve)
- Turbine

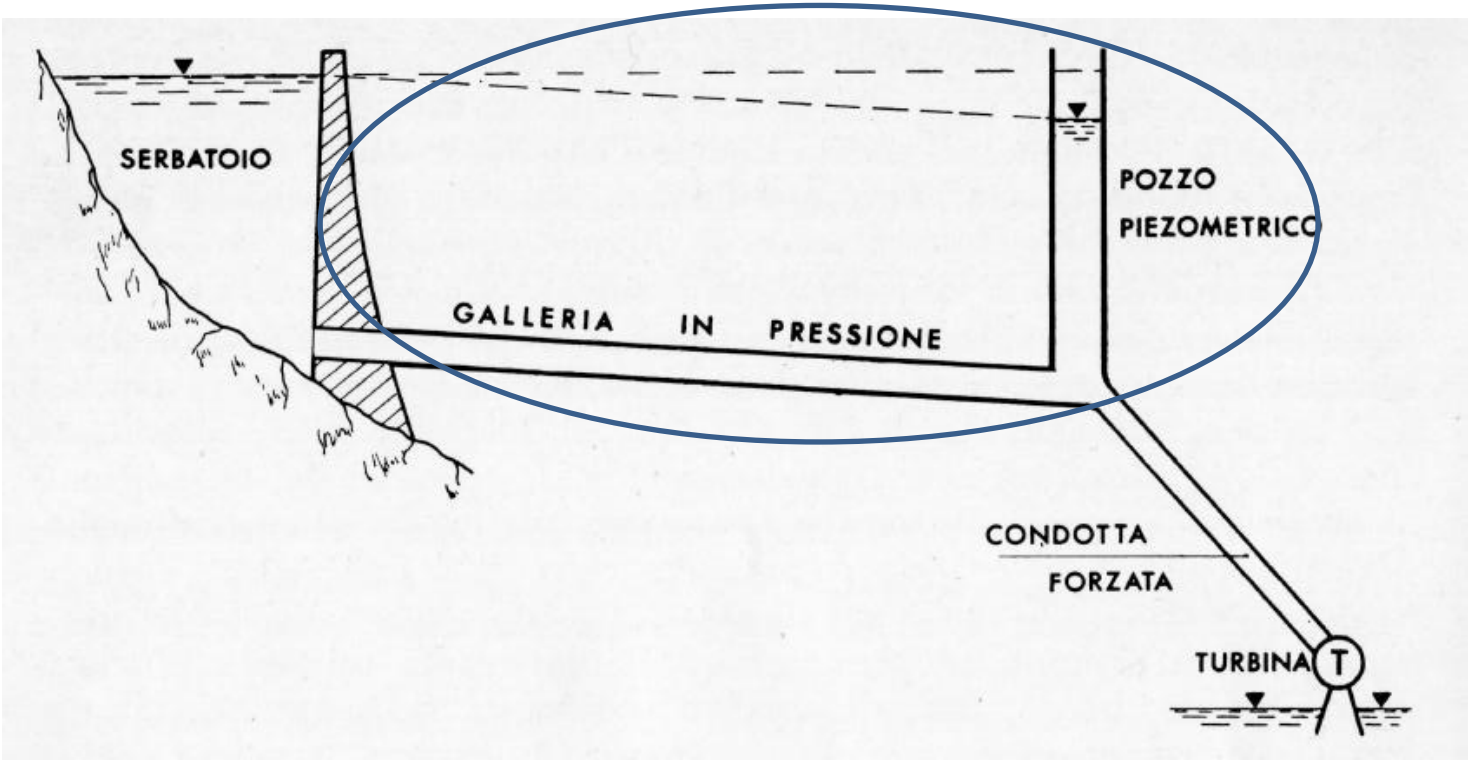


# Introduction – Hydropower plant




The flow regulator generates disturbances that propagate upstream up to the surge tank.
The surge tank behaves like a very large reservoir, keeping the pressure at the inlet section of the penstock constant → negative reflection of the disturbance.
The disturbance that reaches the regulator is reflected and stays in the penstock.
The process continues alternately, producing periodic variations of pressure and velocity

# Introduction – Hydropower plant




Another type of unsteady flow.
The pressure tunnel – surge tank system adapts more slowly to the new discharge in the penstock.
The difference in discharge is absorbed by the surge tank through its liquid level, which in turn generates new disturbances. Propagation times are very small compared to the time scale of the level evolution. Their celerity can be considered infinite → incompressible liquid and rigid pipe. In this case we are dealing with mass oscillations.

# Introduction – Water hammer

Examples of pipe failures caused by water hammer





# Introduction – Water hammer

The phenomenon described in the penstock is called water hammer; it tends to damp out quickly, because of the dissipation in the conversion of kinetic energy into pressure/compression-expansion of fluid and pipe and vice versa.

Because of the celerity and the length of the system, this alternation takes place countless times.

From the hydraulic point of view, the initial time phases of water hammer (order of seconds or fractions of a second) are the relevant ones, when the phenomenon is most intense.

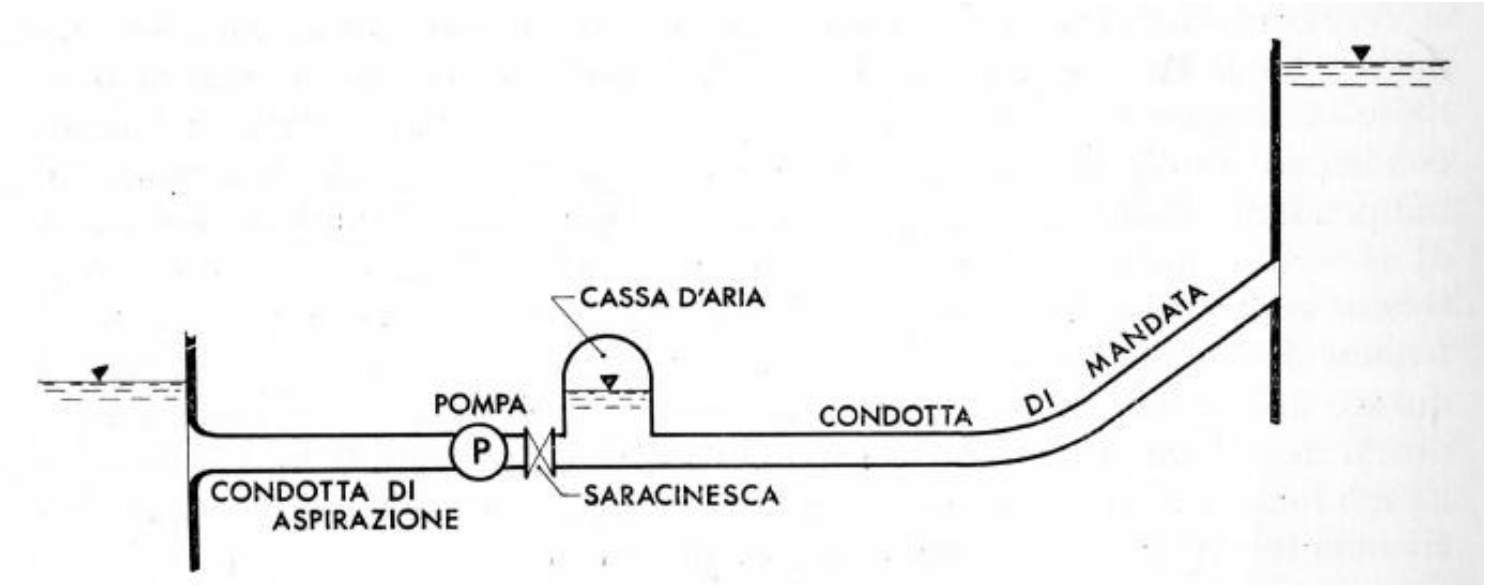
Some remarks:

Short pipes → resistance effects are negligible given the short duration of the phenomenon (e.g. penstock)

Long pipes → resistance effects are not negligible (e.g. oil pipelines)

# Introduction – Pumping plant

Examples of unsteady flow in a pumping plant.




*CONDOTTA DI ASPIRAZIONE = suction pipe;*
*POMPA = pump;*
*SARACINESCA = gate valve;*
*CASSA D'ARIA = air vessel;*
*CONDOTTA DI MANDATA = delivery pipe*

Components:

- Upstream and downstream reservoirs
- Suction and delivery pipes
- Pump
- Flow regulator (valve)
- Air vessel (expansion tank)

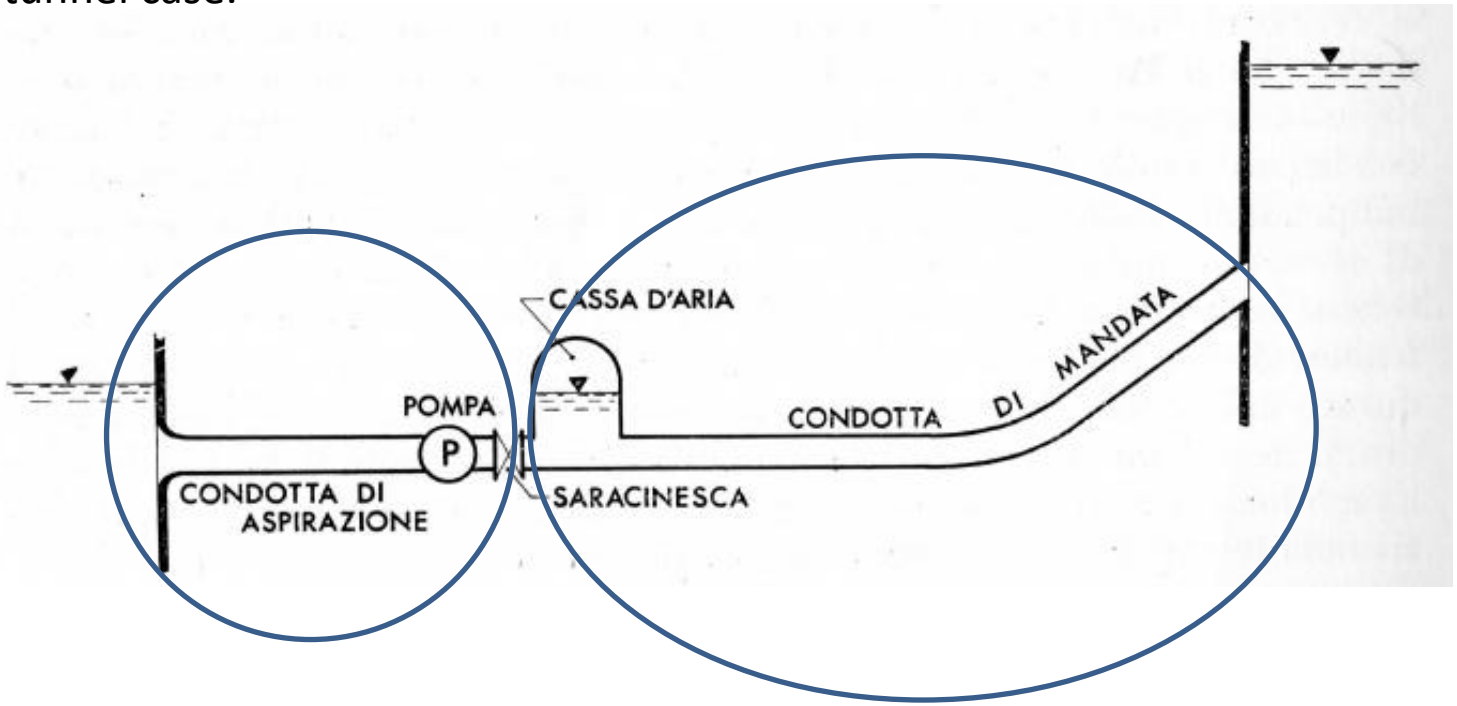
Unsteady flow induced by pump shutdown and start-up.

# Introduction – Pumping plant

Maneuvers are normally performed over long times

→ Propagation times can be considered null, so the liquid is incompressible and the pipe is rigid, but resistances must be taken into account.

In case of sudden shutdown of the system, water hammer occurs in the suction pipe, while in the delivery pipe the unsteady flow is similar to the surge tank – pressure tunnel case.



# Introduction – Types of unsteady flow

Types of unsteady flow in pressure pipes of practical interest:

- Incompressible liquid and rigid pipe. Case of slow discharge variations or short pipes

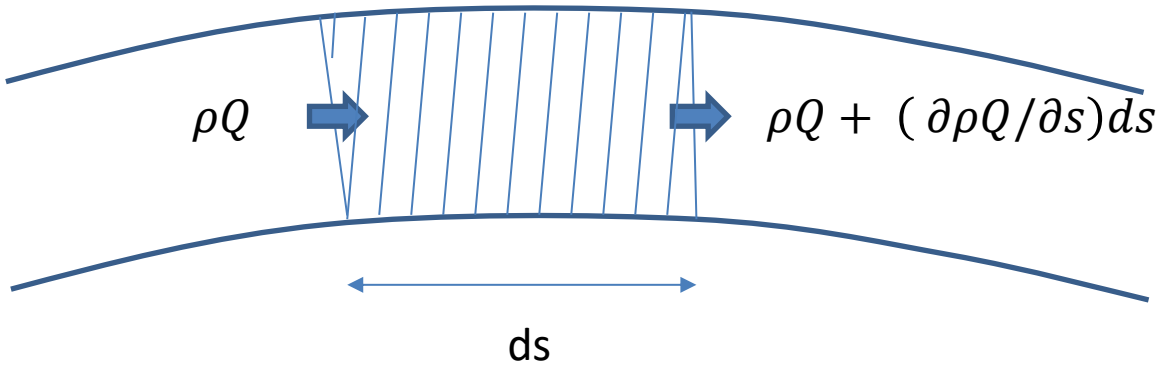
- Compressible liquid and deformable pipe: case of fast discharge variations, where wave propagation times cannot be ignored (water hammer)

- Unsteady flow in a pipe connected to a free-surface (mass oscillations), with incompressible liquid and rigid pipe

# Equations – Continuity

We derive the equations that describe unsteady flow in pressure pipes, considering that the variables (including the geometric ones) vary in space and time. We consider a compressible fluid and an elastic pipe.

## Continuity equation




Let us now consider the continuity equation. Take a fluid reach between two sections at distance ds. Consider the mass flow rate through the inlet and outlet sections in a time interval dt; the difference is:

$$[\rho Q + (\partial \rho Q/\partial s)ds - \rho Q]\, dt = (\partial \rho Q/\partial s)dsdt$$

# Equations – Continuity

We have a difference

$$(\partial \rho Q/\partial s)dsdt$$

By mass conservation, the difference of mass flow rate in a time dt between the two sections must correspond to a decrease of mass in the reach, equal to:

$$-[\partial \rho A/\partial t]dsdt$$

We obtain:

$$\frac{\partial \rho Q}{\partial s}dsdt = -\frac{\partial \rho A}{\partial t}dsdt$$

Hence

$$\boxed{\frac{\partial \rho Q}{\partial s} + \frac{\partial \rho A}{\partial t} = 0}$$

If the fluid is incompressible

$$\frac{\partial Q}{\partial s} + \frac{\partial A}{\partial t} = 0$$

**in steady flow** $\frac{\partial A}{\partial t} = 0$ Hence $\frac{\partial Q}{\partial s} = 0$ The discharge is constant in space

# Equations – Continuity

If I express the discharge in terms of cross-section and velocity

$$\frac{\partial AV}{\partial s} = A\frac{\partial V}{\partial s} + V\frac{\partial A}{\partial s} = 0$$ **In uniform flow** $$\frac{\partial A}{\partial s} = 0$$

Hence $$\frac{\partial V}{\partial s} = 0$$ The velocity remains constant in space

We reconsider the continuity equation in its general form, expressing the discharge Q in terms of A and V, and expand the derivatives:

$$V\frac{\partial \rho A}{\partial s} + \rho A\frac{\partial V}{\partial s} + \rho\frac{\partial A}{\partial t} + A\frac{\partial \rho}{\partial t} = 0$$

The area A and the density ρ vary in space and time as functions of pressure, so A=A(p) and ρ=ρ(p). The equation can then be rewritten as:

$$V\frac{\partial \rho A}{\partial s} + \rho A\frac{\partial V}{\partial s} + \left(\rho\frac{dA}{dp} + A\frac{d\rho}{dp}\right)\frac{\partial p}{\partial t} = 0$$

# Equations – Motion

## Equation of motion

Consider Euler's equation for a fluid point, thus neglecting viscous effects

$$\rho(F - A) = grad\ p$$

where F is the body force and A the acceleration. Considering only weight as body force, we have:

$$-\rho g\ grad\ z - \frac{dV}{dt} = grad\ p$$

Along a trajectory s we have

$$-\rho g \frac{\partial z}{\partial s} - \rho \frac{\partial V}{\partial t} - \frac{\rho}{2}\frac{\partial V^2}{\partial s} = \frac{\partial p}{\partial s}$$

where we have expanded the total derivative dV/dt. Dividing then by the specific weight γ=ρg and collecting terms:

$$\frac{\partial}{\partial s}\left(z + \frac{V^2}{2g}\right) + \frac{1}{\gamma}\frac{\partial p}{\partial s} = -\frac{1}{g}\frac{\partial V}{\partial t}$$

# Equations – Motion

The specific weight must be treated as a variable, so it cannot be taken inside the pressure derivative.

However, we note that:

$$\frac{1}{\gamma}\frac{\partial p}{\partial s} = \frac{\partial}{\partial s}\frac{p}{\gamma} + \frac{p}{\gamma^2}\frac{\partial \gamma}{\partial s}$$

Substituting, and also including the continuous head losses expressed through the friction slope J, we obtain:

$$\boxed{\frac{\partial}{\partial s}\left(z + \frac{V^2}{2g} + \frac{p}{\gamma}\right) + \frac{p}{\gamma^2}\frac{\partial \gamma}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t} + J = 0}$$

The first term on the left is a function of the spatial variation of the hydraulic head.

To recap, the system of equations describing unsteady flow is:

$$\frac{\partial}{\partial s}\left(z + \frac{V^2}{2g} + \frac{p}{\gamma}\right) + \frac{p}{\gamma^2}\frac{\partial \gamma}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t} + J = 0$$

$$V\frac{\partial \rho A}{\partial s} + \rho A\frac{\partial V}{\partial s} + \left(\rho\frac{dA}{dp} + A\frac{d\rho}{dp}\right)\frac{\partial p}{\partial t} = 0$$

$$A = A(p)$$

$$\rho = \rho(p)$$

# Equations – Simplified system

We consider the system of equations just derived for unsteady flow in practical applications, such as those seen in the introduction. The equations can be simplified through the following assumptions:

1) Slightly compressible liquids with a high bulk modulus ϵ

$$\epsilon = \rho \frac{dp}{d\rho}$$

which is assumed constant as pressure varies. If large pressure variations correspond to small density variations, ϵ is very large.

2) Circular pipes of constant thickness and limited deformability, hence high values of the elastic modulus E (analogous to point 1, but for the section A)

3) The mean velocity is of the order of a few meters per second, so kinetic heads are negligible compared to piezometric ones

4) Except for long pipes, the characteristic times of water hammer are short, so resistances can be neglected

In particular, points 1 and 2 imply $\frac{\partial \rho}{\partial s} = 0$ $\frac{\partial A}{\partial s} = 0$

# Equations – Simplified system

We simplify the equation of motion according to the assumptions made. By point 1, large pressure variations correspond to small variations of ρ

$$\frac{\partial}{\partial s}\left(z+\frac{p}{\gamma}+\frac{V^2}{2g}\right)+\frac{p}{\gamma^2}\frac{\partial \gamma}{\partial s}+\frac{1}{g}\frac{\partial V}{\partial t}+J=0$$

Considering point 3

$$\frac{\partial}{\partial s}\left(z+\frac{p}{\gamma}+\frac{V^2}{2g}\right)+\frac{1}{g}\frac{\partial V}{\partial t}+J=0$$

Considering point 4

$$\frac{\partial}{\partial s}\left(z+\frac{p}{\gamma}\right)+\frac{1}{g}\frac{\partial V}{\partial t}+J=0$$

therefore

$$\frac{\partial h}{\partial s}+\frac{1}{g}\frac{\partial V}{\partial t}=0 \quad \text{with} \quad h=z+\frac{p}{\gamma}$$ The piezometric head

# Equations – Simplified system

We now consider the continuity equation

$$V\frac{\partial \rho A}{\partial s} + \rho A\frac{\partial V}{\partial s} + \left(\rho\frac{dA}{dp} + A\frac{d\rho}{dp}\right)\frac{\partial p}{\partial t} = 0$$

For points 1 and 2

$$V\frac{\partial \rho A}{\partial s} + \rho A\frac{\partial V}{\partial s} + \left(\rho\frac{dA}{dp} + A\frac{d\rho}{dp}\right)\frac{\partial p}{\partial t} = 0$$

Divide by ρA and recall that ϵ=ρ dp/dρ

$$\frac{\partial V}{\partial s} + \left(\frac{1}{A}\frac{dA}{dp} + \frac{1}{\epsilon}\right)\frac{\partial p}{\partial t} = 0$$

Express p as a function of h

$$\frac{\partial h}{\partial t} = \frac{\partial z}{\partial t} + \frac{\partial}{\partial t}\left(\frac{p}{\gamma}\right) = \frac{1}{\gamma}\frac{\partial p}{\partial t} - \frac{p}{\gamma^2}\frac{\partial \gamma}{\partial t} = \frac{1}{\gamma}\left(1 - \frac{p}{\gamma}\frac{d\gamma}{dp}\right)\frac{\partial p}{\partial t} = \frac{1}{\gamma}\left(1 - \frac{p}{\epsilon}\right)\frac{\partial p}{\partial t}$$

Having assumed for point 1 that p/ϵ << 1. Substituting

$$\frac{\partial V}{\partial s} + \gamma\left(\frac{1}{A}\frac{dA}{dp} + \frac{1}{\epsilon}\right)\frac{\partial h}{\partial t} = 0$$

# Equations – Simplified system

Recapping, the system becomes

$$\begin{cases} \dfrac{\partial h}{\partial s} + \dfrac{1}{g}\dfrac{\partial V}{\partial t} = 0 \\ \dfrac{\partial V}{\partial s} + \gamma\left(\dfrac{1}{A}\dfrac{dA}{dp} + \dfrac{1}{\epsilon}\right)\dfrac{\partial h}{\partial t} = 0 \end{cases}$$

If we take as positive direction for s the one opposite to the flow

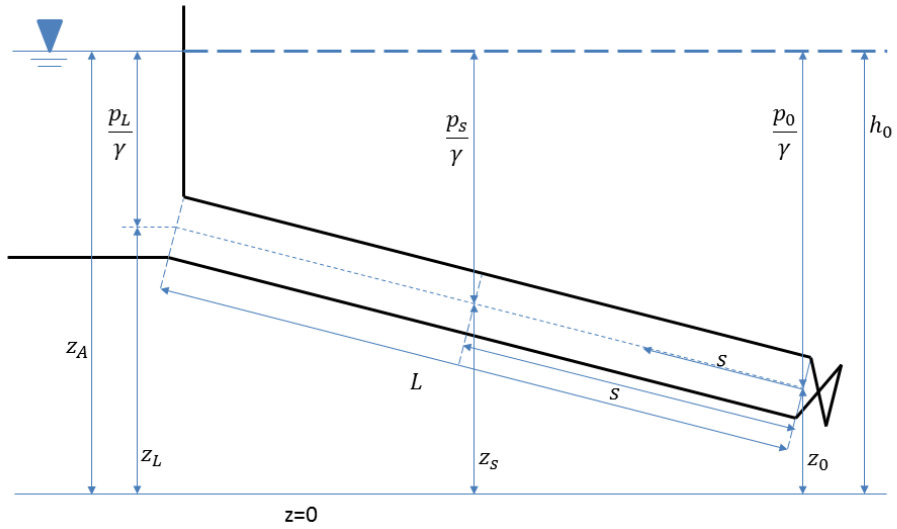
$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial V}{\partial s} = \gamma\left(\dfrac{1}{A}\dfrac{dA}{dp} + \dfrac{1}{\epsilon}\right)\dfrac{\partial h}{\partial t} \end{cases}$$

Plus the relation A=A(p)

# Water hammer – Valve closure

## Valve maneuvers

Before using the system of equations just derived, we carry out a global analysis of the phenomenon on a simplified system, made of a reservoir, a pipe and a valve. This system is representative of the surge tank – penstock – valve system, since the surge tank behaves like a reservoir.




We assume that:

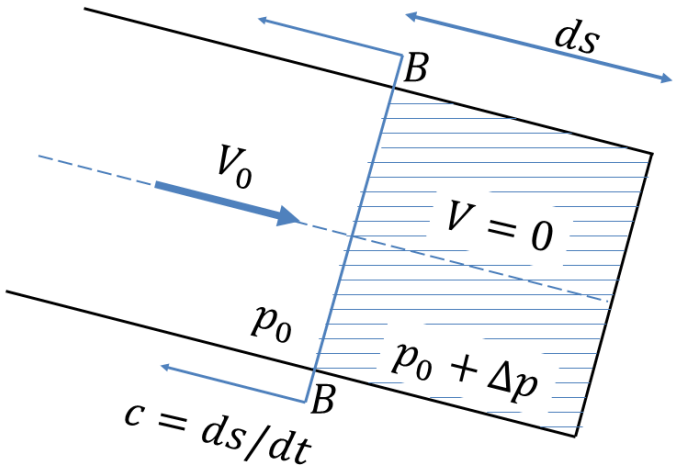
- 1) the reservoir elevation $Z_A$ remains constant in time
- 2) the pipe of length L has constant cross-section and is made of a rigid material, hence **rigid**
- 3) Head losses and kinetic heads are neglected.

The first assumption implies that the head at the pipe inlet remains constant and equal to $h_0$; the second implies, among other things, that any volume variation is only due to the compressibility of the fluid; finally, neglecting continuous and local losses implies that, in steady flow, the relative piezometric line is horizontal, at the reservoir level.

# Water hammer – Valve closure

Assume that, with the valve open, a steady flow with velocity $V_0$ exists. At a certain instant, taken as $t=0$, the valve is closed instantaneously.
By the impulse theorem, the change of momentum must balance the impulse, i.e. the forces (pressure × area) integrated in time. So the change in time of the momentum per unit area is balanced by pressure variations. If the fluid stopped instantaneously, an infinite pressure increase would follow, while practical evidence shows that the increase is finite.

Consider the valve zone (see the figure on the side). At an instant $dt$ after the instantaneous closure, only a volume $Ads$ of fluid will have stopped. There is therefore a section BB dividing the pipe into two reaches: on the right the velocity is zero, while on the left the fluid still moves with the steady-flow velocity $V_0$. Section BB then moves upstream by an amount $V_0dt$, giving a reduction of the volume of stopped fluid equal to $AV_0dt$. This volume reduction causes a pressure increase $\Delta p$, which we can compute with the impulse theorem.



# Water hammer – Valve closure

Applying the impulse theorem we have:

$$\underbrace{\underbrace{\rho A ds}_{\text{volume}} V_0}_{\text{mass}} = \underbrace{A \Delta p}_{\text{force}}\, dt$$

Change of momentum from $V_0$ a 0

therefore

$$\Delta p = \rho \frac{ds}{dt} V_0$$

$$\Delta p = \rho c\, V_0$$

Having defined

$$c = ds/dt$$

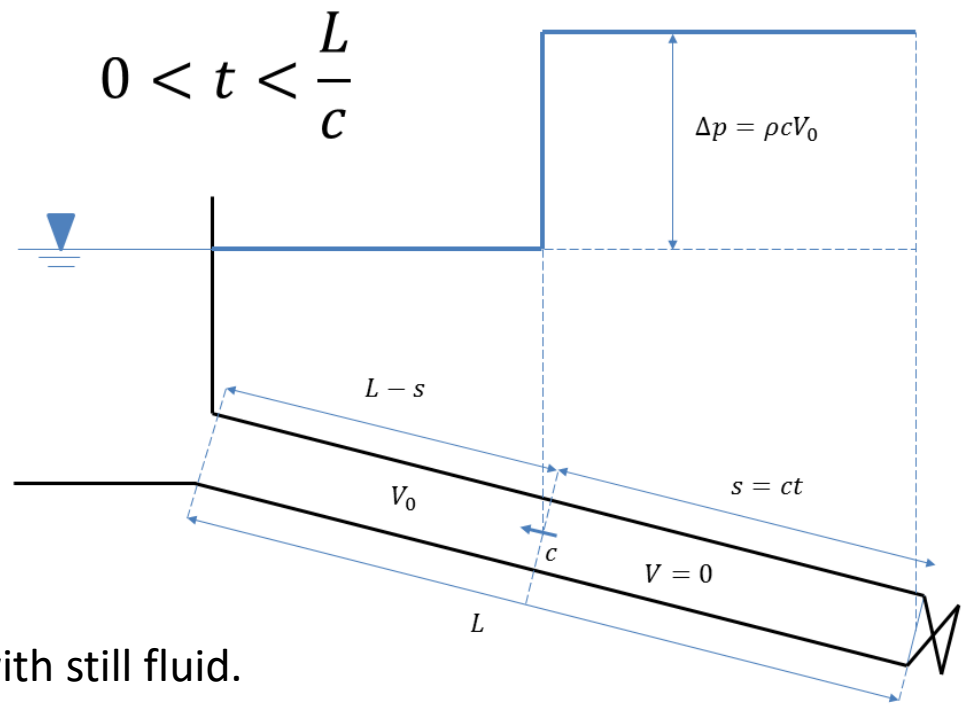
# Water hammer – Phases of the surge

At a later instant another zone of fluid will have stopped, and the interface between the fluid moving with velocity $V_0$ and the zone with $V=0$ will have moved upstream. This interface travels along the pipe with velocity $c$, called celerity. As we will see, it depends on the deformability of the pipe and on the compressibility of the fluid.

After a finite time $t$we have the situation in the figure: a zone of length $s$ with still fluid and overpressure $\Delta p$, and another zone of length $L$-$s$ still in steady-flow conditions. Note that the overpressure does not depend on the initial head conditions.
The front covers the entire length $L$ in a time:

$$t_1 = \frac{L}{c}$$

PHASE 1: positive upstream-travelling disturbance




At time $t_1$ the entire length $L$ is filled with still fluid.

# Water hammer – Phases of the surge

The new condition in the pipe, still fluid with overpressure $\Delta p$ is not a stable condition. Consider an infinitesimal fluid volume at the inlet section.

The initial assumption of a reservoir at elevation $Z_A$ constant implies that the pressure at the inlet with the reservoir remains constant and equal to $p_L$.

At the downstream section BB we have instead $p_L+\Delta p$

At time $t=t_1+dt$ the impulse $A\Delta pdt$ must correspond to a change of momentum $\rho AdsV_0$ so a velocity $-V_0$ is established, while in the reach $ds$ the pressure returns to $p_L$

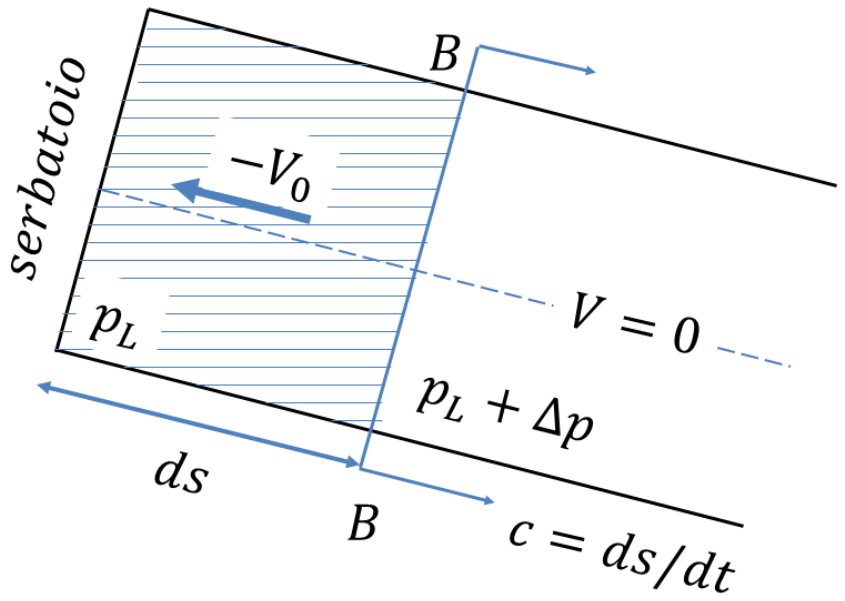
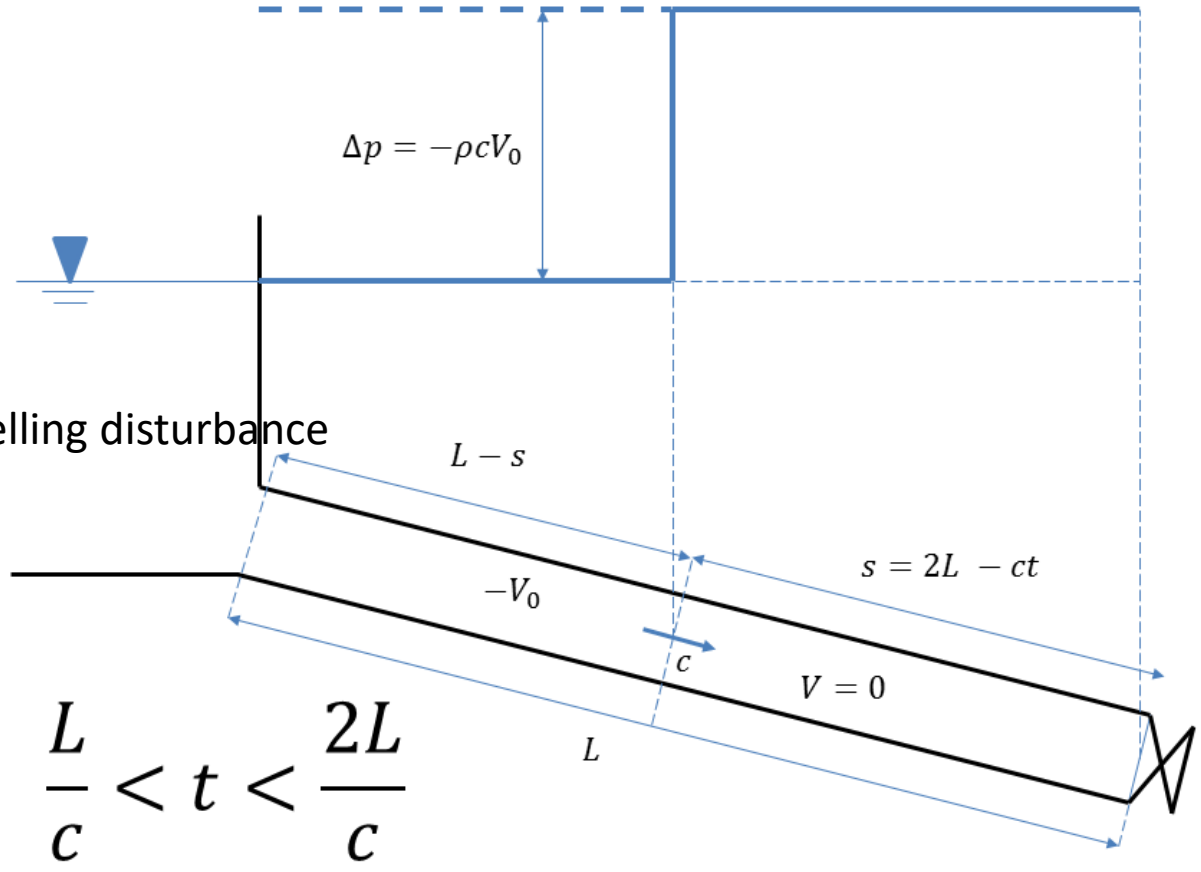



*Figure label: serbatoio = reservoir*

# Water hammer – Phases of the surge

At the upstream section we therefore have a reflection of the disturbance that had travelled up the pipe, giving a negative propagation travelling downstream.
It is characterized by a negative pressure $-\Delta p$ that brings the head back to the steady-flow value, i.e. the undisturbed value, while the velocity becomes negative. At time $t_2=2L/c$ the whole column is in steady motion with velocity $-V_0$

PHASE 2: negative downstream-travelling disturbance

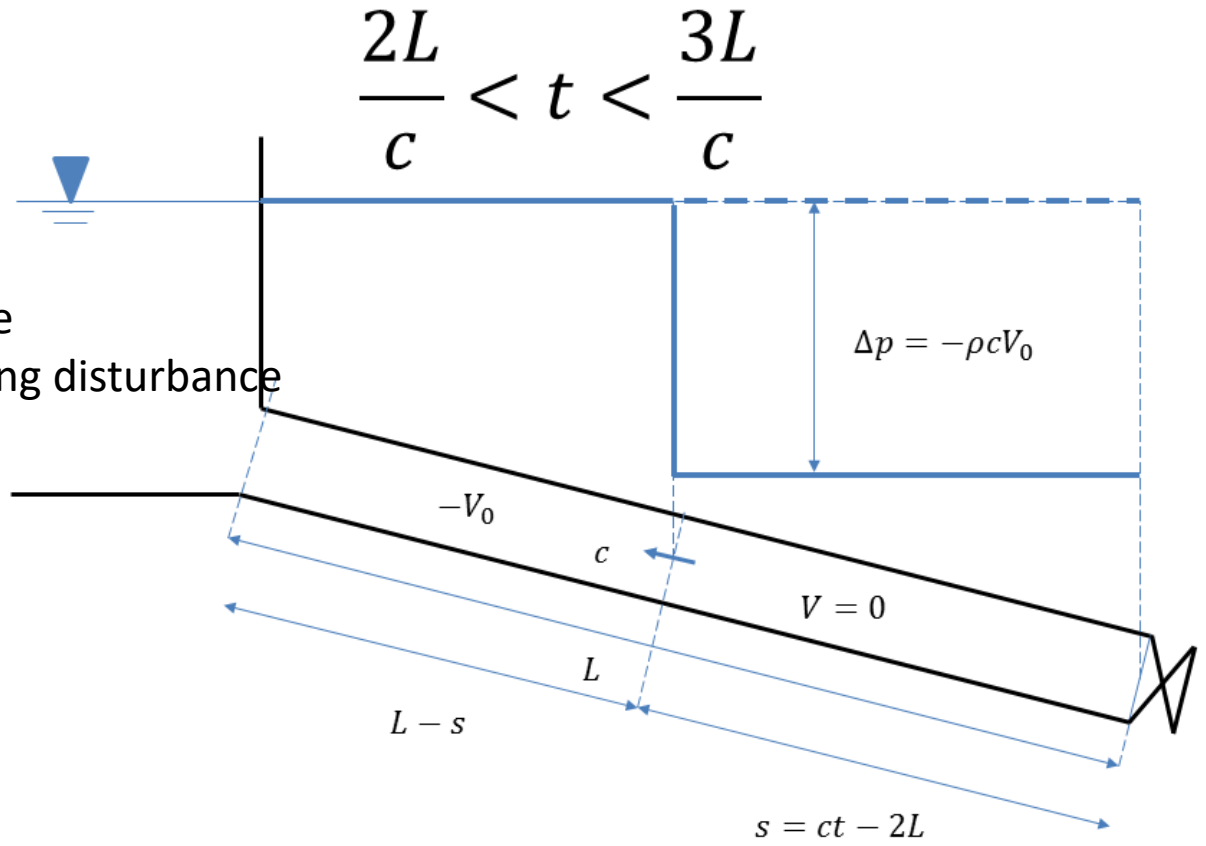



$$\frac{L}{c} < t < \frac{2L}{c}$$

# Water hammer – Phases of the surge

At time $t_2=2L/c$ the whole column moves with velocity $-V_0$ and undisturbed pressure. At the valve, however, the condition is zero velocity, so a situation analogous to phase 1 is generated; in this case, however, since the velocity is - $V_0$ we obtain a pressure drop $-\rho c V_0$. At the valve section a new reflection takes place, i.e. a negative disturbance travelling upstream that propagates in the pipe.

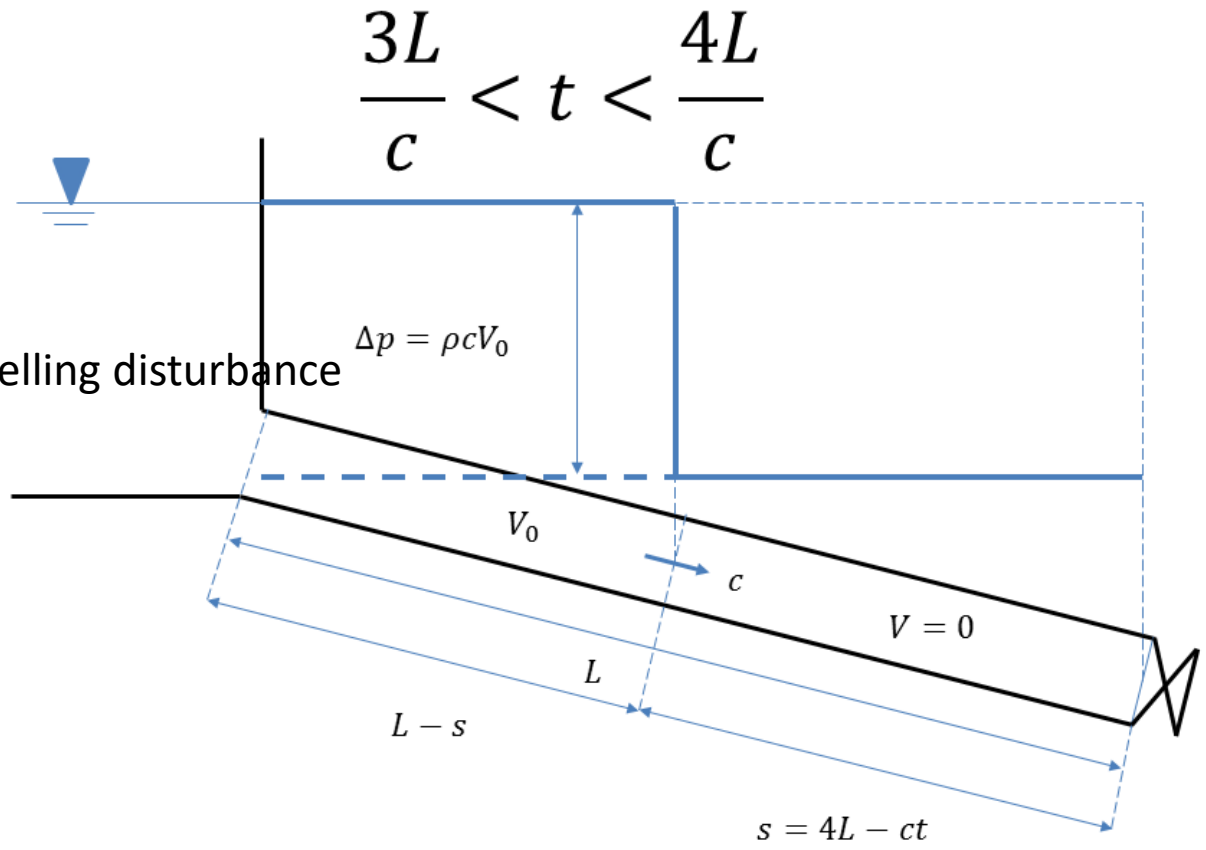
PHASE 3: negative upstream-travelling disturbance



# Water hammer – Phases of the surge

At time $t_3=3L/c$ the whole column is at rest with pressure drop $-\rho c V_0$
At the upstream section for $t>t_3$ , given the constant head imposed by the reservoir, a positive downstream-travelling disturbance forms and moves with celerity $c$ bringing the flow back to steady-flow conditions with velocity $V_0$ and undisturbed pressure. At time $t_4=4L/c$ we are back to the initial conditions and the phenomenon just described repeats periodically.

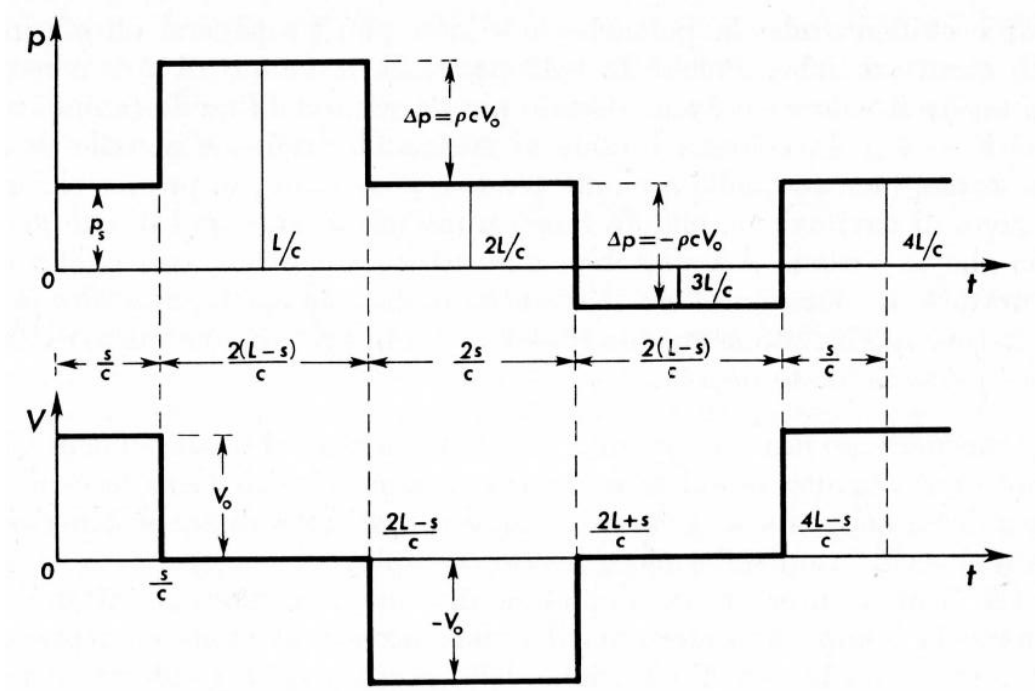
PHASE 4: positive downstream-travelling disturbance



# Water hammer – Phases of the surge

The unsteady flow just described repeats itself periodically with period $T=4L/c$. At a generic section $s$, if we observe pressure and velocity in time, we obtain a graph like the one in the figure.

Up to time $s/c$ the section stays in the undisturbed steady-flow conditions. When the positive upstream-travelling disturbance arrives, the pressure rises by $\Delta p$ and the velocity becomes zero. This lasts for a time $2(L-s)/c$, i.e. the time needed by the disturbance to travel up the rest of the pipe ($L$-$s$), be reflected and come back, bringing pressure to the undisturbed value and velocity to a negative value.

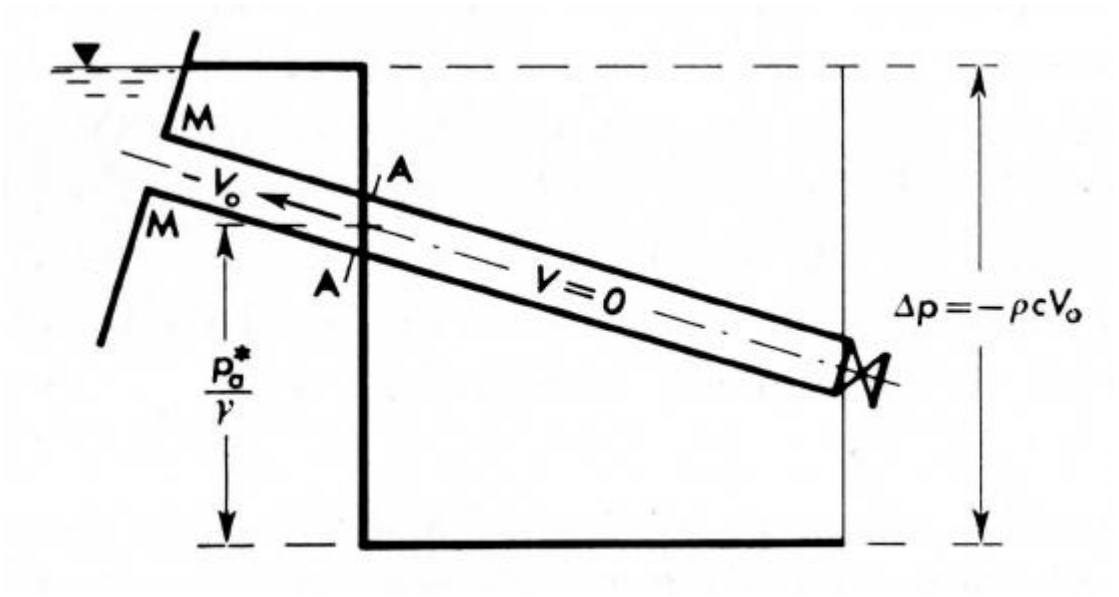



In this new condition, the section remains for a time $2s/c$, i.e. the time needed by the downstream-travelling wave to reach the valve, be reflected and return to the section. The following phases will be characterized by the time intervals just described.

# Water hammer – Phases of the surge

The oscillations are damped in time by the energy losses in the transformation from elastic to kinetic energy and vice versa. Experimental tests show that the first oscillations behave as predicted by the theoretical model.

Remark on the third phase, characterized by a negative disturbance




Special case in which the pressure drop brings the relative pressure in a section to $-p_a$ i.e. zero absolute pressure.
The disturbance cannot propagate any further.
Reach AA-O: still fluid
Reach MM-AA: the flow continues
A cavitation condition develops, with large energy dissipation

# Water hammer – Wave celerity

## Celerity of the disturbance

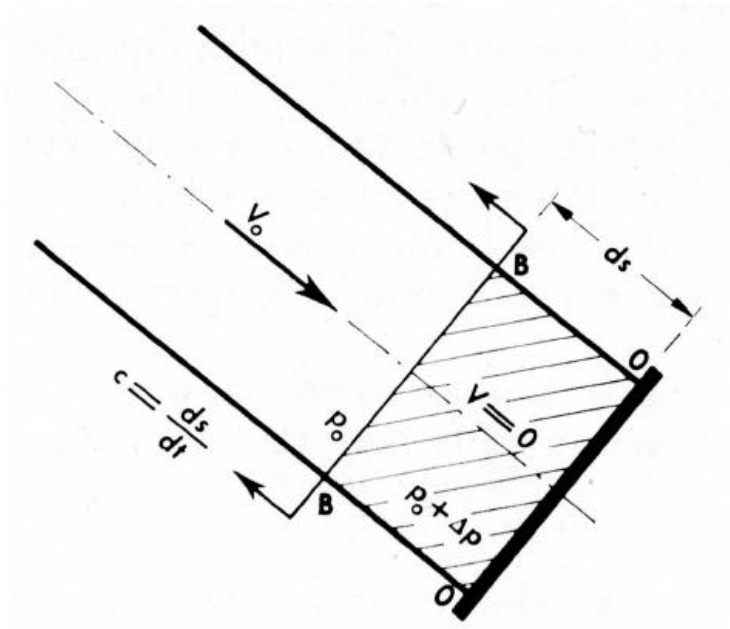
It is fundamental in water hammer processes; we define how it depends on the elastic properties of fluid and pipe.

### 1) Rigid pipe

A *dt* after closure, section BB moves upstream by $V_0 dt$, and the volume $W=Ads$ decreases by:

$$dW = -AV_0 dt$$

The change of momentum in the control volume of still fluid corresponds to a pressure increase $\Delta p=\rho c V_0$ which causes a reduction of the fluid volume. We consider the elastic properties of the fluid.




$$\epsilon = \rho \frac{dp}{d\rho}$$

If the volume *W* varies, the mass $\rho W$ remains constant anyway ($d(\rho W)=0$), hence

$$\rho dW + W d\rho = 0 \quad ; \quad \frac{\rho}{d\rho} = -\frac{W}{dW} \qquad dW = -\frac{W\Delta p}{\epsilon} = -\frac{A\rho c V_0}{\epsilon} ds$$

# Water hammer – Wave celerity

Equating the two volume decrements obtained (same phenomenon observed in different ways) we get:

$$AV_0 dt = \frac{A\rho c V_0}{\epsilon} ds \qquad 1 = \frac{\rho c}{\epsilon}\frac{ds}{dt}$$

But by definition

$$c = \frac{ds}{dt}$$

therefore

$$\frac{\rho c^2}{\epsilon} = 1$$

$$\boxed{c = \sqrt{\frac{\epsilon}{\rho}} = \sqrt{\frac{dp}{d\rho}}}$$

The celerity is the speed of sound in the fluid, and has values of the order of $10^3$ m/s

# Water hammer – Wave celerity

2) **Deformable pipe**

Similarly to the previous case, from the displacement of section BB

$$dW = -AV_0 dt$$

For the pressure increment we have a component related to the compressibility of the fluid

$$dW_1 = -\frac{A\rho c V_0}{\epsilon} ds$$

And a component related to the dilation of the pipe (opposite sign)

$$dW_2 = dA\, ds$$

Considering that the area variations are functions of the pressure variations we can write

$$dW_2 = \frac{dA}{dp} \Delta p\, ds = \rho c V_0 \frac{dA}{dp} ds$$

Equating the two expressions for the volumes $dW = dW_1 - dW_2$

$$AV_0 dt = \frac{A\rho c V_0}{\epsilon} ds + \rho c V_0 \frac{dA}{dp} ds$$

# Water hammer – Wave celerity

Dividing by $AV_0dt$

$$1 = \frac{\rho c}{\epsilon}\frac{ds}{dt} + \rho c \frac{1}{A}\frac{dA}{dp}\frac{ds}{dt}$$

Since $c=ds/dt$

$$\frac{1}{c^2} = \rho\left[\frac{1}{A}\frac{dA}{dp} + \frac{1}{\epsilon}\right] = \frac{\rho}{\epsilon}\left[\frac{\epsilon}{A}\frac{dA}{dp} + 1\right]$$

$$c^2 = \frac{1}{\frac{\rho}{\epsilon}\left[\frac{\epsilon}{A}\frac{dA}{dp} + 1\right]} = \frac{\frac{\epsilon}{\rho}}{\left[\frac{\epsilon}{A}\frac{dA}{dp} + 1\right]}$$

therefore

$$c = \frac{c^*}{\sqrt{\frac{\epsilon}{A}\frac{dA}{dp} + 1}}$$

where $C^*$ is the celerity for a rigid pipe. Let us analyze the denominator: $dp>0$ corresponds to $dA>0$ (overpressure → dilation), while $dp<0$ corresponds to $dA<0$ (pressure drop → contraction), so the ratio $dA/dp>0$, as is the ratio $\varepsilon/A$. Hence the denominator is always >1 and we deduce that $C<C^*$: the celerity in a deformable pipe is lower than in a rigid one.

# Water hammer – Wave celerity

The dilation of the pipe has effects similar to an increase in the compressibility of the fluid, i.e. a reduction of $\epsilon$, and hence a reduction of the celerity

From this analogy a fictitious elastic modulus can be defined

$$\frac{1}{\epsilon'} = \frac{1}{\epsilon} + \frac{1}{A}\frac{dA}{dp}$$

$$\epsilon' = \frac{\epsilon}{1 + \frac{\epsilon}{A}\frac{dA}{dp}}$$

Dividing by the density and taking the root

$$c = \frac{\sqrt{\frac{\epsilon}{\rho}}}{\sqrt{1 + \frac{\epsilon}{A}\frac{dA}{dp}}} = \sqrt{\frac{\epsilon'}{\rho}}$$

This means that the analysis carried out previously for water hammer in a rigid pipe remains valid for a deformable pipe as well: it is enough to consider a fluid with bulk modulus ϵ’ instead of ϵ

# Water hammer – Wave celerity

We reformulate the celerity for circular pipes of diameter *D* and thickness *e* made of an elastic material obeying Hooke's law, with E the elastic modulus

$$E = D\frac{d\sigma}{dD}$$

The stress *σ* per unit surface can be expressed through Mariotte's formula, assuming that *e<<D*

$$\sigma = \frac{pD}{2e}$$

Substituting and expressing in terms of *dD* we obtain

$$dD = \frac{D^2}{2eE}dp$$

Considering the area deformation ratio with respect to pressure, and expressing the area as a function of the diameter

$$\frac{dA}{dp} = \frac{dA}{dD}\frac{dD}{dp} = \frac{d}{dD}\left(\frac{\pi D^2}{4}\right)\frac{D^2}{2eE} = \frac{2\pi D}{4}\frac{D^2}{2eE} = \frac{\pi D^2}{4}\frac{D}{eE} = A\frac{D}{eE}$$

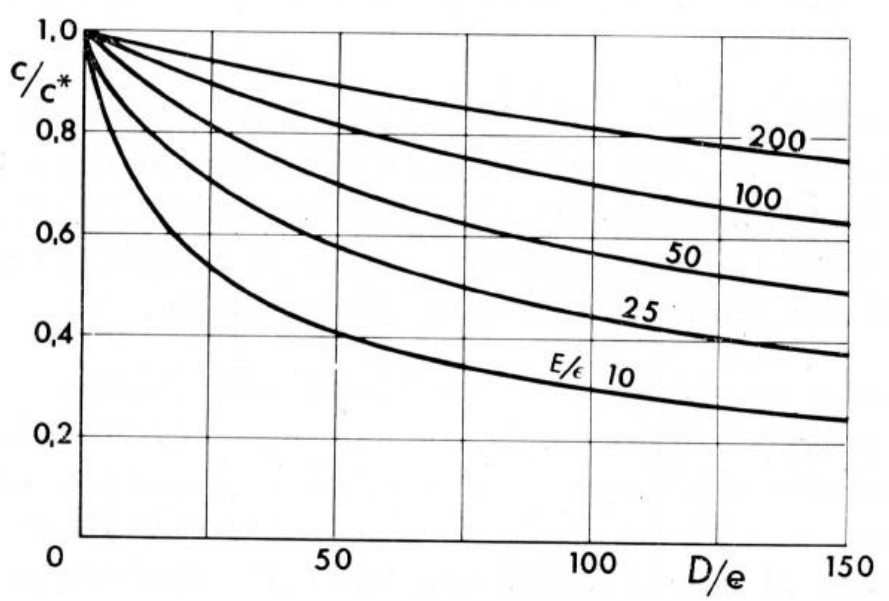
# Water hammer – Wave celerity

The bulk modulus $\epsilon'$ becomes

$$\epsilon' = \frac{\epsilon}{1 + \frac{\epsilon}{A}\frac{dA}{dp}} = \frac{\epsilon}{1 + \frac{\epsilon D}{eE}}$$

The celerity

$$c = \frac{c^*}{\sqrt{1 + \frac{\epsilon D}{eE}}}$$



# Water hammer – Typical celerity values

## Celerity in a deformable pipe

$$c = c_0 / \sqrt{(1 + \varepsilon \cdot D / (E \cdot e))}$$

$c_0 = \sqrt{(\varepsilon/\rho)} \approx 1440$ m/s for water at 20 °C ($\varepsilon \approx 2.07$ GPa); D = diameter, e = wall thickness, E = Young modulus of the pipe.

| Pipe material | E [GPa] | c [m/s] (typical) |
|---|---|---|
| Steel | 200–210 | 1000–1300 |
| Ductile iron | 100–170 | 1000–1250 |
| Reinforced concrete | 25–40 | 900–1100 |
| PVC | ≈ 3 | 300–500 |
| HDPE | ≈ 1 | 200–400 |

## Worked example

Steel pipe, D = 1 m, e = 10 mm, E = 206 GPa

$\varepsilon \cdot D / (E \cdot e) = 2.07 \cdot 10^9 \cdot 1 / (2.06 \cdot 10^{11} \cdot 0.01) \approx 1.0$

**$c = 1440 / \sqrt{2.0} \approx 1020$ m/s**

Joukowsky, head form: $\Delta h = c \cdot \Delta V / g$

**$\Delta h \approx 1020 \cdot 1 / 9.81 \approx 104$ m for every 1 m/s of velocity change**

Rule of thumb: in steel pipes a stopped flow of 1 m/s produces about 10 bar of overpressure; plastic pipes give 3–5 times less.

# Water hammer – Wave equation

## Equations of motion as a function of celerity

The system of equations for unsteady flow, for the study of water hammer, was determined as

$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial V}{\partial s} = \gamma\left(\dfrac{1}{A}\dfrac{dA}{dp} + \dfrac{1}{\epsilon}\right)\dfrac{\partial h}{\partial t} \end{cases}$$

During the analysis of the celerity we found the following relation

$$\frac{1}{c^2} = \rho\left[\frac{1}{A}\frac{dA}{dp} + \frac{1}{\epsilon}\right]$$

Recalling that $\gamma=\rho g$ and and substituting

$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial V}{\partial s} = \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} \end{cases}$$

# Water hammer – Wave equation

## Analysis of water hammer through the equations of motion

We want to derive a form of the equations of motion that allows us to study water hammer

$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial V}{\partial s} = \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} \end{cases}$$

Differentiate in s

Differentiate in t

$$\begin{cases} \dfrac{\partial^2 h}{\partial s^2} = \dfrac{1}{g}\dfrac{\partial^2 V}{\partial t \partial s} \\ \dfrac{\partial^2 V}{\partial s \partial t} = \dfrac{g}{c^2}\dfrac{\partial^2 h}{\partial t^2} \end{cases}$$

$$\begin{cases} g\dfrac{\partial^2 h}{\partial s^2} = \dfrac{\partial^2 V}{\partial t \partial s} \\ \dfrac{\partial^2 V}{\partial s \partial t} = \dfrac{g}{c^2}\dfrac{\partial^2 h}{\partial t^2} \end{cases} \qquad g\frac{\partial^2 h}{\partial s^2} = \frac{g}{c^2}\frac{\partial^2 h}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial s^2}$$

2nd-order differential equation in h

$$\begin{cases} \dfrac{\partial h}{\partial s} = \dfrac{1}{g}\dfrac{\partial V}{\partial t} \\ \dfrac{\partial V}{\partial s} = \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} \end{cases}$$

Differentiate in t

Differentiate in s

$$\begin{cases} \dfrac{\partial^2 h}{\partial s \partial t} = \dfrac{1}{g}\dfrac{\partial^2 V}{\partial t^2} \\ \dfrac{\partial^2 V}{\partial s^2} = \dfrac{g}{c^2}\dfrac{\partial^2 h}{\partial t \partial s} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 h}{\partial s \partial t} = \dfrac{1}{g}\dfrac{\partial^2 V}{\partial t^2} \\ \dfrac{c^2}{g}\dfrac{\partial^2 V}{\partial s^2} = \dfrac{\partial^2 h}{\partial t \partial s} \end{cases} \qquad \frac{1}{g}\frac{\partial^2 V}{\partial t^2} = \frac{c^2}{g}\frac{\partial^2 V}{\partial s^2}$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial t^2} = c^2\frac{\partial^2 V}{\partial s^2}$$

2nd-order differential equation in V

# Water hammer – Wave equation

We have moved from a system of two first-order partial differential equations in both *V* and *h*to two second-order partial differential equations, each in a single variable, *V* o *h*. Both equations have the same form of the type

$$\frac{\partial^2 \beta}{\partial t^2} = c^2 \frac{\partial^2 \beta}{\partial s^2}$$

This type of equation is known as the D'Alembert equation and describes the propagation of waves. In a non-dispersive medium and for small-amplitude waves it admits the following solution:

$$\beta = f(s - ct) + g(s + ct)$$

With *f* and *g* arbitrary functions. To verify that the above is indeed a solution of the D'Alembert equation, it is enough to consider the change of variable $\eta$=*s-ct* and $\xi$=*s+ct*, which gives:

$$\frac{\partial^2 \beta}{\partial t^2} = c^2 \frac{\partial^2 \beta}{\partial s^2} \quad \Rightarrow \quad \frac{\partial^2 \beta}{\partial \eta \partial \xi} = 0$$

Integrating first in $\eta$ keeping $\xi$ constant, and then integrating in $\xi$ we get

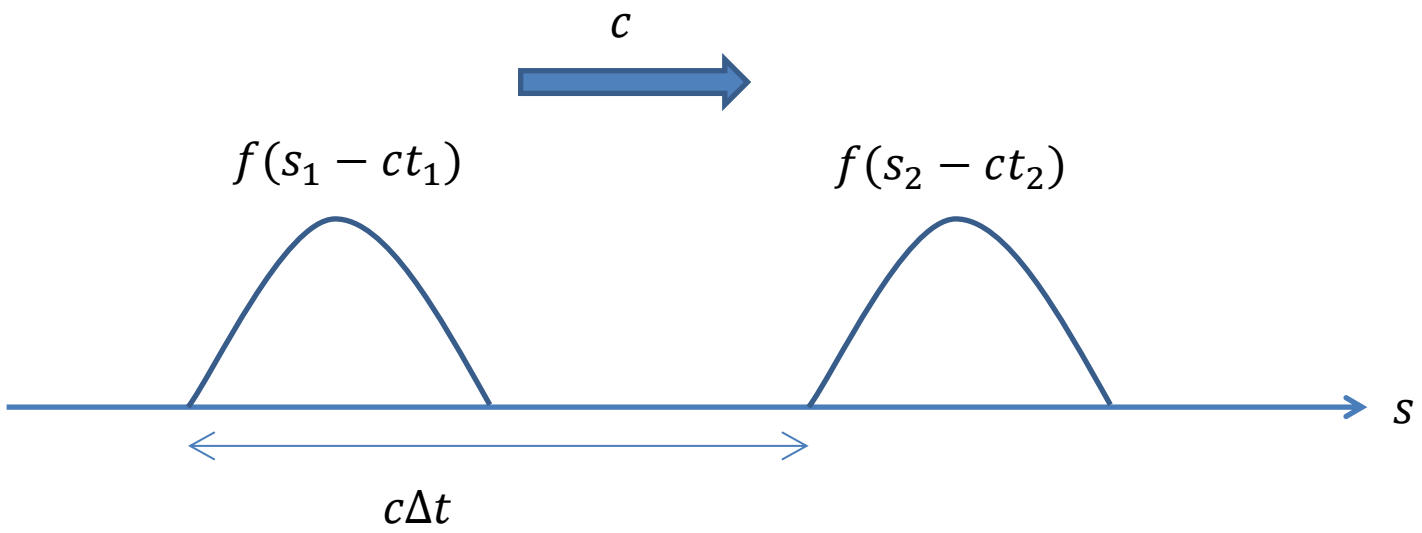
$$\beta = f(\eta) + g(\xi) = f(s - ct) + g(s + ct)$$

# Water hammer – Wave equation

*f* represents a disturbance travelling in the positive direction of *s* with velocity *c*, while *g* is a disturbance travelling in the negative direction *s* with velocity always *c*.
The *f* and the *g* do not change shape while propagating in the medium with celerity *c*.
If we consider, for example, the function *f* at time $t_1$ at position $s_1$, at time $t_2$ it will have to be found at position $s_2$ such that

$$s_1 - ct_1 = s_2 - ct_2$$

$$s_2 - s_1 = c(t_2 - t_1)$$



# Water hammer – Wave equation

For the first equation

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial s^2}$$

We therefore have

$$\Delta h = h - h_0 = f(s - ct) + g(s + ct)$$

Where $h_0$ derives from the initial condition. The trend of the piezometric line is therefore defined by the functions *f* and *g*, while for the pressure one can simply consider $\Delta p = \gamma \Delta h$, since there will be no variations of elevation *z*.
The functions *f* and *g* are to be defined according to the initial and boundary conditions.

Initial condition:

- At time *t=0* the steady-flow conditions hold $V=V_0$ and $h=h_0$

Boundary conditions:

- *s=L* inlet section at the reservoir, valid for every *t* that $h=h_0=constant$
- *s=0* valve section, we consider the law with which the mean velocity *V=V(t)*varies as a function of the valve closure.

We now want to show that the same functions *f* and *g* are also solutions of the 2nd equation

# Water hammer – Wave equation

$$\Delta h = h - h_0 = f(s - ct) + g(s + ct)$$

Differentiate in $s$

$$\frac{\partial h}{\partial s} = \frac{\partial f}{\partial s} + \frac{\partial g}{\partial s}$$

$$\frac{\partial f}{\partial s} = \frac{df}{d(s-ct)}\frac{\partial (s-ct)}{\partial s} = \frac{df}{d(s-ct)}$$

$$\frac{\partial f}{\partial t} = \frac{df}{d(s-ct)}\frac{\partial (s-ct)}{\partial t} = -c\frac{df}{d(s-ct)}$$

$$\frac{\partial f}{\partial s} = -\frac{1}{c}\frac{\partial f}{\partial t}$$

$$\frac{\partial g}{\partial s} = \frac{dg}{d(s+ct)}\frac{\partial (s+ct)}{\partial s} = \frac{dg}{d(s+ct)}$$

$$\frac{\partial g}{\partial t} = \frac{dg}{d(s+ct)}\frac{\partial (s+ct)}{\partial t} = c\frac{dg}{d(s+ct)}$$

$$\frac{\partial g}{\partial s} = \frac{1}{c}\frac{\partial g}{\partial t}$$

We have found two relations linking the spatial derivatives to the temporal ones; substituting:

$$\frac{\partial h}{\partial s} = -\frac{1}{c}\left(\frac{\partial f}{\partial t} - \frac{\partial g}{\partial t}\right) \quad \text{being} \quad \frac{\partial h}{\partial s} = \frac{1}{g}\frac{\partial V}{\partial t} \quad \Rightarrow \quad -\frac{c}{g}\frac{\partial V}{\partial t} = \frac{\partial f(s-ct)}{\partial t} - \frac{\partial g(s+ct)}{\partial t}$$

# Water hammer – Wave equation

Integrating in time and considering $\Delta V = V_0 - V$ with $V_0$ deriving from the initial conditions

$$\frac{c}{g}(V_0 - V) = \frac{c}{g}\Delta V = f(s - ct) - g(s + ct)$$

We have therefore obtained the solution for the velocity with the same arbitrary functions *f* and *g*, which are solutions for the head *h*.

To recap, *f(s-ct)* is a disturbance propagating along the positive *s* with velocity *c*. At a time $t_2 > t_1$ the distance $s_2$ must be larger than $s_1$ to guarantee the same value of (*s-ct*), i.e. the argument of the function. Along the *s* the function *f* keeps the same shape but is displaced by a quantity *ct*. *f* is the disturbance from the valve to the inlet, *g* from the inlet to the valve.

Let us now evaluate some important relations that can be derived from the initial and boundary conditions. Consider the solution for the head:

$$\Delta h = h - h_0 = f(s - ct) + g(s + ct)$$

At the inlet *s=L* we have $h=h_0=const$ and $\Delta h$ so it is null, therefore:

$$f(L - ct) + g(L + ct) = 0 \qquad g(L + ct) = -f(L - ct)$$

So *f* and *g* take equal values with opposite sign at every instant: when an upstream-travelling disturbance *f* arrives at the inlet section, a disturbance *g* of equal absolute value but opposite sign is generated, which propagates downstream.

# Water hammer – Wave equation

Consider now the generic section *s* reached by a downstream-travelling disturbance *g(s+ct)*; this had passed through the inlet at time

$$t_1 = t - (L - s)/c$$

Without changing shape, therefore:

$$g(s + ct) = g(L + ct_1)$$

But at the inlet, the disturbance *g* was created by reflection of an upstream-travelling function *f* ; we found this relation earlier and we have

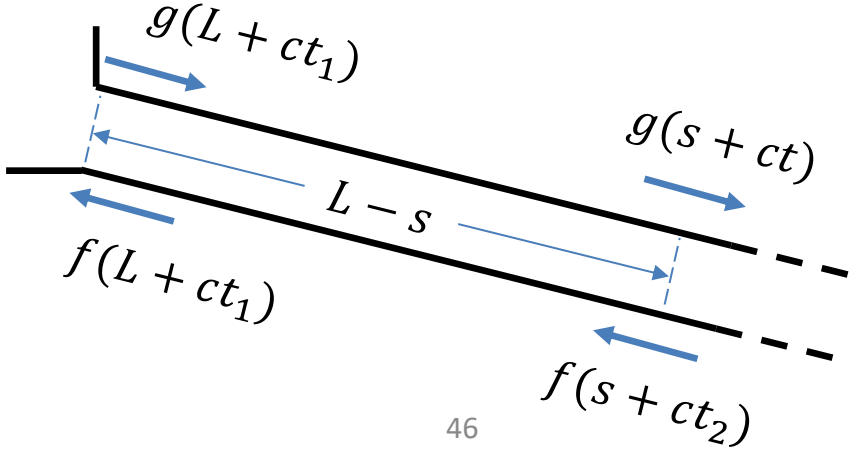
$$g(L + ct_1) = -f(L - ct_1)$$

The function f that passed through $t_1$ at the inlet was at section *s* at a time $t_2$ earlier than $t_1$

$$t_2 = t_1 - \frac{L - s}{c} = t - \frac{2(L - s)}{c}$$

therefore

$$f(L - ct_1) = f(s - ct_2)$$



# Water hammer – Wave equation

Recapping the relations found, at the generic section *s* we have:

$$g(s+ct) = g(L+ct_1) = -f(L-ct_1) = -f(s-ct_2)$$

therefore

$$g(s+ct) = -f\left[s - c\left(t - \frac{2(L-s)}{c}\right)\right]$$

Given a downstream-travelling wave *g* that passed through *s* at time *t*, it will be equal, but of opposite sign, to the upstream-travelling one *f* which passed through section *s* earlier by

$$\tau_s = \frac{2(L-s)}{c}$$

But using this relation, the expressions found for *h* and *V* as functions of *f* and *g* can be rewritten as functions of the upstream-travelling disturbance only *f*

$$\Delta h = f(s-ct) + g(s+ct) \quad \Rightarrow \quad \Delta h = f(s-ct) - f(s-c(t-\tau_s))$$

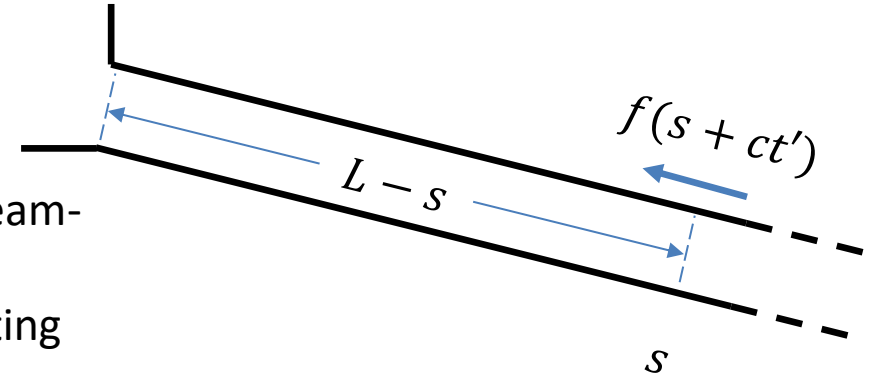
$$\frac{c}{g}\Delta V = f(s-ct) - g(s+ct) \quad \Rightarrow \quad \frac{c}{g}\Delta V = f(s-ct) + f(s-c(t-\tau_s))$$

# Water hammer – Direct phase

## Direct water hammer phase

Consider now a valve maneuver that leads, in a finite time, to a change of velocity in the pipe (for complete closure the final velocity will be zero); as already seen, this generates a series of upstream-travelling waves.
At the generic section *s*, the first upstream-travelling disturbances arrive at time:

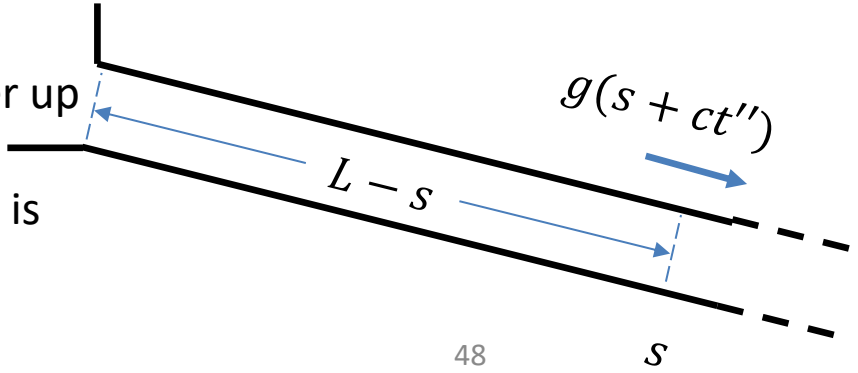
$$t' = s/c$$




The section remains affected only by the upstream-travelling wave *f*for the time needed by the disturbance to cover twice the distance separating it from the inlet (time $\tau_s$), and then returns, reflected, to section s at time:

$$t'' = t' + \tau_s = \frac{s}{c} + \frac{2(L-s)}{c} = \frac{2L - s}{c}$$

This means that, from the start of the maneuver up to time *t''*, the section is affected only by the upstream-travelling wave for an interval $\tau_s$. This is called the direct water hammer phase.



# Water hammer – Direct phase

Then in the interval *0<t<t''* we can simplify the solution, neglecting the downstream-travelling waves

$$\Delta h = f(s - ct) - \cancel{f(s - c(t - t_s))}$$

$$\frac{c}{g}\Delta V = f(s - ct) + \cancel{f(s - c(t - t_s))}$$

Combining them, since the terms on the right of the equal sign are identical, we obtain:

$$\Delta h = \frac{c}{g}\Delta V$$

Or in terms of pressure

$$\Delta h = \frac{\Delta p}{\rho g}\ ;\ \Delta p = \ \rho c \Delta V = \ \rho c (V_0 - V)$$

Direct water hammer phase: the pressure variation in a section is proportional to the velocity variation occurring in that section, and does not depend on how that variation took place. The direct water hammer phase at section *s=0*, i.e. at the valve, is

$$\tau_0 = t'' \Big|_{s=0} = \frac{2(L - s)}{c} = \frac{2L}{c}$$

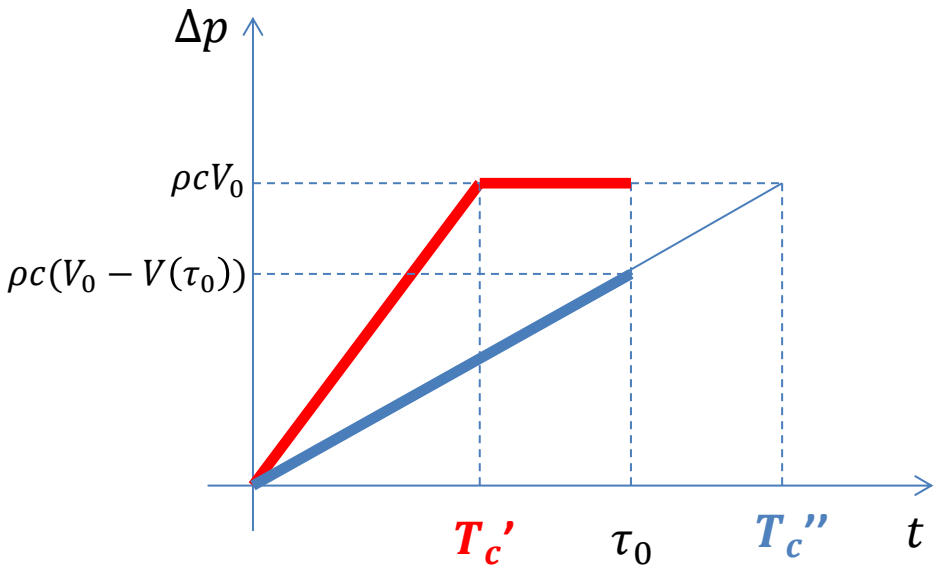
# Water hammer – Direct phase

Consider now that the complete closure of the valve takes place linearly in a time $T_c$, bringing the velocity from $V_0$ a $V=0$. The velocity will therefore follow the relation below.

$$V(t) = V_0 \left(1 - \frac{t}{T_c}\right)$$

At time $t=0$ we have the undisturbed velocity $V_0$, at time $t=T_c$ the velocity will be $V=0$. At each interval $\Delta t$ we have a linear variation of $V$ and therefore a linear variation of $\Delta p$.

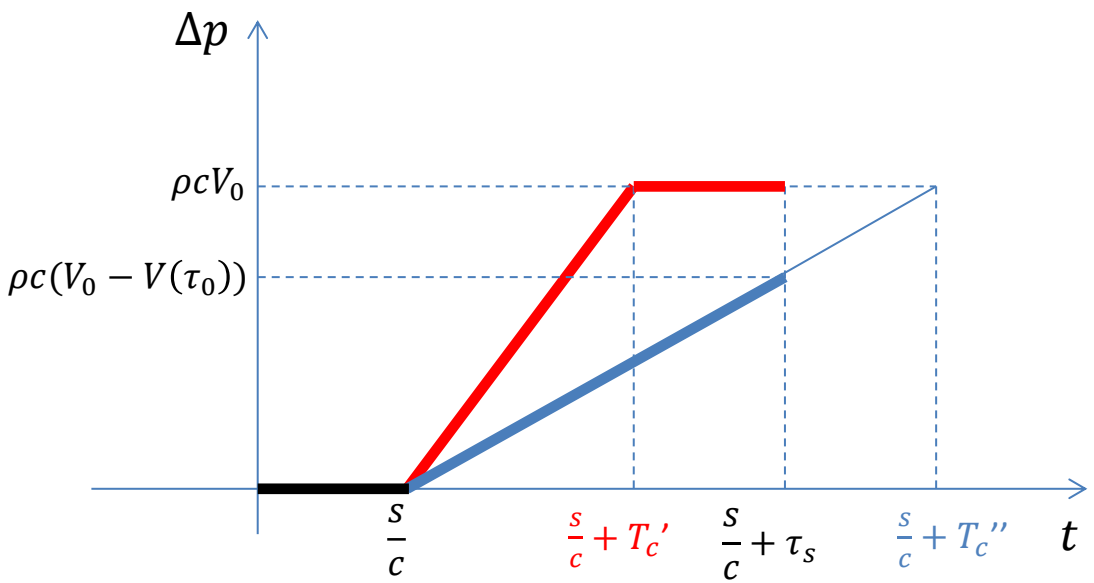
Let us now evaluate the direct water hammer phase at the valve section, depending on the closure time $T_c$ with respect to $\tau_0$.
If $T_c' < \tau_0$ at time $T_c'$ the valve is closed, and we reach a maximum overpressure $\Delta p = \rho c V_0$, so the pressure remains constant since there are no more velocity variations. Once time $\tau_0$ is reached, the section is hit by the downstream-travelling wave with pressure drop and by its reflection (also with pressure drop), with a consequent decrease of $\Delta p$.



# Water hammer – Direct phase

If instead $T_c'' > \tau_0$, the variation of velocity, and consequently of pressure, takes place with a smaller slope than in the previous case. At time $\tau_0$ the velocity at the valve has not yet become zero, so a smaller overpressure than in the previous case is reached. For $t>\tau_0$, the overpressure decreases because of the arrival of the downstream-travelling disturbances. In both cases, the maximum overpressure is reached within the time $\tau_0$ ($T_c$ in the first case, $\tau_0$ in the second), after which this value can at most be equaled, but never exceeded, because of the superposition with the effects of the other waves.
The case $T_c < \tau_0$ which gives the maximum possible overpressure $\rho c V_0$ is called **sudden closure**.

An analogous argument holds at a generic section *s*, but in this case, apart from the initial time *s/c*, the comparison is made between the time $T_c$ and $\tau_s$.



# Water hammer – Reflected-wave phase

## Reflected-wave phase (contraccolpo)

It is the phase following direct water hammer, in which upstream-travelling waves and downstream-travelling ones overlap.

$$\Delta h = f(s - ct) - f(s - c(t - t_s))$$

$$\frac{c}{g}\Delta V = f(s - ct) + f(s - c(t - t_s))$$

Consider the valve section

$$f(s - ct)\Big|_{s=0} = f(-ct)$$

Since it is a function of time only, for simplicity I introduce a function $f_0$

$$f_0(t) = f(-ct)$$

We assume that the velocity variation at the valve is a function of time *V=V(t)*. We first evaluate the velocity disturbances; in particular we want to determine them at a generic time $t_n$, characterized by a $V_n$. For *s=0*, $\tau_s = \tau_0$ therefore

$$\frac{c}{g}(V_0 - V_n) = f_0(t_n) + f_0(t_n - \tau_0) = f_0(t_n) + f_0(t_{n-1})$$

# Water hammer – Reflected-wave phase

Having defined

$$t_{n-1} = t_n - \tau_0$$

Hence

$$f_0(t_n) = \frac{c}{g}(V_0 - V_n) - f_0(t_{n-1})$$

We have therefore found a relation linking the value of $f_0$ a $t_n$, to a $f_0$ at time $t_{n-1}$ as a function of a time delay equal to $\tau_0$

But the same procedure can be applied to evaluate the $f_0$ at time $t_{n-1}$

$$\frac{c}{g}(V_0 - V_{n-1}) = f_0(t_{n-1}) + f_0(t_{n-1} - \tau_0) = f_0(t_{n-1}) + f_0(t_{n-2})$$

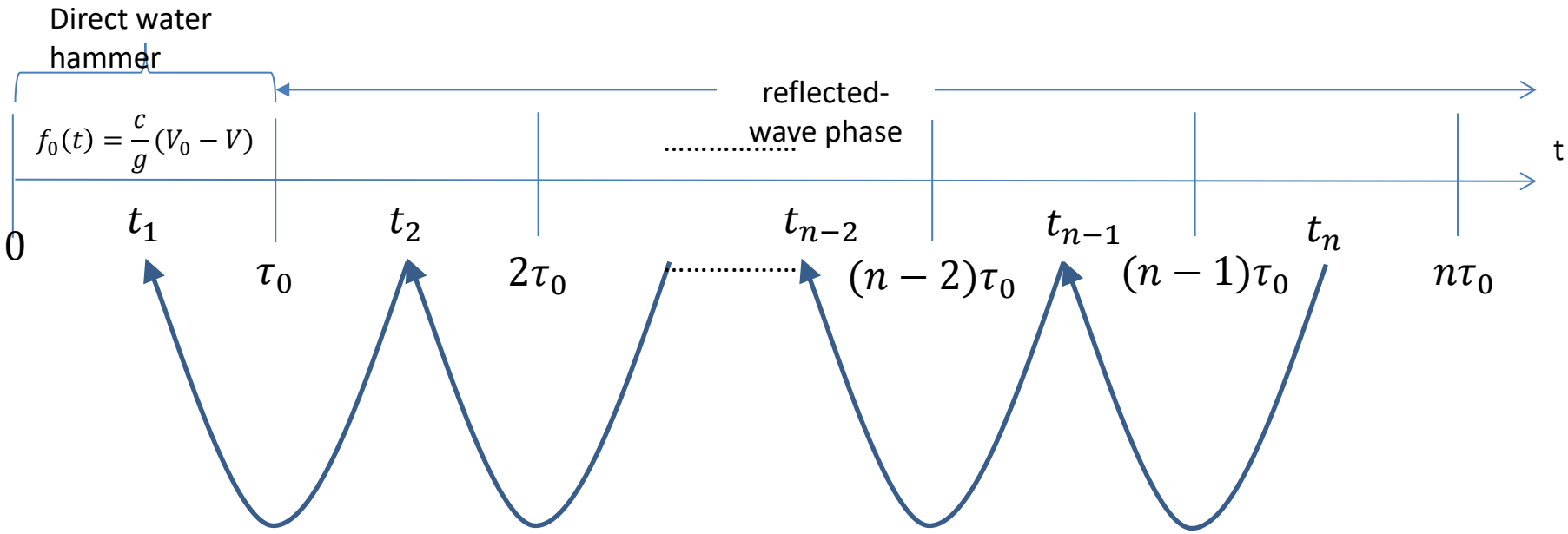
$$f_0(t_{n-1}) = \frac{c}{g}(V_0 - V_{n-1}) - f_0(t_{n-2})$$

Having defined

$$t_{n-2} = t_{n-1} - \tau_0$$

# Water hammer – Reflected-wave phase

In this way we proceed by linking, step by step, the function $f_0$ to an earlier time, with an offset of $\tau_0$. Let us then consider time as made up of a series of intervals $\tau_0$; the time $t_n$ will fall in one of these intervals, and in succession so will the various $t_{n-1}$, $t_{n-2}$ etc. The goal is then to reach a time $t_1$, which falls in the first interval $0<t_1<\tau_0$, i.e. the direct water hammer phase, where the function $f_0$ is known



# Water hammer – Reflected-wave phase

Proceeding in this way we have

$$f_0(t_n) = \frac{c}{g}(V_0 - V_n) - f_0(t_{n-1})$$

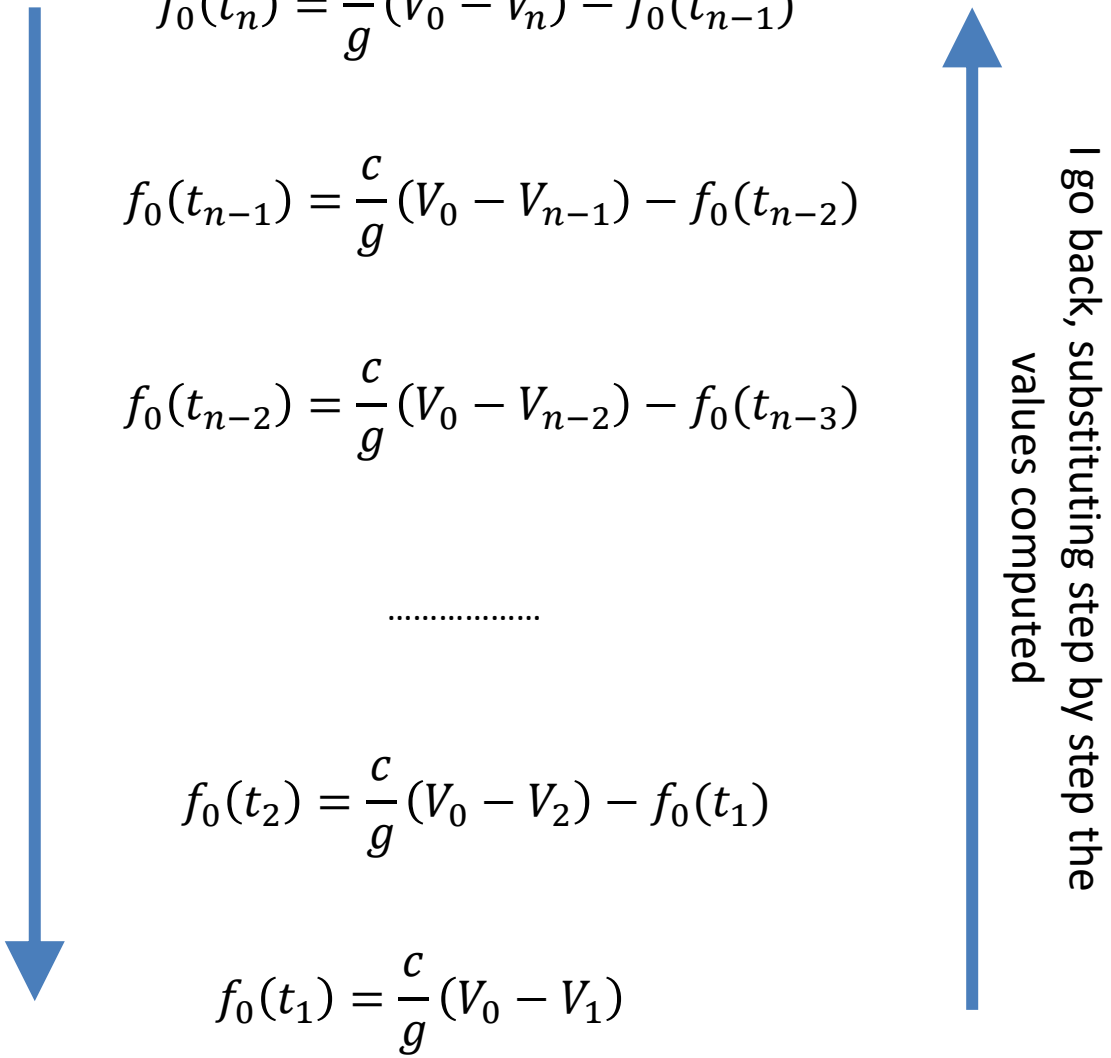
$$f_0(t_{n-1}) = \frac{c}{g}(V_0 - V_{n-1}) - f_0(t_{n-2})$$

$$f_0(t_{n-2}) = \frac{c}{g}(V_0 - V_{n-2}) - f_0(t_{n-3})$$

.....

$$f_0(t_2) = \frac{c}{g}(V_0 - V_2) - f_0(t_1)$$

$$f_0(t_1) = \frac{c}{g}(V_0 - V_1)$$



I go back, substituting step by step the values computed

# Water hammer – Reflected-wave phase

Collecting terms:

$$f_0(t_n) = \frac{c}{g}\left[(V_{n-1} - V_n) + (V_{n-3} - V_{n-2}) + \cdots + (V_0 - V_1)\right]$$

For an even number of intervals

$$f_0(t_n) = \frac{c}{g}\left[(V_{n-1} - V_n) + (V_{n-3} - V_{n-2}) + \cdots + (V_1 - V_2)\right]$$

For an odd number of intervals

$$f_0(t_{n-1}) = \frac{c}{g}\left[(V_{n-2} - V_{n-1}) + (V_{n-4} - V_{n-3}) + \cdots + (V_0 - V_1)\right]$$

For an even number of intervals

$$f_0(t_{n-1}) = \frac{c}{g}\left[(V_{n-2} - V_{n-1}) + (V_{n-4} - V_{n-3}) + \cdots + (V_1 - V_2)\right]$$

For an odd number of intervals

For the head

$$\Delta h = f(s - ct) - f(s - c(t - t_s))$$

$$\Delta_0 p = p - p_0 = \gamma(f_0(t_n) - f_0(t_{n-1}))$$

From the previous relations

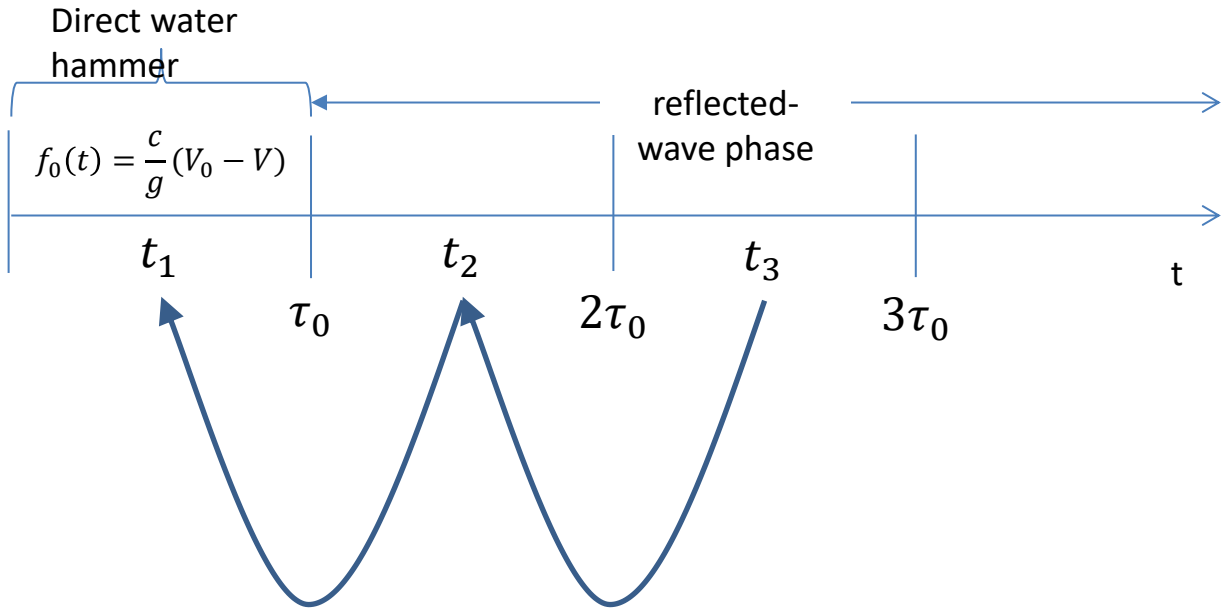
$$\Delta_0 p = \rho c[(V_{n-1} - V_n) - (V_{n-2} - V_{n-1}) + \cdots \pm (V_0 - V_1)]$$

+ for odd n

Which, with *V(t)* known, allows the computation of the overpressures at the valve

# Water hammer – Reflected-wave phase

**Example**
considering n=3




$$f_0(t_3) = \frac{c}{g}(V_0 - V_3) - f_0(t_2)$$

$$t_2 = t_3 - \tau_0 \qquad f_0(t_2) = \frac{c}{g}(V_0 - V_2) - f_0(t_1)$$

$$t_1 = t_2 - \tau_0 = t_3 - 2\tau_0 \qquad f_0(t_1) = \frac{c}{g}(V_0 - V_1)$$

# Water hammer – Reflected-wave phase

substituting

$$f_0(t_1) = \frac{c}{g}(V_0 - V_1)$$

$$f_0(t_2) = \frac{c}{g}(V_0 - V_2) - \frac{c}{g}(V_0 - V_1)$$

$$f_0(t_3) = \frac{c}{g}(V_0 - V_3) - \frac{c}{g}(V_0 - V_2) + \frac{c}{g}(V_0 - V_1) = \frac{c}{g}[(V_2 - V_3) + (V_0 - V_1)]$$

If, for example, $T_c < t_1$, $V_1=V_2=V_3=0$

$$f_0(t_1) = \frac{c}{g}V_0$$

$$f_0(t_2) = 0$$

$$f_0(t_3) = \frac{c}{g}V_0$$

$$\Delta_0 p = p - p_0 = \rho g\big(f_0(t_3) - f_0(t_2)\big) = \rho c V_0$$

# Water hammer – Reflected-wave phase

At the generic section *s* at time $t_n$

$$\Delta_s p = \gamma[f(s - ct_n) - f\big(s - c(t_n - t_s)\big)]$$

$$\frac{c}{g}\Delta_s V = f(s - ct_n) + f(s - c(t_n - t_s))$$

It has the same *f* that is present at the valve at time $t_n$-*s*/*c*

$$f(s - ct_n) = f_0(t_n - \frac{s}{c})$$

$$f(s - \mathrm{c}(\mathrm{t_n} - \mathrm{t_s})) = f_0\left(t_n - \frac{s}{c} - \frac{2(L-s)}{c}\right) = f_0\left(t_n - \frac{2L}{c} + \frac{s}{c}\right) = f_0\left(t_n - \tau_0 + \frac{s}{c}\right)$$

$$= f_0\left(t_{n-1} + \frac{s}{c}\right)$$

substituting

$$\Delta_s p = \gamma[f_0\left(t_n - \frac{s}{c}\right) - f_0\left(t_{n-1} + \frac{s}{c}\right)]$$

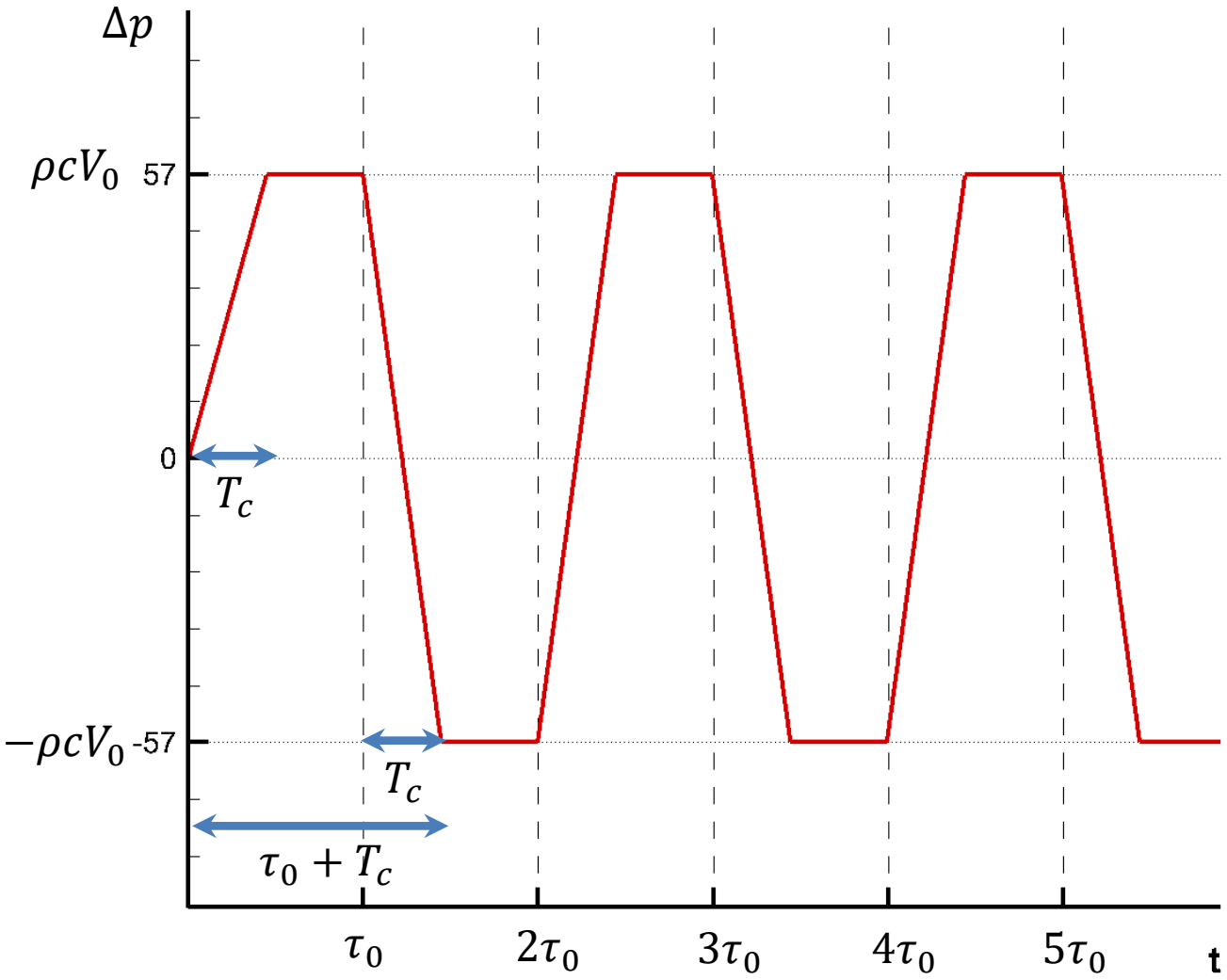
$$\frac{c}{g}\Delta_s V = f_0\left(t_n - \frac{s}{c}\right) + f_0(t_{n-1} + \frac{s}{c})$$

And an analogous methodology to the previous one can be used

# Water hammer – Examples

Below are some examples of pressure-surge profiles in time, computed with the method of characteristics. Exercise 9.2 Citrini-Noseda: pressure at the valve section with $T_c<\tau_0$. We assume that the velocity varies as

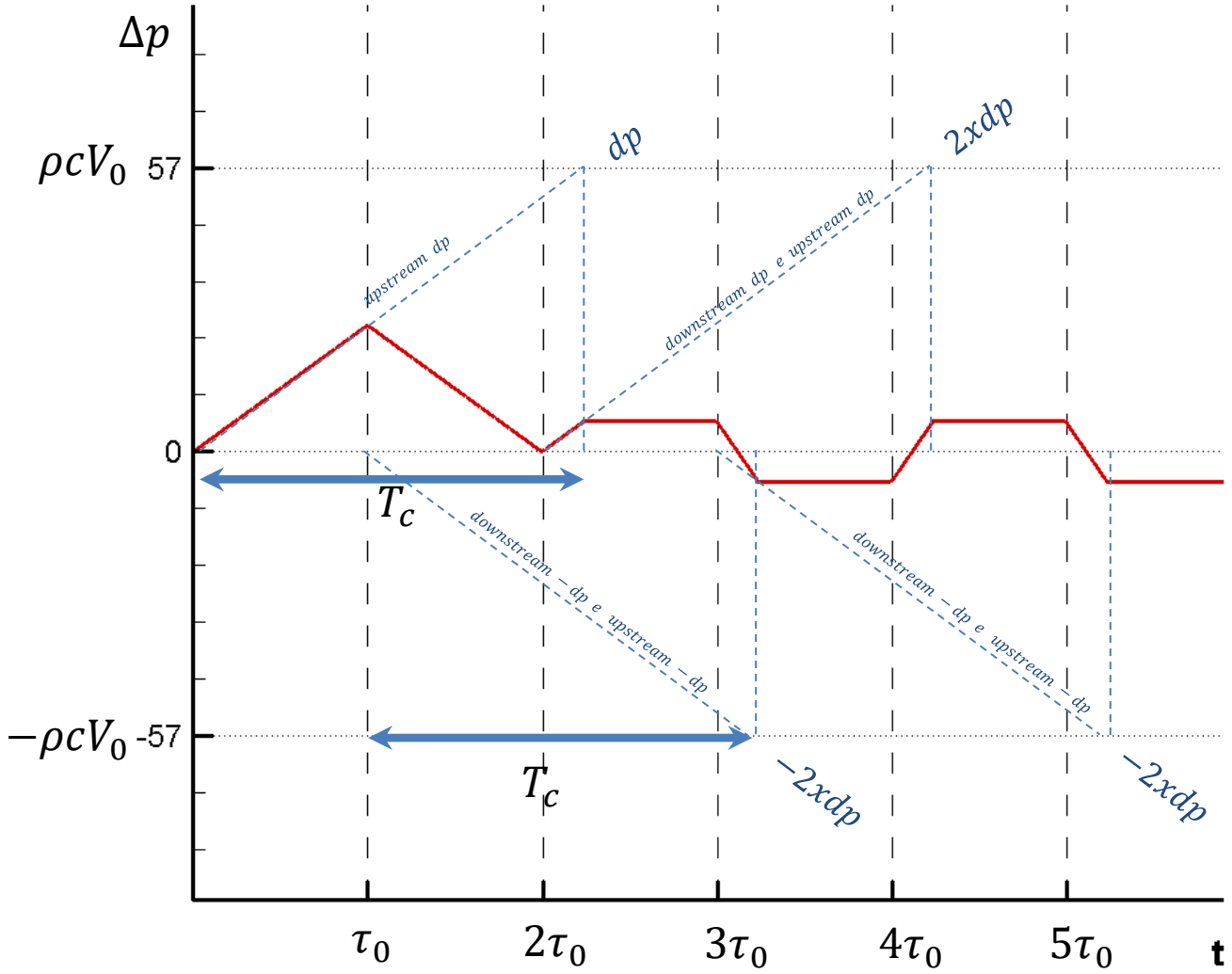
$$V(t) = V_0\left(1 - \frac{t}{T_c}\right)$$




The pressure grows until time $T_c$, then, since there are no more velocity variations, the pressure remains constant until $\tau_0$. For $t> \tau_0$ the section starts to feel, for a time interval $T_c$, the negative downstream-travelling wave and its reflection at the valve, decreasing with double slope. The process continues with analogous reasoning.

# Water hammer – Examples

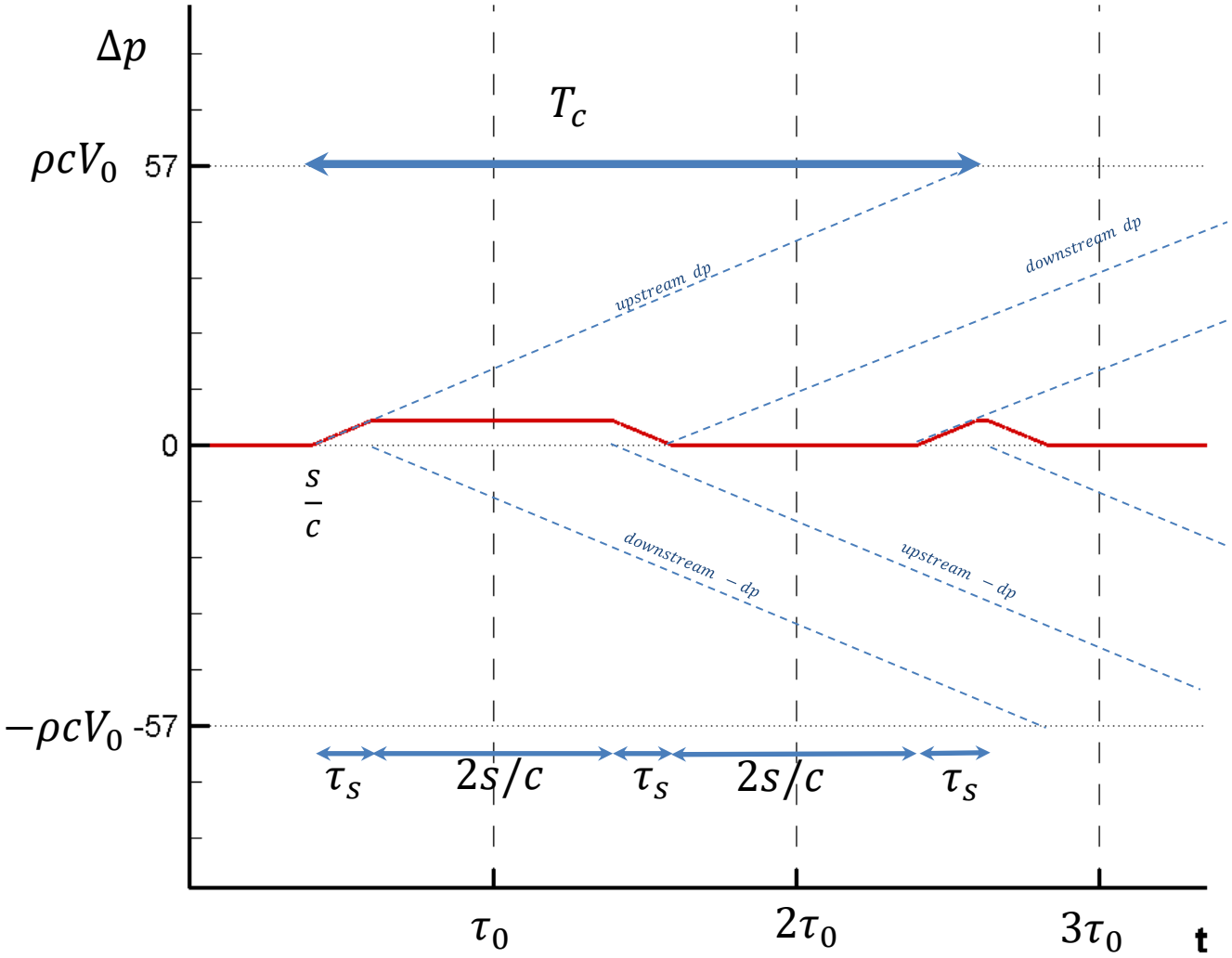
Example (ex. 9.3 Citrini-Noseda) of the pressure at the valve section with $T_c > \tau_0$. In this case the pressure grows up to $\tau_0$, because of the initial upstream-travelling overpressure generated by the valve closure. Up to $T_c$ the section feels this first train of overpressures. At time $\tau_0$, however, the section starts to feel,




for a time interval $T_c$, the negative downstream-travelling wave and its reflection at the valve. So, in the face of an increment *dp* it feels a decrement *2dp*, hence the general reduction of the overpressure. At each characteristic time interval $\tau_0$, for a period $T_c$, the section feels a new series of disturbances, hence the graph in the figure.

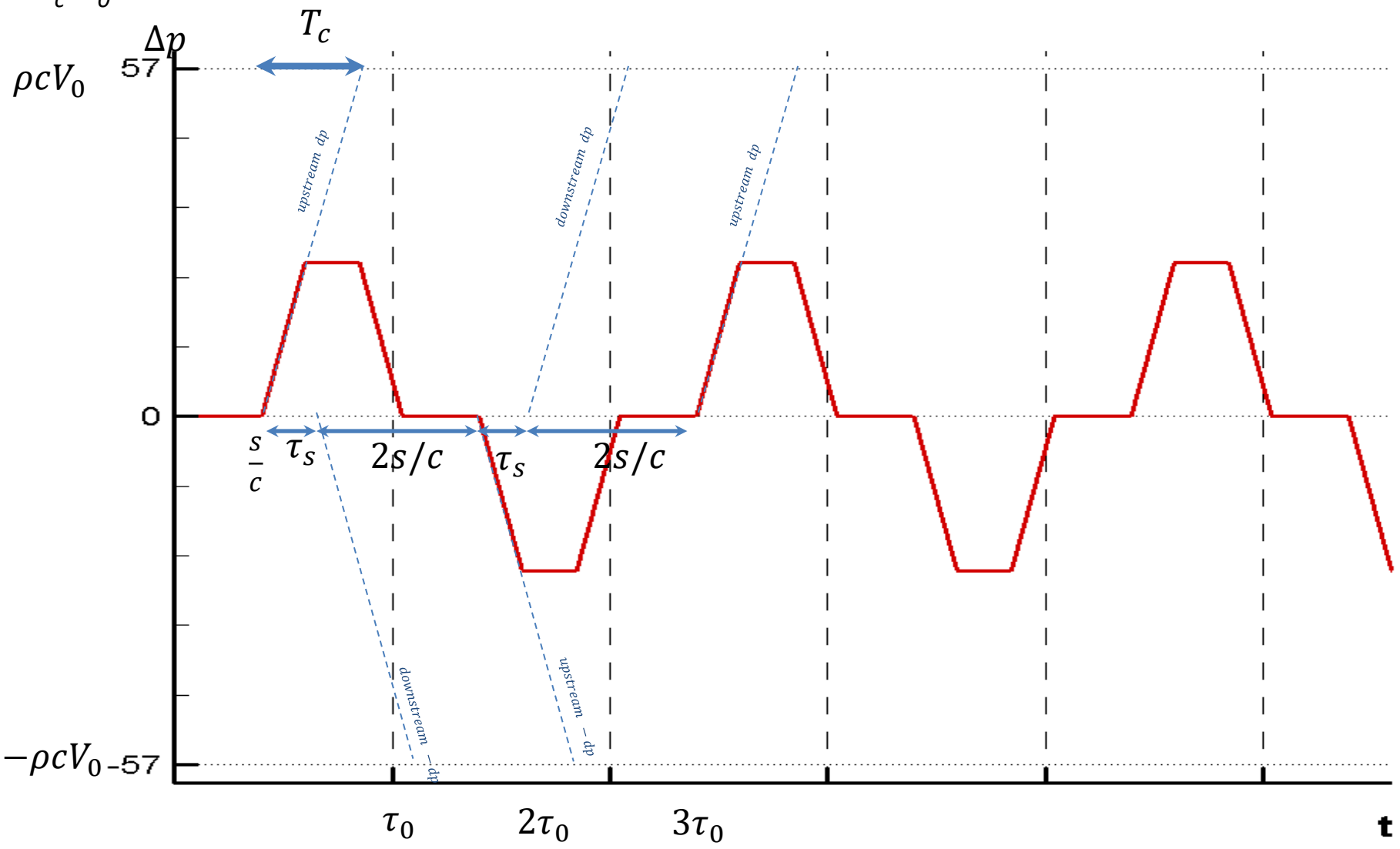
# Water hammer – Examples

Example (ex. 9.3 Citrini-Noseda) of the pressure at an intermediate section (s=800m) with $T_c > \tau_0$. The pressure remains unchanged until time *s/c*, i.e. when the first upstream-travelling wave produced at the valve reaches the section considered, and then grows for an interval $\tau_s$. In this case, besides the duration $T_c$ of the wave train, the section, in terms of pressure, is affected by two characteristic times: $\tau_s$ and *2s/c*. The first is the time interval after which an upstream-travelling wave passing through the section reaches the inlet, is reflected and returns to hit the section. The second is the time interval after which a downstream-travelling wave passing through the section reaches the valve, is reflected and returns to the section, hence the graph in the figure.



# Water hammer – Examples

Example (ex. 9.2 Citrini-Noseda) of the pressure at an intermediate section (s=800m) with $T_c<\tau_0$.



# Water hammer – Examples

In the graph of the previous page, the pressure remains unchanged until time *s/c*, when it feels the upstream-travelling wave coming from the valve for an interval $T_c$. After an interval $\tau_s$, however, the section feels the downstream-travelling wave with pressure drop *–dp*, so the pressure no longer increases but, by superposition of the effects, remains constant. At time $s/c+T_c$ the first upstream-travelling wave has ended and the section feels only the downstream-travelling one with *–dp*; when this ends at time $s/c+ \tau_s+T_c$ the pressure is at an unaltered value with respect to the initial conditions. Starting from time $s/c+ \tau_s$, after an interval *2s/c* the negative downstream-travelling wave has been reflected at the valve with the same sign, producing a negative upstream-travelling wave, and has reached the section: the pressure drops, and so on.

---

In the case of overpressure at the valve, with $T_c > \tau_0$, the maximum overpressure is reached during the direct water hammer phase, and can be evaluated at the end of this phase, i.e. at time $\tau_0$:

$$\Delta p = \rho c(V_0 - V) = \rho c\left(V_0 - V_0\left(1 - \frac{\tau_0}{T_c}\right)\right) = \rho c V_0 \frac{\tau_0}{T_c} = \rho c V_0 \frac{2L}{c}\frac{1}{T_c} = \frac{2\rho L V_0}{T_c}$$

Known as Michaud's formula.

For intermediate sections, in this case too the maximum overpressure is evaluated at the end of the direct water hammer phase, i.e. at $s/c+ \tau_s$. At this time, the velocity variation at the section is the same that characterized the valve at an earlier time, smaller by an amount *s/c* so that

$$\Delta p = \rho c(V_0 - V) = \rho c\left(V_0 - V_0\left(1 - \frac{\tau_s}{T_c}\right)\right) = \rho c V_0 \frac{\tau_s}{T_c}$$

# Water hammer – Closure time and surge size

## Characteristic time

2L/c = time for the wave to go to the reservoir and back

## Rapid closure (Tc ≤ 2L/c)

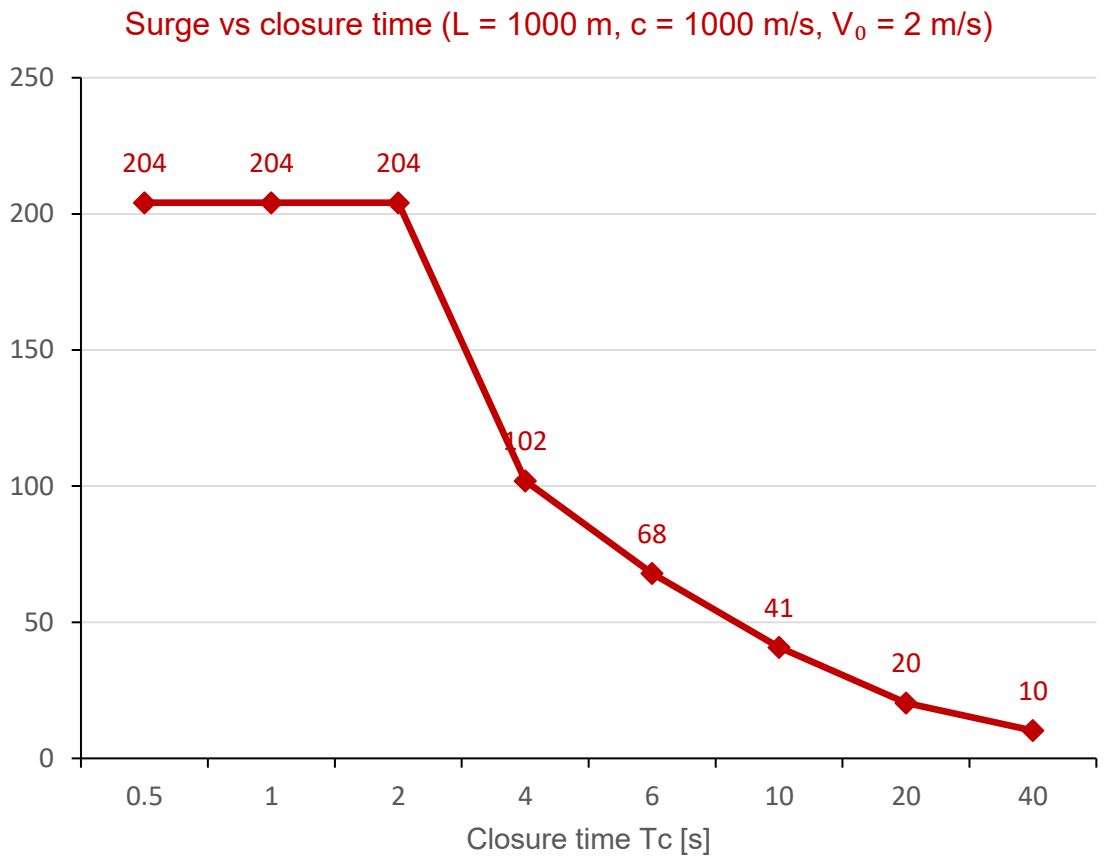
$\Delta p = \rho \cdot c \cdot V_0 \rightarrow \Delta h = c \cdot V_0 / g$ (Joukowsky)

## Slow closure (Tc > 2L/c), linear closure

$\Delta p \approx 2 \cdot \rho \cdot L \cdot V_0 / Tc \rightarrow \Delta h \approx 2 \cdot L \cdot V_0 / (g \cdot Tc)$ (Michaud)

Example: L = 1000 m, c = 1000 m/s, $V_0$ = 2 m/s, so 2L/c = 2 s

**Tc = 1 s → Δh ≈ 204 m (≈ 20 bar); Tc = 20 s → Δh ≈ 20 m**




Closing the valve more slowly than 2L/c is the cheapest way to reduce the overpressure; the Michaud formula gives an estimate of the maximum surge for linear closures.

# Water hammer – Method of characteristics

## Solution of water hammer with the method of characteristics

We reconsider the equations of motion, but this time with sign concordant with the direction of steady flow

$$\begin{cases} \dfrac{\partial h}{\partial s} + \dfrac{1}{g}\dfrac{\partial V}{\partial t} = 0 \\ \dfrac{\partial V}{\partial s} + \dfrac{g}{c^2}\dfrac{\partial h}{\partial t} = 0 \end{cases}$$

We consider a combination of them through an operator $\lambda$

$$\lambda\left[\frac{\partial h}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t}\right] + \frac{\partial V}{\partial s} + \frac{g}{c^2}\frac{\partial h}{\partial t} = 0$$

We collect the terms depending on whether they are velocity *V* or head *h*

$$\left[\frac{\partial V}{\partial s} + \frac{\lambda}{g}\frac{\partial V}{\partial t}\right] + \left[\lambda\frac{\partial h}{\partial s} + \frac{g}{c^2}\frac{\partial h}{\partial t}\right] = 0$$

# Water hammer – Method of characteristics

i.e.

$$\frac{\lambda}{g}\left[\frac{\partial V}{\partial t}+\frac{g}{\lambda}\frac{\partial V}{\partial s}\right]+\frac{g}{c^2}\left[\frac{\partial h}{\partial t}+\frac{\lambda c^2}{g}\frac{\partial h}{\partial s}\right]=0$$

We recall the expressions of the total derivative for *V* and *h*

$$\frac{dV}{dt}=\frac{\partial V}{\partial t}+\frac{ds}{dt}\frac{\partial V}{\partial s}$$

$$\frac{dh}{dt}=\frac{\partial h}{\partial t}+\frac{ds}{dt}\frac{\partial h}{\partial s}$$

The terms in square brackets are similar in form to the respective total derivatives, apart from the term *ds/dt*. For them to be identical, we would need:

$$\frac{g}{\lambda}=\frac{ds}{dt}$$

for the part in *V*, while for the part in *h*:

$$\frac{\lambda c^2}{g}=\frac{ds}{dt}$$

# Water hammer – Method of characteristics

Equating the two relations

$$\frac{g}{\lambda} = \frac{\lambda c^2}{g}$$

we obtain

$$\lambda = \pm \frac{g}{c}$$

Therefore, **<u>if and only if</u>**, the operator introduced takes the values $\lambda=g/c$ o $\lambda=-g/c$ we can assume that the terms inside the square brackets are total derivatives, and we can rewrite the equation as

$$\frac{\lambda}{g}\frac{dV}{dt} + \frac{g}{c^2}\frac{dh}{dt} = 0$$

We now substitute the two possible values found for $\lambda$; if $\lambda=g/c$, then:

$$\frac{ds}{dt} = \frac{g}{\lambda} = g\frac{c}{g} = c \qquad \text{and} \qquad \frac{dV}{dt} + \frac{g}{c}\frac{dh}{dt} = 0$$

While in the other case ($\lambda=-g/c$)

$$\frac{ds}{dt} = -c \qquad \text{and} \qquad \frac{dV}{dt} - \frac{g}{c}\frac{dh}{dt} = 0$$

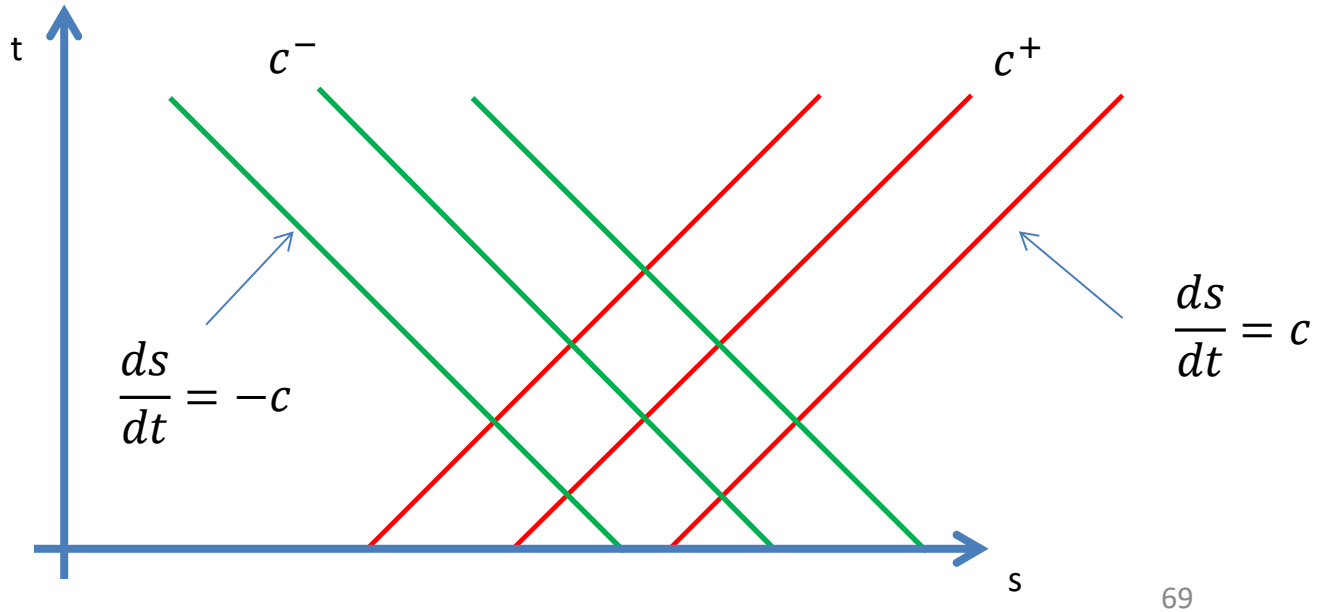
# Water hammer – Method of characteristics

To recap, we have obtained two ordinary equations in *V* and *h*, which I will call equations $C^+$ and $C^-$

equation $C^+$ $\quad \frac{dV}{dt} + \frac{g}{c}\frac{dh}{dt} = 0 \quad$ Valid only if $\quad \frac{ds}{dt} = c$

equation $C^-$ $\quad \frac{dV}{dt} - \frac{g}{c}\frac{dh}{dt} = 0 \quad$ Valid only if $\quad \frac{ds}{dt} = -c$

The equations *ds/dt = +- c* are called characteristic lines; if we observe them in a Cartesian plane *s-t*, they form a family of straight lines with slope *+c* o *–c*, along which the equations $C^+$ and $C^-$ hold, where the intercept on the *s* axis depends on the initial conditions

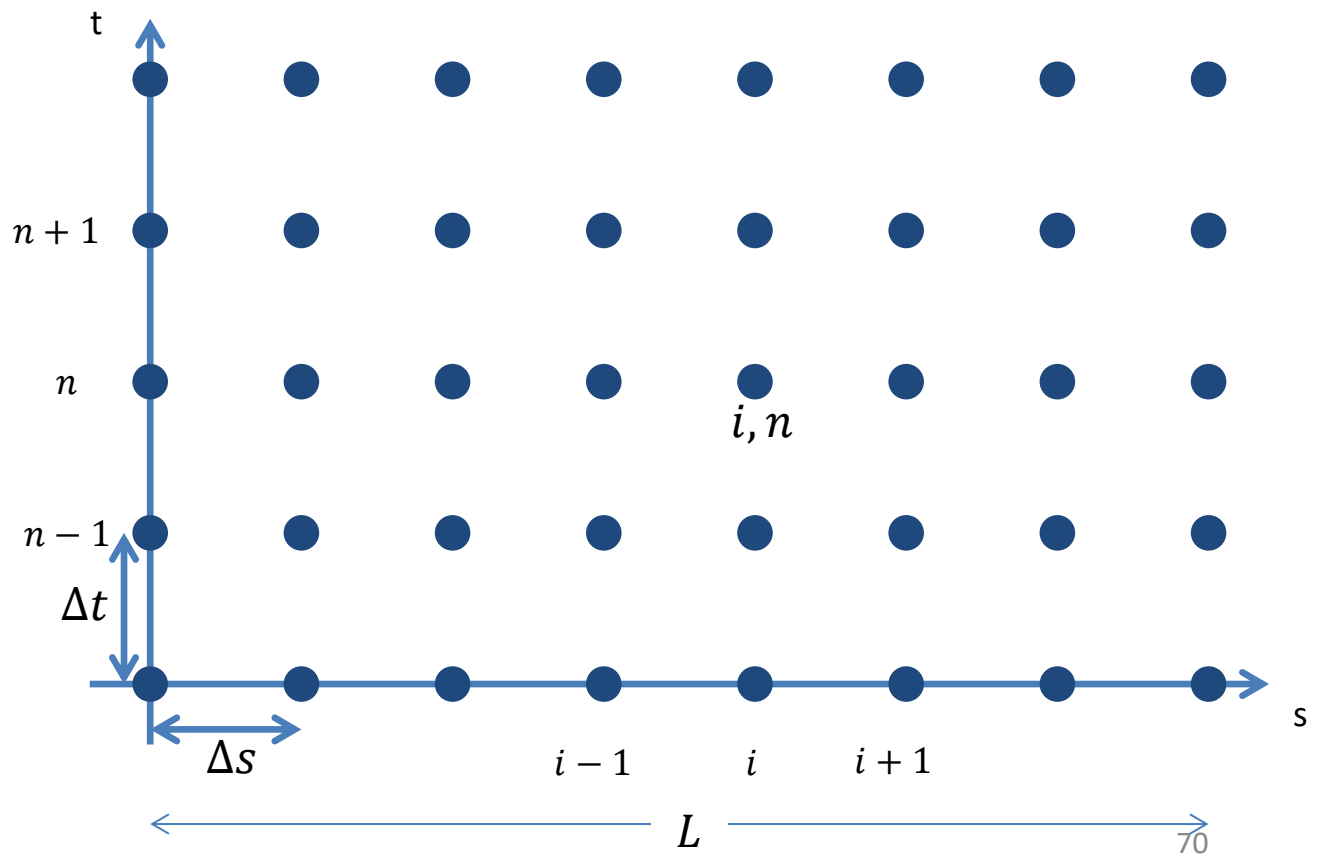


# Water hammer – Method of characteristics

Let us now consider numerically what we have just seen, with a discrete approach. We take a discretization in space with a step $\Delta s$ for the pipe length $L$, and in time a time step $\Delta t$; their values are not chosen arbitrarily, but by forcing the nodes of the discretization to lie on the characteristic lines. Therefore:

If I fix $\Delta t$ it must be $\Delta s = c\Delta t$

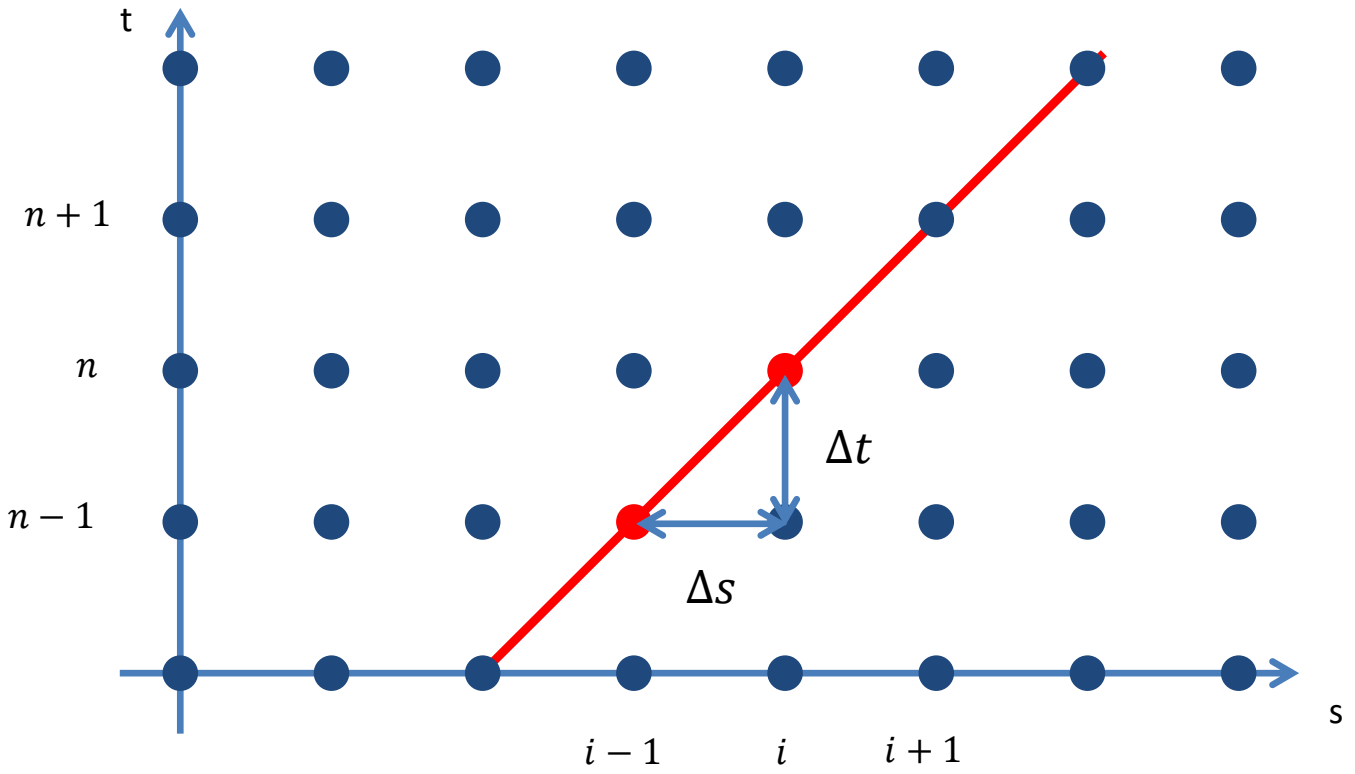
If I fix $\Delta s$ it must be $\Delta t = \Delta s / c$



# Water hammer – Method of characteristics

Fixing an index *i* to move in space and an index *n* to move in time, each node is identified by a pair *(i,n)*. I want to evaluate at node *(i,n)* my equation $C^+$ , considering that the characteristic line through the node necessarily passes through other nodes of our discretization

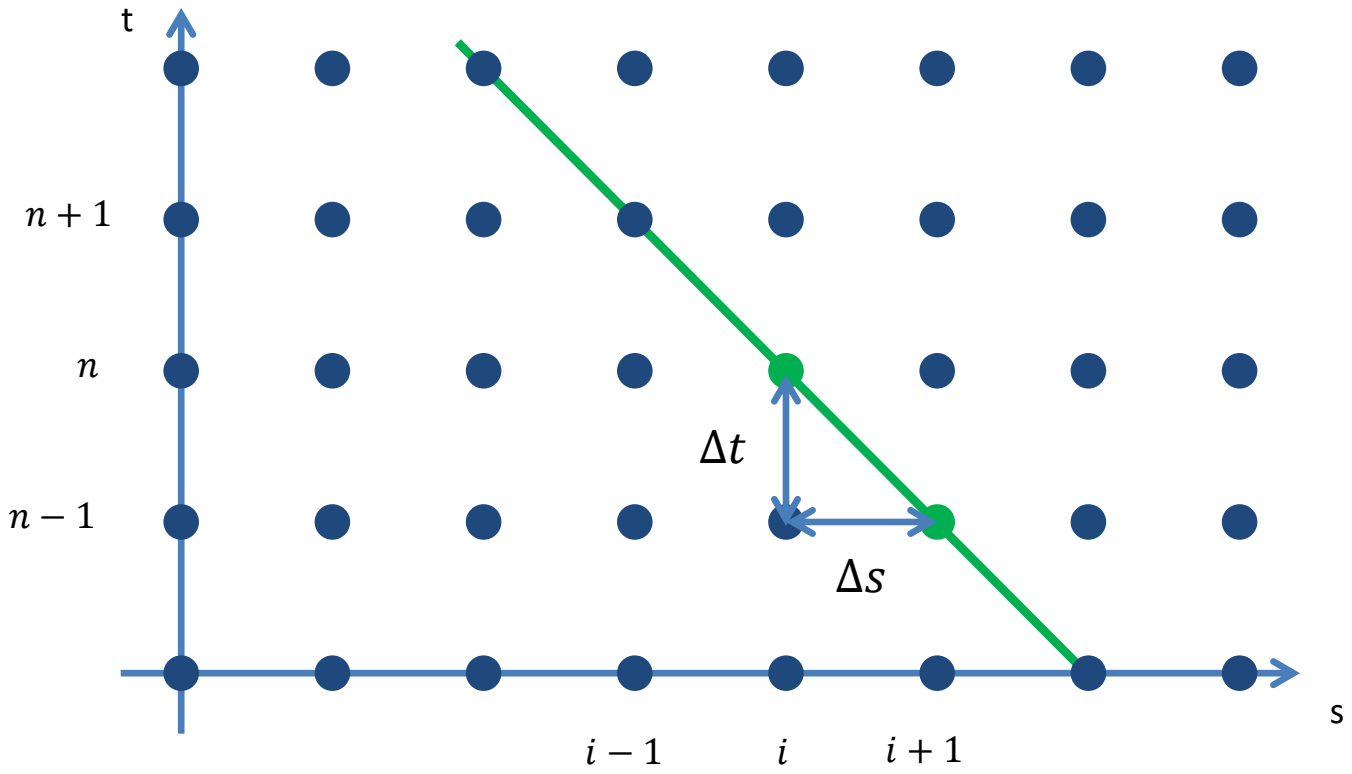
$$\frac{dV}{dt} + \frac{g}{c}\frac{dh}{dt} = 0 \quad \Longrightarrow \quad \frac{V_i^n - V_{i-1}^{n-1}}{\Delta t} + \frac{g}{c}\frac{h_i^n - h_{i-1}^{n-1}}{\Delta t} = 0$$



# Water hammer – Method of characteristics

But node *(i,n)* will also be the intersection with a line $C^-$ for which the following also holds

$$\frac{dV}{dt} - \frac{g}{c}\frac{dh}{dt} = 0 \quad \Longrightarrow \quad \frac{V_i^n - V_{i+1}^{n-1}}{\Delta t} - \frac{g}{c}\frac{h_i^n - h_{i+1}^{n-1}}{\Delta t} = 0$$



# Water hammer – Method of characteristics

To recap, at node *(i,n)* the following hold simultaneously

$$\frac{V_i^n - V_{i-1}^{n-1}}{\Delta t} + \frac{g}{c}\frac{h_i^n - h_{i-1}^{n-1}}{\Delta t} = 0 \qquad \frac{V_i^n - V_{i+1}^{n-1}}{\Delta t} - \frac{g}{c}\frac{h_i^n - h_{i+1}^{n-1}}{\Delta t} = 0$$

Each of which depends simultaneously on *V* and *h* at time *n*. Considering a combination of them, from the first equation plus the second I obtain:

$$V_i^n = \frac{1}{2}\left[V_{i-1}^{n-1} + V_{i+1}^{n-1} + \frac{g}{c}\left(h_{i-1}^{n-1} - h_{i+1}^{n-1}\right)\right]$$

In which the velocity at step *n*depends only on the variables at step *n-1*. Taking now the first equation minus the second I obtain:

$$h_i^n = \frac{1}{2}\left[h_{i-1}^{n-1} + h_{i+1}^{n-1} + \frac{c}{g}\left(V_{i-1}^{n-1} - V_{i+1}^{n-1}\right)\right]$$

In which the head at step *n*depends only on the variables at step *n-1*. Through these two expressions, once step *n-1*is known, we can compute the values of *V* and *h* at the generic node *(i,n)*

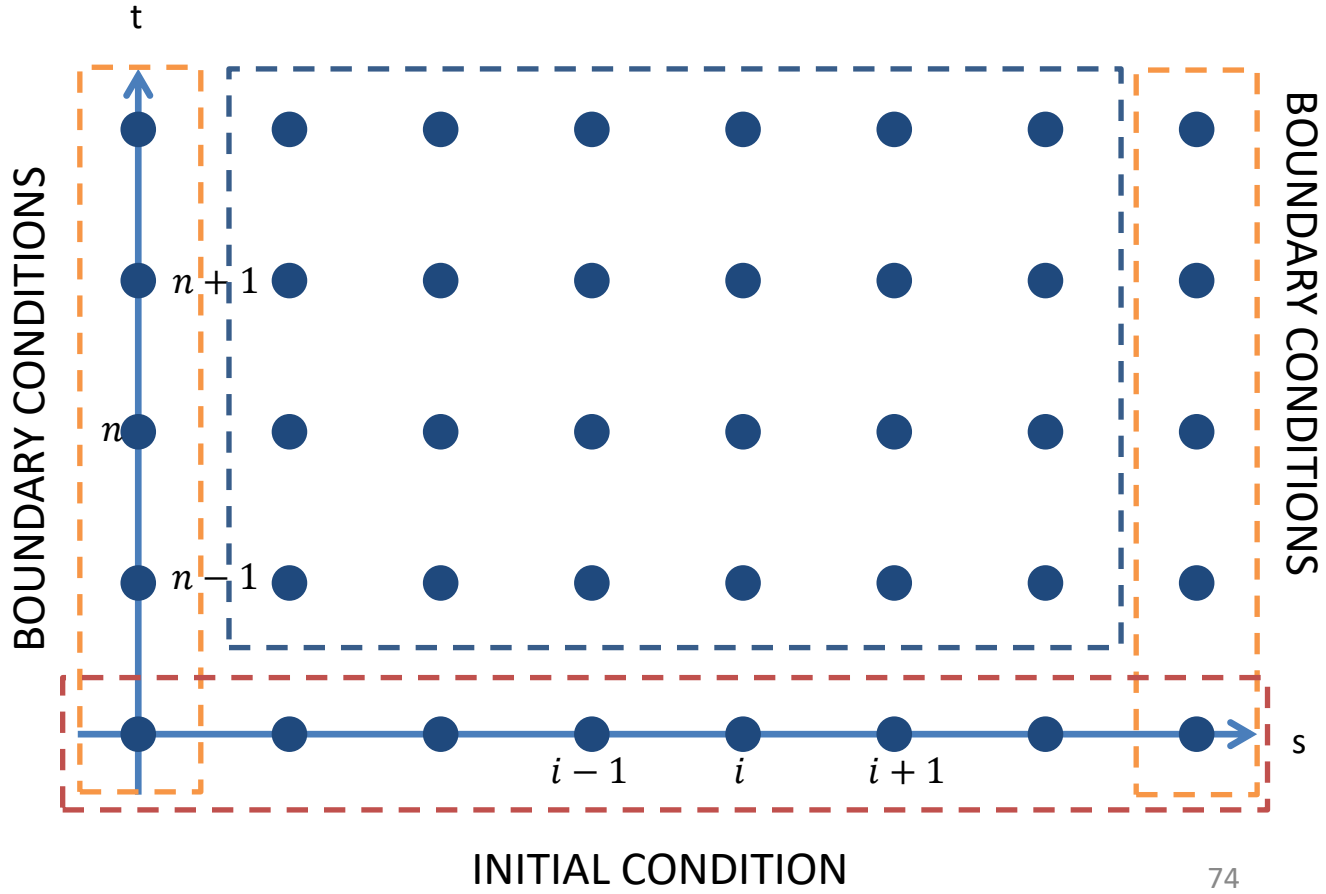
The equations just seen apply only to the central part of the grid, given the required spatial stencil, which implies *i-1* and *i+1*, and the dependence on the previous time step.
Therefore if $i=0, i_{tot}$ and $n=0, n_{tot}$, they will be applied in the range $i=1, i_{tot}-1$ and $n=1, n_{tot}$

# Water hammer – Method of characteristics

To obtain their evolution in space and time, both an initial condition and a boundary condition must be defined.

**The initial condition** will be the steady flow one, at time *t=0 (n=0)* $h=h_0$ and $V=V_0$

**The boundary condition** will be: at the inlet *s=0 (i=0)* $h=h_0$, at the valve *s=L* $(i=i_{tot})$ $V=V(t)$
Where *V(t)* is the known law of velocity variation at the valve; if it is completely closed at time $T_c$, then for $t>T_c$ $V=0$



# Water hammer – Method of characteristics

It remains to define the boundary conditions for *V* at *s=0* and for *h* at *s=L*; these are evaluated at time *n* as a function of what has been computed inside the spatial domain at the same time

At *s=0*

$$V_0^n = V_1^n + \frac{g}{c}\,(h_0^n - h_1^n)$$

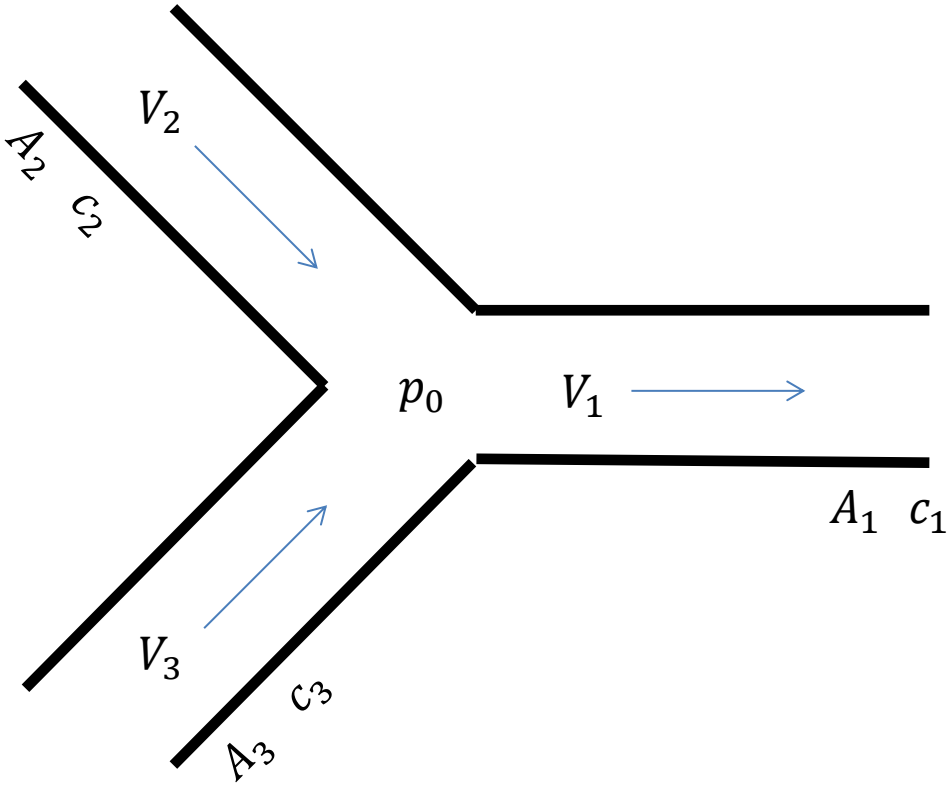
At *s=L*

$$h_{i_{tot}}^n = h_{i_{tot}-1}^n - \frac{c}{g}\left(V_{i_{tot}}^n - V_{i_{tot}-1}^n\right)$$

## Pipe systems

The case of a pipe with constant diameter and thickness is an ideal condition. Normally diameter and thickness vary, or pipes meet at junction nodes, producing discontinuities that modify the pressure waves.

**Case of a node with three pipes**.
Consider a steady-flow condition, with velocities $V_1$, $V_2$ and $V_3$, while the pipes are defined by their area $A_1$, $A_2$, $A_3$ and a characteristic celerity $c_1$, $c_2$,$c_3$.
We assume that head losses and kinetic heads are negligible, and let $p_0$ be the pressure at the node and hence in the branches. At the node, continuity gives:



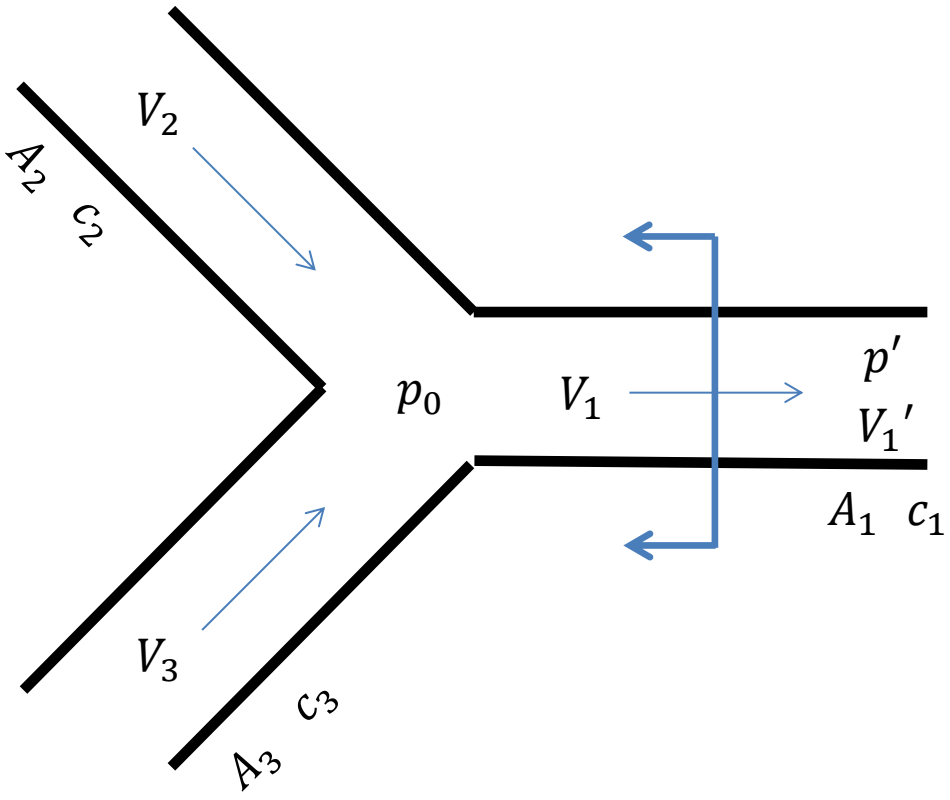

$$A_1V_1 = A_2V_2 + A_3V_3$$

# Water hammer – Pipe systems

Assume that branch 1 is reached by an upstream-travelling disturbance, which moves up towards the node. This causes a velocity variation $V_1 \rightarrow V_1'$ to which corresponds a pressure variation $p_0 \rightarrow p'$, linked to each other by the relation:

$$p' - p_0 = \rho c_1 (V_1 - V_1')$$

When the disturbance reaches the node, the whole branch 1 will be characterized by $p'$ and $V_1'$, while at the node, by head compatibility, we will have a new pressure $p''$. At this point the disturbance is transmitted along branches 2 and 3 as an upstream-travelling wave, and is partly reflected, producing a downstream-travelling wave along branch 1. In branches 2 and 3 a new velocity condition is established $V_2'$ and $V_3'$ with pressure $p''$, while in branch 1, $p''$ and $v_1''$



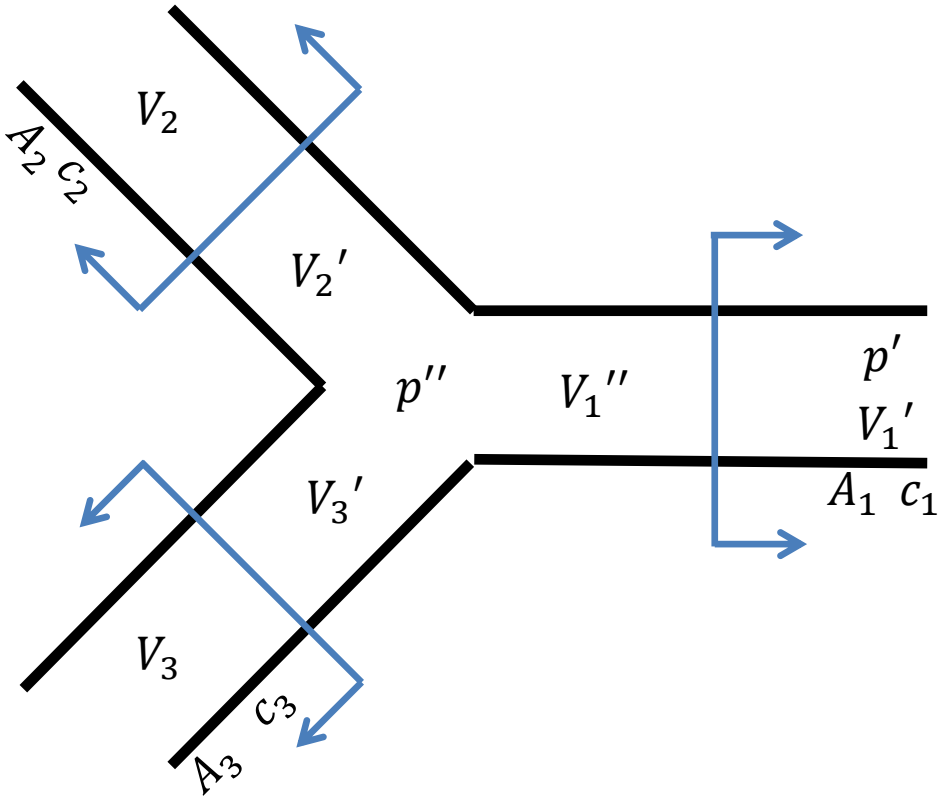
# Water hammer – Pipe systems

In this phase we have

$$p'' - p_0 = \rho c_2 (V_2 - V_2') \quad \text{upstream-travelling}$$

$$p'' - p_0 = \rho c_3 (V_3 - V_3') \quad \text{upstream-travelling}$$

$$p' - p'' = \rho c_1 (V_1' - V_1'') \quad \text{downstream-travelling}$$




While the continuity equation reads

$$A_1 V''_1 = A_2 V'_2 + A_3 V'_3$$

To solve the resulting system of equations we eliminate the dependence on velocity; first of all, we consider the difference between the continuity equations obtained:

$$A_1 (V_1 - V_1'') = A_2 (V_2 - V_2') + A_3 (V_3 - V_3')$$

# Water hammer – Pipe systems

For branches 2 and 3 I have

$$p'' - p_0 = \rho c_2 (V_2 - V_2') \qquad (V_2 - V_2') = \frac{p'' - p_0}{\rho c_2}$$

$$p'' - p_0 = \rho c_3 (V_3 - V_3') \qquad (V_3 - V_3') = \frac{p'' - p_0}{\rho c_3}$$

For branch 1

$$\left.\begin{aligned} p' - p_0 &= \rho c_1 (V_1 - V_1') \\ p' - p'' &= \rho c_1 (V_1' - V_1'') \end{aligned}\right\} \quad p' - p_0 + p' - p'' = \rho c_1 (V_1 - V_1' + V_1' - V_1'')$$

$$(V_1 - V_1'') = \frac{(2p' - 2p_0) - (p'' - p_0)}{\rho c_1}$$

Obtained by adding $+p_0$ and $-p_0$, I substitute the expressions obtained for the velocity into the equation derived from the continuity equations

# Water hammer – Pipe systems

I obtain

$$\frac{A_1}{\rho c_1}(2p' - 2p_0) - \frac{A_1}{\rho c_1}(p'' - p_0) = \frac{A_2}{\rho c_2}(p'' - p_0) + \frac{A_3}{\rho c_3}(p'' - p_0)$$

$$\frac{A_1}{\rho c_1}(2p' - 2p_0) = \frac{A_1}{\rho c_1}(p'' - p_0) + \frac{A_2}{\rho c_2}(p'' - p_0) + \frac{A_3}{\rho c_3}(p'' - p_0)$$

Hence

$$s = \frac{p'' - p_0}{p' - p_0} = \frac{\frac{2A_1}{c_1}}{\frac{A_1}{c_1} + \frac{A_2}{c_2} + \frac{A_2}{c_3}}$$

I have therefore defined a coefficient *s*, the transmission coefficient of the disturbances, a function of the areas and celerities of the branches only, which gives the ratio between the upstream-travelling wave transmitted into the branches and the upstream-travelling wave incident on the node.

$$r = \frac{p'' - p'}{p' - p_0} = s - 1$$

The reflection coefficient *r* , defined starting from *s*, identifies instead the ratio between the reflected downstream-travelling wave and the upstream-travelling wave incident on the node

# Water hammer – Pipe systems

**Pipe – abrupt change of cross-section**.

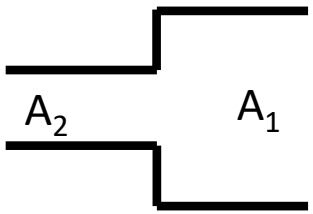
Under the assumption of constant celerity, it is reduced to the case of a node where two pipes of different diameter meet. From the relations seen previously we obtain:

$$s = \frac{2A_1}{A_1 + A_2} \qquad r = \frac{A_1 - A_2}{A_1 + A_2}$$

**Case A2<A1** narrowing of the section: *s>1* and *0<r<1*

The transmitted wave is larger than the incident one; the reflected wave has the same sign as the incident one and worsens the conditions already present in the branch.

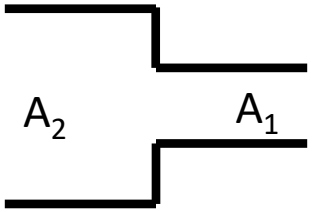
In the limit case, if *A2 = 0*, it is as if we were facing a wall, the reflection coefficient takes the value *r=1*, and the reflected wave equals the incident one (note: case of the valve section in water hammer)




**Case A2>A1** widening of the section: *0<s<1* and *r<0*

The transmitted wave is smaller than the incident one, while the reflected wave changes sign.

In the limit case of *A2>>A1*, tending to infinity, we obtain *s=0* and *r=-1* the transmitted wave is null and the reflected one is equal and opposite to the incident one. (note: this configuration can be seen as a pipe connected to a reservoir; see what was seen for water hammer at the inlet section)



# Water hammer – Pipe systems

Recall that

$$c = \frac{c^*}{\sqrt{1 + \frac{\epsilon D}{eE}}}$$

Having assumed constant celerity with pipes of different diameter therefore implies that they must necessarily have different thickness.

Thickness increases → celerity increases
Diameter increases → celerity decreases

**Pipe – constant diameter but varying thickness**.
In this case the dependence on the area vanishes from the coefficients, while the celerity takes different values thanks to the use of different thicknesses. We obtain

| $c_2$ | $c_1$ |
|---|---|

$$s = \frac{2c_2}{c_1 + c_2} \qquad r = \frac{c_2 - c_1}{c_1 + c_2}$$

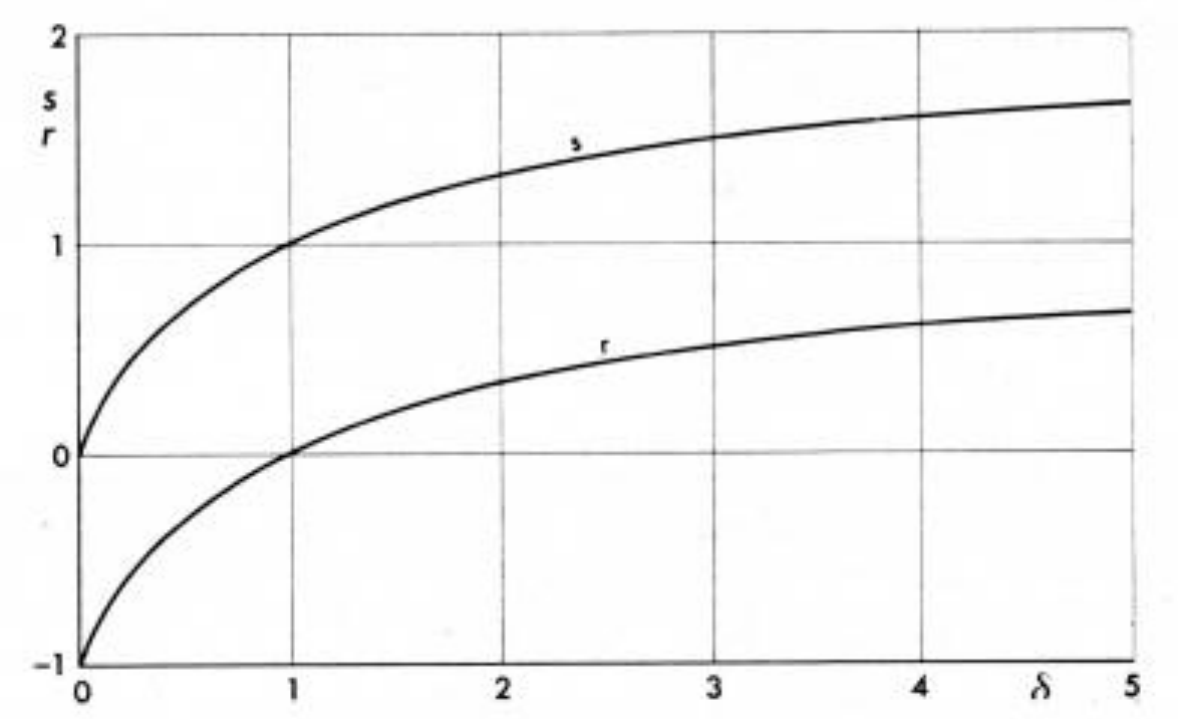
from

$$s = \frac{\frac{2}{c_1}}{\frac{1}{c_1} + \frac{1}{c_2}} = \frac{2}{c_1}\frac{c_1 c_2}{c_1 + c_2}$$

# Water hammer – Pipe systems

s and r have the same trend, whether we consider $\delta = \frac{A_1}{A_2}$

or $\delta = \frac{c_2}{c_1}$



# Water hammer – Protection measures

## Reduce the cause

- Slower or staged valve closure ($T_c > 2L/c$)
- Flywheel on the pump or soft-start / variable-speed drive
- Slow-closing non-return valves

## Absorb the surge

- Air vessel (hydropneumatic tank) at the pump
- Surge tank / one-way tank where the profile allows
- Surge-relief and pressure-reducing valves

## Prevent low pressures

- Air-admission (vacuum-breaker) valves at high points
- Bypass with non-return valve around the pump
- Keep the pipe below the hydraulic grade line

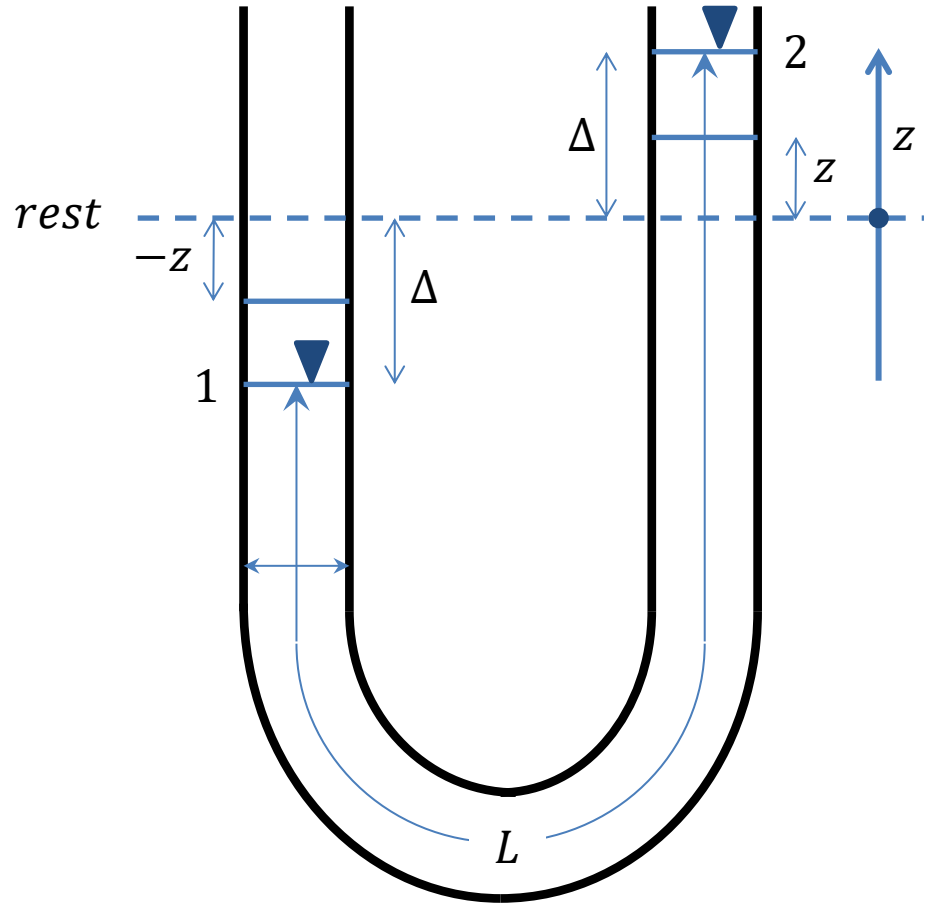
## Resist the surge

- Pipe class chosen for static head + surge
- Anchor blocks at bends and valves
- Check $p_{max}$ and $p_{min}$ along the whole profile, e.g. with the method of characteristics

# Mass oscillations – U-tube

## Mass oscillations in a U-tube

Consider a U-shaped pipe of length L, as in the figure, with both ends in contact with the atmosphere. Take an initial condition with an imposed level difference equal to *2Δ*, with respect to the rest condition.
The condition is not stable, and the fluid will tend to return to the rest level through an unsteady-flow process.




The process is slow (compare it with the water hammer case seen above), so the fluid can be considered incompressible and the pipe rigid.

# Mass oscillations – U-tube

The equations governing the unsteady flow are those we have already derived

$$\frac{\partial}{\partial s}\left(z+\frac{V^2}{2g}+\frac{p}{\gamma}\right)+\frac{p}{\gamma^2}\frac{\partial \gamma}{\partial s}+\frac{1}{g}\frac{\partial V}{\partial t}+J=0$$

$$\frac{\partial \rho Q}{\partial s}+\frac{\partial \rho A}{\partial t}=0$$

Which, under the assumptions of rigid pipe and incompressible fluid, reduce to

$$\frac{\partial}{\partial s}\left(z+\frac{V^2}{2g}+\frac{p}{\gamma}\right)+\frac{1}{g}\frac{\partial V}{\partial t}+J=0$$

$$\frac{\partial Q}{\partial s}=0$$

From the continuity equation, expressed in terms of area and velocity, we obtain that

$$\frac{\partial AV}{\partial s}=0 \qquad \frac{\partial V}{\partial s}=0$$

That is, along the spatial coordinate the velocity remains constant, and is a function of time only

# Mass oscillations – U-tube

Integrating the equation of motion along s we have

$$\int \frac{\partial h}{\partial s} ds + \int \frac{1}{g} \frac{\partial V}{\partial t} \, ds + \int J ds = 0$$

The first term is simply the head difference between level 2 and level 1

$$\int \frac{\partial h}{\partial s} ds = h_2 - h_1 = \left( z + \frac{V^2}{2g} + \frac{p_a}{\gamma} \right) - \left( -z + \frac{V^2}{2g} + \frac{p_a}{\gamma} \right) = 2z$$

The second term, given the continuity equation, does not depend on *s*, and can be written as

$$\int \frac{1}{g} \frac{\partial V}{\partial t} \, ds = \frac{L}{g} \frac{dV}{dt}$$

The third term, given the expressions of *J* in terms of velocity, can be written, following the same reasoning as for the previous term, as:

$$\int J ds = JL$$

# Mass oscillations – U-tube

collecting terms

$$2z + \frac{L}{g}\frac{dV}{dt} + JL = 0$$

The equation is a function of *V* and *z*, our variables of interest; to obtain a solution we need an additional equation. From the continuity equation we can write

$$\frac{dz}{dt} = V$$

The variation of the free surface in time is necessarily a function of the velocity that is established

**Case of a perfect (frictionless) liquid**

From the system of equations above I obtain an analytical solution under the assumption of null head losses, *J=0*.

from $\frac{dz}{dt} = V$ $\qquad \frac{d^2z}{dt^2} = \frac{dV}{dt}$

from $2z + \frac{L}{g}\frac{dV}{dt} = 0$ $\qquad \frac{dV}{dt} = -\frac{2g}{L}\,z$

$$\frac{d^2z}{dt^2} + \omega^2 z = 0 \quad with \quad \omega = \sqrt{\frac{2g}{L}}$$

# Mass oscillations – U-tube

Where $\omega$ has the dimensions of a frequency ($s^{-1}$), the equation just obtained admits a solution of the type:

$$z = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Where $c_1$ and $c_2$ are constants to be determined with the boundary conditions. We know that at time $t=0$, $z=\Delta$, $v=0$. I then evaluate the solution at time $t=0$:

$$z\Big|_{t=0} = c_1 \sin(0) + c_2 \cos(0) = \Delta \qquad \rightarrow \quad c_2 = \Delta$$

$$V\Big|_{t=0} = \frac{dz}{dt}\Big|_{t=0} = c_1 \omega \cos(0) - c_2 \omega \sin(0) = 0 \qquad \rightarrow \qquad c_1 = 0$$

therefore

$$z = \Delta \cos(\omega t)$$

$$V = -\omega \Delta \sin(\omega t)$$

The motion is oscillatory with period $T=\pi/\omega$ and amplitude $\Delta$ for the $z$, while the amplitude is $\omega\,\Delta$ for the velocity. Since head losses have been neglected, the motion is not damped in time and never reaches the rest condition.

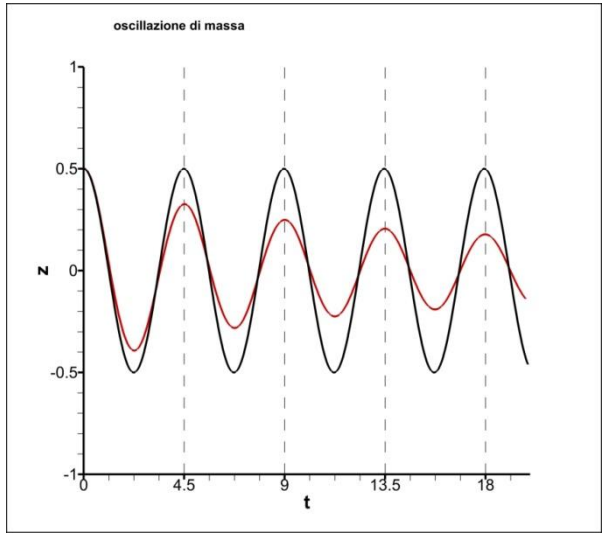
# Mass oscillations – U-tube

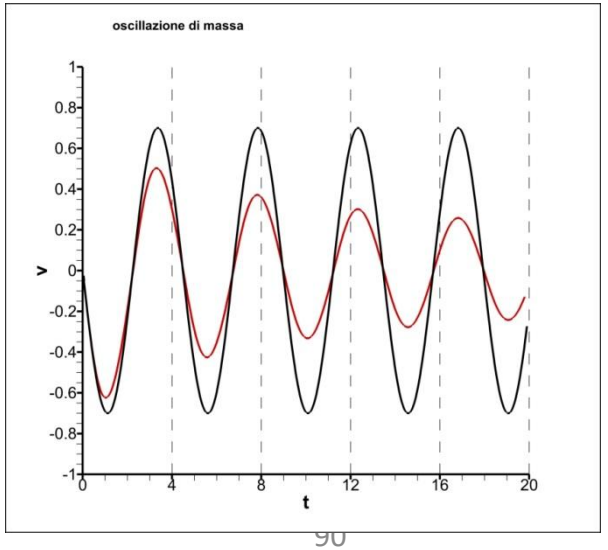
**Case of a real (viscous) liquid**

In this case head losses cannot be neglected. The equations can be solved numerically, using an explicit approach, i.e. linking the advancement of the variables only to the previous step; we have:

$$\frac{V^n - V^{n-1}}{\Delta t} = -\frac{g}{L}\left[2z + kL\frac{V^{n-1}|V^{n-1}|}{2g}\right] \qquad \frac{z^n - z^{n-1}}{\Delta t} = V^{n-1}$$

From which the values $V^n$ and $z^n$ can be obtained. We have expressed $J=V^2/2g$ with $|V|$ to account for the velocity taking different signs during the motion. In the graphs below, the black line is the solution for the perfect liquid, the red one the solution including head losses, where the damping effects can be observed when the continuous losses are non-null



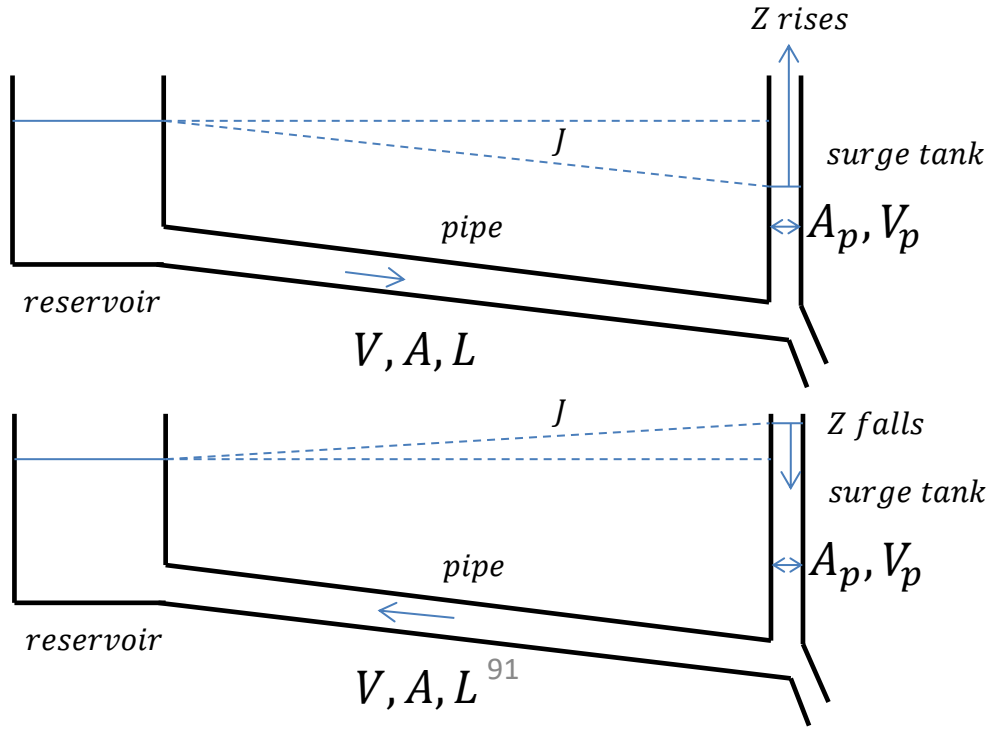
# Mass oscillations – Surge tank

## Reservoir – long pipe – surge tank

What we saw for the U-tube configuration makes it easier to understand the dynamics arising in the reservoir – long pipe – surge tank system. As already pointed out, the surge tank prevents the disturbances generated in the penstock from travelling up the long pipe. At the instant of valve closure, the pipe will still be in steady-flow conditions, with velocity $V_0$.
Instead of flowing in the penstock, the flow will start to flow into the surge tank, where the free surface will begin to rise, increasing the head at the end of the pipe and consequently reducing the discharge flowing there. When the level in the tank reaches that of the reservoir, a static equilibrium condition would be reached.

In reality, because of inertia, the level in the tank will keep rising up to a maximum height, reached when the flow in the pipe becomes null. Then, with the fluid at rest, the level in the tank will be higher than that of the reservoir, an unstable condition.

A flow in the opposite direction, from the tank to the reservoir, will therefore start, with the free surface in the tank gradually lowering.



# Mass oscillations – Surge tank

In this case too a minimum level will be reached, different from the rest condition, and hence not stable. This oscillatory motion will repeat in time, until the oscillations are damped by viscous effects.

To analyze the problem we can make some considerations that allow us to simplify the study. First of all, we can observe that the process identified here evolves with a characteristic velocity of the same order of magnitude as the steady-flow velocity, unlike, for example, water hammer, where the characteristic velocity of the problem, i.e. the celerity of the disturbances, is several orders of magnitude larger than the steady-flow velocity.

The oscillation process can therefore be seen as a slow phenomenon, in which fluid and pipe have time to adapt to the new conditions being established. We can therefore assume that the fluid is incompressible and the pipe rigid.
The system of equations describing the unsteady flow can then be reduced to the form:

$$\frac{\partial h}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t} + J = 0$$

$$\frac{\partial V}{\partial s} = 0$$

# Mass oscillations – Surge tank

We can also assume that:

1) The reservoir is very large, and its free-surface level remains constant in time.
2) Kinetic heads in the surge tank, given the low velocities involved, are negligible compared to the free-surface elevation
3) Local losses at the inlet from the reservoir and at the inlet to the tank*, as well as continuous losses inside the tank, are negligible compared to those characterizing the long pipe. Inertial effects, too, are neglected except in the long pipe.

The assumptions made allow us to state that, below the surge tank, i.e. at the end of the long pipe, the head is equal to the hydrostatic one, linked to the free-surface height inside the tank.

The goal of our analysis is therefore to find the maximum height reached by the free surface in the tank, since this tells us the heads to which the final part of the pipe is subjected. The minimum level is also of interest, because if it were lower than the pipe elevation, air would enter the system, which is meant to be pressurized, with obvious consequences.

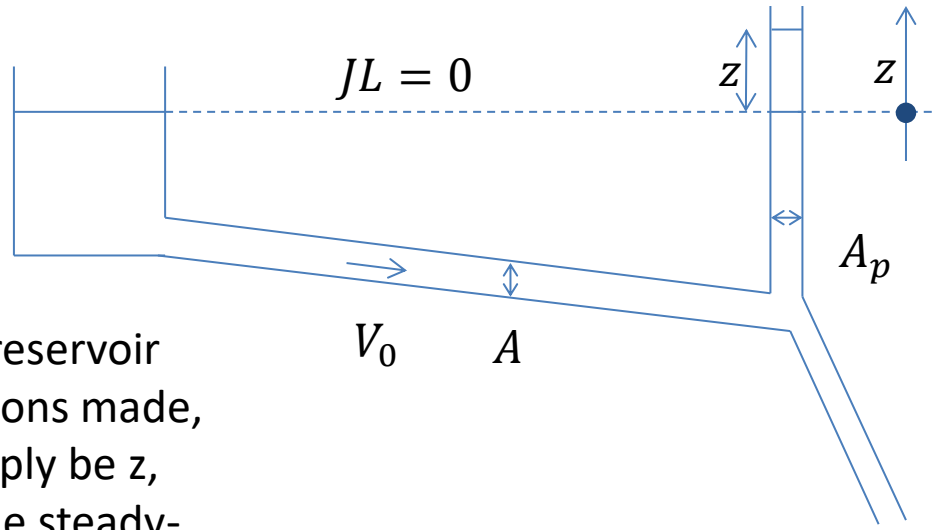
* This assumption will be reconsidered later.

# Mass oscillations – Surge tank

To obtain an analytical solution of the problem, I consider the case in which the continuous losses can also be neglected.

$$\frac{\partial h}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t} = 0$$

$$\frac{\partial V}{\partial s} = 0$$




Integrating the equation of motion in space, taking the reservoir level as reference (so that $h_s$=0), and given the assumptions made, the head difference between tank and reservoir will simply be z, the difference of elevation in the tank with respect to the steady-flow condition.

$$z = -\frac{L}{g}\frac{dV}{dt}$$

The inertial term has been derived with the same considerations made for the U-tube oscillations, besides the assumptions made in this specific case.

# Mass oscillations – Surge tank

The continuity equation, instead, can be rewritten as

$$AV = A_p V_p + Q_c$$

The discharge flowing in the long pipe must equal the sum of the discharge flowing in the surge shaft (of area $A_p$ and velocity $V_p$) and that flowing in the penstock.

The discharge $Q_c$ will depend on the valve closure time, which in the present case is much shorter than the period of the oscillatory motion; it is therefore fully acceptable to consider the closure as instantaneous $Q_c = 0$ (or a constant new value). Hence:

$$AV = A_p V_p = A_p \frac{dz}{dt}$$

Where the velocity in the tank has been related to the velocity with which the free-surface height varies. Combining the equations obtained we get:

$$\frac{dz}{dt} = \frac{A}{A_p} V \qquad \frac{d^2z}{dt^2} = \frac{A}{A_p}\frac{dV}{dt}$$

$$z + \frac{L}{g}\frac{dV}{dt} = 0 \qquad \frac{dV}{dt} = -\frac{g}{L} z$$

$$\frac{d^2z}{dt^2} + \omega^2 z = 0 \quad with \quad \omega = \sqrt{\frac{Ag}{A_p L}}$$

# Mass oscillations – Surge tank

Where $\omega$ has the dimensions of a frequency ($s^{-1}$), the equation just obtained admits a solution of the type:

$$z = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

Where $c_1$ and $c_2$ are constants to be determined with the boundary conditions. We know that at time $t=0$, $z=0$, $Vp=(A/Ap)V_0$. I then evaluate the solution at time $t=0$:

$$z\Big|_{t=0} = c_1 \sin(0) + c_2 \cos(0) = \Delta \qquad \rightarrow \quad c_2 = 0$$

$$V_p\Big|_{t=0} = \frac{dz}{dt}\Big|_{t=0} = c_1\, \omega \cos(0) = \frac{A}{A_p} V_0 \qquad \rightarrow \qquad c_1 = \frac{AV_0}{A_p \omega}$$

therefore

$$z = \frac{AV_0}{A_p \omega} \sin(\omega t) = \; V_0 \frac{A}{A_p} \sqrt{\frac{A_p L}{Ag}} \sin(\omega t) = V_0 \sqrt{\frac{AL}{A_p g}} \sin(\omega t)$$

The level in the tank has a sinusoidal trend with maximum and minimum amplitude:

$$z_{max,min} = \pm V_0 \sqrt{\frac{AL}{A_p g}}$$

# Mass oscillations – Throttled surge tank

In the analysis we have neglected both continuous and local head losses. In practice, concentrated losses are deliberately induced at the base of the tank by introducing a throttle (orifice). This throttle induces concentrated losses that can be expressed as:

$$K_p = \beta Q_p^2$$

Where $Q_p$ is the discharge flowing through the tank. In steady-flow conditions there is no flow through the tank, so there will be no head loss induced by the throttle. At valve closure, the discharge that previously flowed in the pipe will start to flow into the surge shaft and will be:

$$K_0 = \beta Q_0^2 = \beta A^2 V_0^2 \qquad for\ t = 0$$

The head loss will then have its maximum value at time t=0; then, as the level rises in the shaft, reducing the flow, the velocity will decrease and consequently so will the head loss.
The basic idea is that upstream of the throttle, in the pipe, a head equal to the free-surface height in the shaft plus the local head loss is created. The increase of pressure in the tunnel causes a slowing down of the velocity in the tunnel itself
→ reduction of the surge in the shaft.

# Mass oscillations – Throttled surge tank

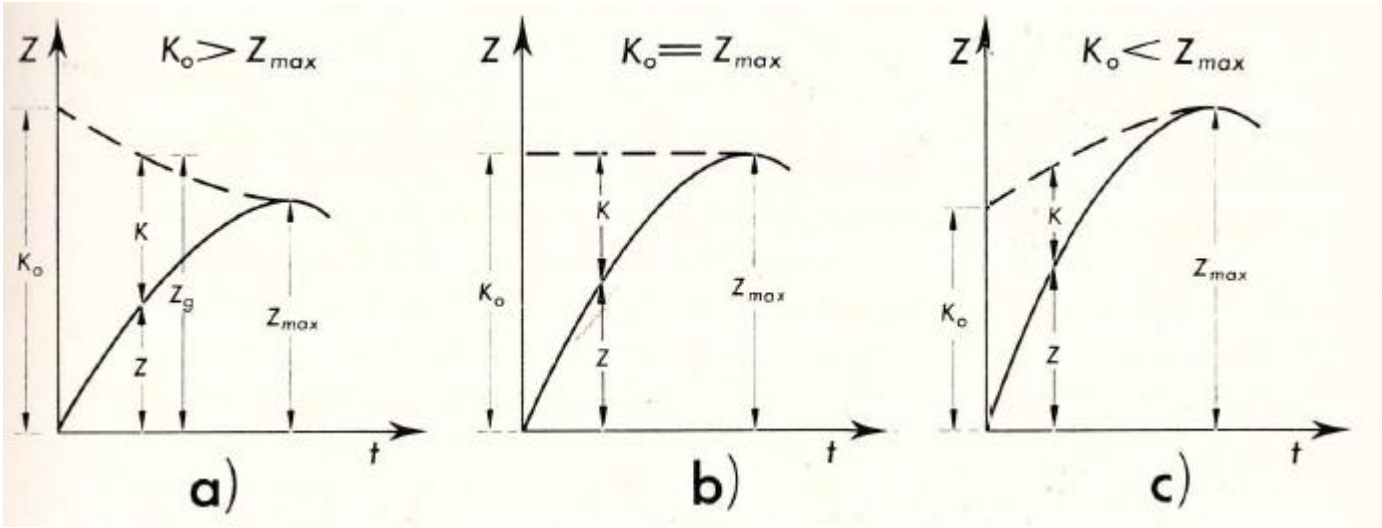
For t=0 we still have Z=0, while in the tunnel, apart from the piezometric head of the reservoir, we have

$$Z_g = K_0$$

When the shaft has reached its maximum level, the incoming discharge is null and

$$Z_g = Z_{max}$$

Depending on how strong the chosen throttle is, there will be a consequent effect on the maximum surge and on the pressure in the tunnel; see the figures.




A very strong throttle, figure a), would induce at time t=0 a pressure in the tunnel higher than the one related to the surge z, an undesirable condition. A weak throttle, figure c, would instead not achieve the purpose of the throttle.

# Mass oscillations – Throttled surge tank

The ideal configuration is the one with

$$K_0 = Z_{max} = K + Z$$

The system, considering the local head loss, will be:

$$\frac{L}{g}\frac{dV}{dt} + z + K = 0$$

Which, under optimal throttle conditions, becomes

$$\frac{L}{g}\frac{dV}{dt} + z_{max} = 0$$

Integrating in time between t and t=0

$$\frac{L}{g}\left[V - V_0\right] + Z_{max}t = 0$$

With the condition that at t= 0 V=$V_0$

# Mass oscillations – Throttled surge tank

Recalling the continuity equation

$$\frac{A_p}{A}\frac{dz}{dt} = V$$

I substitute V in the equation just derived and obtain

$$\frac{L}{g}\frac{A_p}{A}\frac{dZ}{dt} - \frac{L}{g}V_0 + Z_{max}t = 0$$

Integrating again in time

$$\frac{L}{g}\frac{A_p}{A}[Z - Z_0] - \frac{L}{g}V_0 t + \frac{1}{2}Z_{max}t^2 = 0 \qquad \text{t= 0 } Z_0\text{=0}$$

$$\frac{L}{g}\frac{A_p}{A}Z - \frac{L}{g}V_0 t + \frac{1}{2}Z_{max}t^2 = 0$$

We obtain

$$Z = \frac{A}{A_p}\left[V_0 - \frac{g}{2L}Z_{max}t\right]t$$

# Mass oscillations – Throttled surge tank

The maximum occurs when dz/dt=0, hence

$$\frac{A}{A_p}V_0 \; - \frac{A}{A_p}\frac{g}{2L}\; Z_{max}2t = 0$$

The expression vanishes when the time t takes a value equal to

$$t^* = \; V_0 \frac{L}{gZ_{max}}$$

Therefore at time t=t* we know that my function has a maximum condition and we will have $Z=Z_{max}$ substituting

$$Z_{max} = \frac{A}{A_p}\left[V_0 \; - \frac{g}{2L}\; Z_{max}t^*\right]\; t^*$$

$$Z_{max} = \frac{A}{A_p}\left[V_0 \; - \frac{g}{2L}\; Z_{max}V_0\frac{L}{gZ_{max}}\right]V_0\frac{L}{gZ_{max}}$$

# Mass oscillations – Throttled surge tank

$$Z_{max} = \frac{A}{A_p}\left[V_0 - \frac{g}{2L} Z_{max} V_0 \frac{L}{g Z_{max}}\right] V_0 \frac{L}{g Z_{max}}$$

$$Z_{max} = \frac{A}{A_p}\left[V_0 - \frac{V_0}{2}\right] V_0 \frac{L}{g Z_{max}}$$

$$Z_{max} = \frac{A}{A_p} \frac{V_0^2}{2} \frac{L}{g Z_{max}}$$

$$Z_{max}^2 = \frac{AL}{A_p g} \frac{V_0^2}{2}$$

We obtain

$$Z_{max} = \frac{1}{\sqrt{2}} V_0 \sqrt{\frac{AL}{A_p g}}$$

As without the throttle, but with a multiplicative factor of about 0.7

# Mass oscillations – Worked example

## Surge tank without throttle

Tunnel: L = 2000 m, D = 2 m (A = 3.14 $m^2$), $V_0$ = 3 m/s

Tank: D = 10 m ($A_p$ = 78.5 $m^2$)

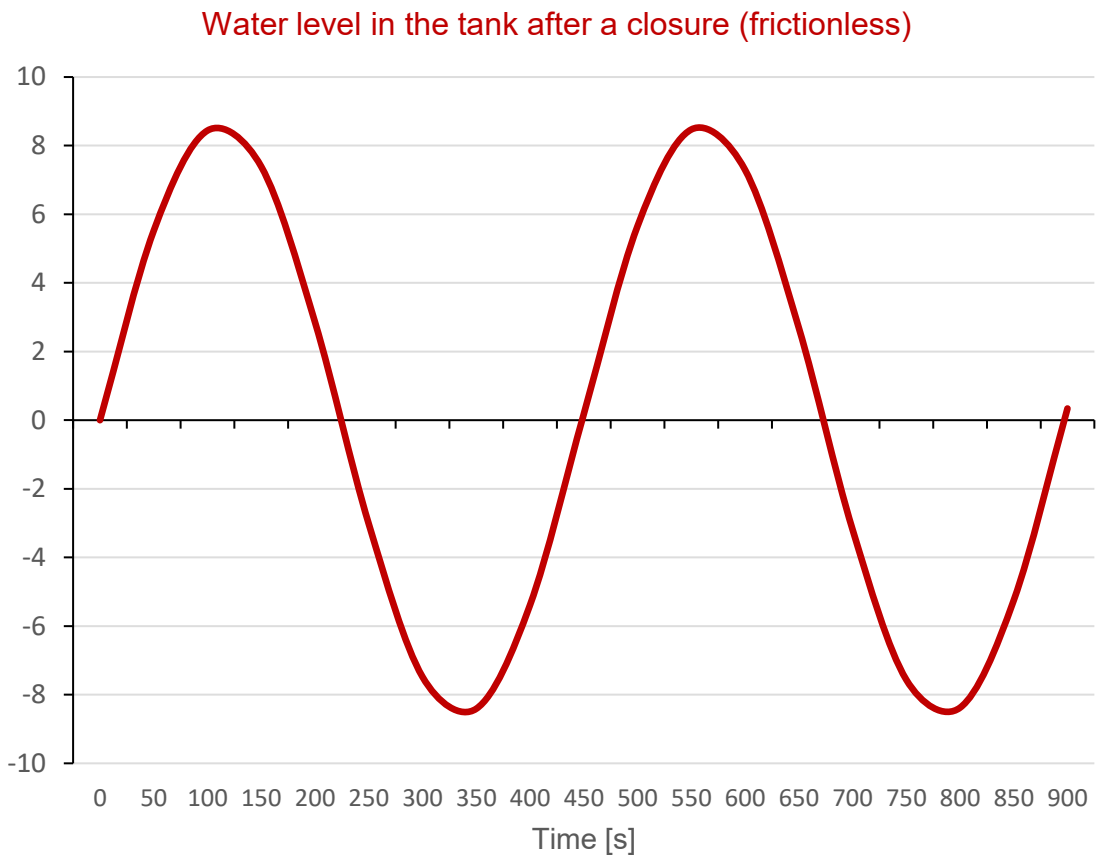
$$z_max = V_0 \sqrt{(A \cdot L / (g \cdot A_p))} = 3 \cdot 2.86 \approx 8.6\ m$$

$$\omega = \sqrt{(g \cdot A / (L \cdot A_p))} \approx 0.014\ s^{-1}$$

**T = 2π/ω ≈ 450 s (7.5 min); first maximum after T/4 ≈ 110 s**

With an optimal throttle the maximum surge drops by a factor ≈ 0.7 (z_max ≈ 6.1 m).

In practice friction damps the oscillation; the tank must be wide enough for it to be stable (Thoma criterion).



# Mass oscillations – Numerical solution

If we instead want to solve the system by numerical integration, also considering the continuous head losses, we will solve the system:

$$\frac{dV}{dt} = -\frac{g}{L}[z + JL]$$

$$\frac{dz}{dt} = \frac{A}{A_p}V - \frac{Q_c}{A_p}$$

Here I take into account the discharge flowing in the penstock, since it represents a boundary condition of the system, in case a closure or start-up operation is considered. Using an explicit time approach

$$\frac{V^n - V^{n-1}}{\Delta t} = -\frac{g}{L}\left[z + \frac{f}{D}L\frac{V^{n-1}|V^{n-1}|}{2g}\right]$$

$$\frac{z^n - z^{n-1}}{\Delta t} = \frac{A}{A_p}V^{n-1} - \frac{Q_c^{n-1}}{A_p}$$

From which the expressions for V and z at time n are obtained, as a function of the previous step only.

# Mass oscillations – Numerical solution

Let us first consider a start-up maneuver. The free surfaces in the reservoir and in the shaft are at the same level, and the fluid is at rest. We have:

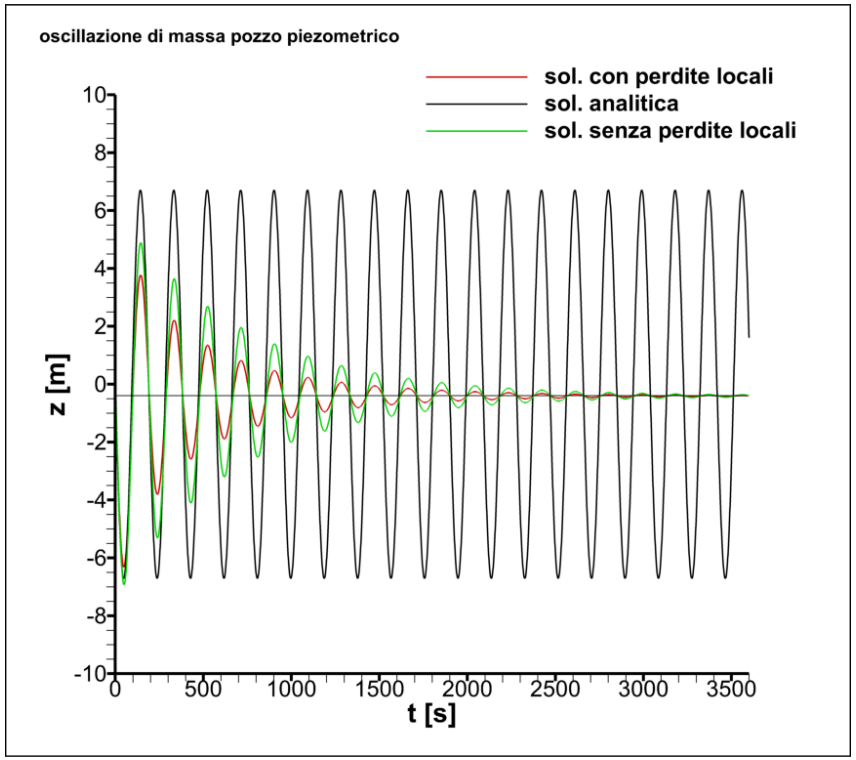
$$t = 0 \;\rightarrow\; z = 0\,;\; V = 0\,;\; Q_c^0 = 0$$

While for t>0 a condition for $Q_c$=$AV_0$.
The graph below shows a comparison, for the start-up maneuver, between the proposed numerical solution and the analytical one derived; in addition, in one case a concentrated loss at the tank has been considered besides the continuous losses.

We note that the analytical solution oscillates without damping, since head losses were not considered.

The numerical one, instead, shows damping in time, larger when local losses are also considered.

As the flow reaches the steady (permanent) regime, the level z will be negative, because in steady flow the tank will sit lower than the reservoir by an amount JL.



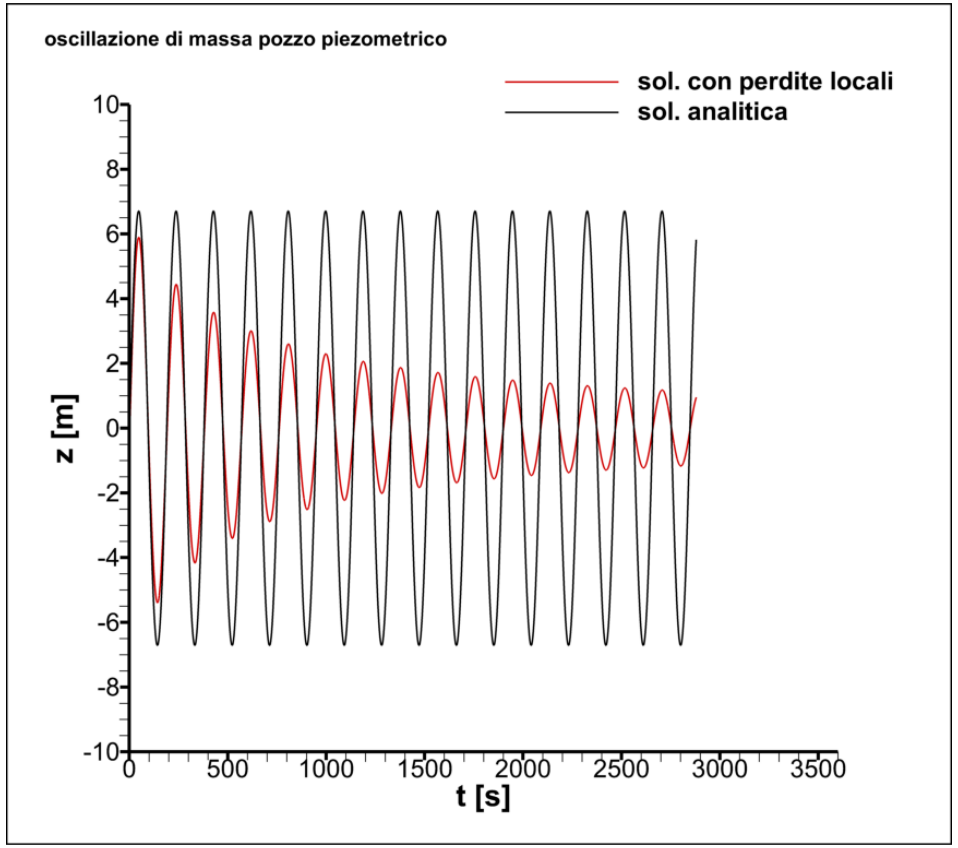
# Mass oscillations – Numerical solution

Starting from the result obtained, closing the valve we have as initial conditions

$$t = 0 \rightarrow \quad z = z_0 \; ; \; V = V_0 \; ; Q_c^0 = AV_0$$

While for t>0 a condition for $Q_c$=0.

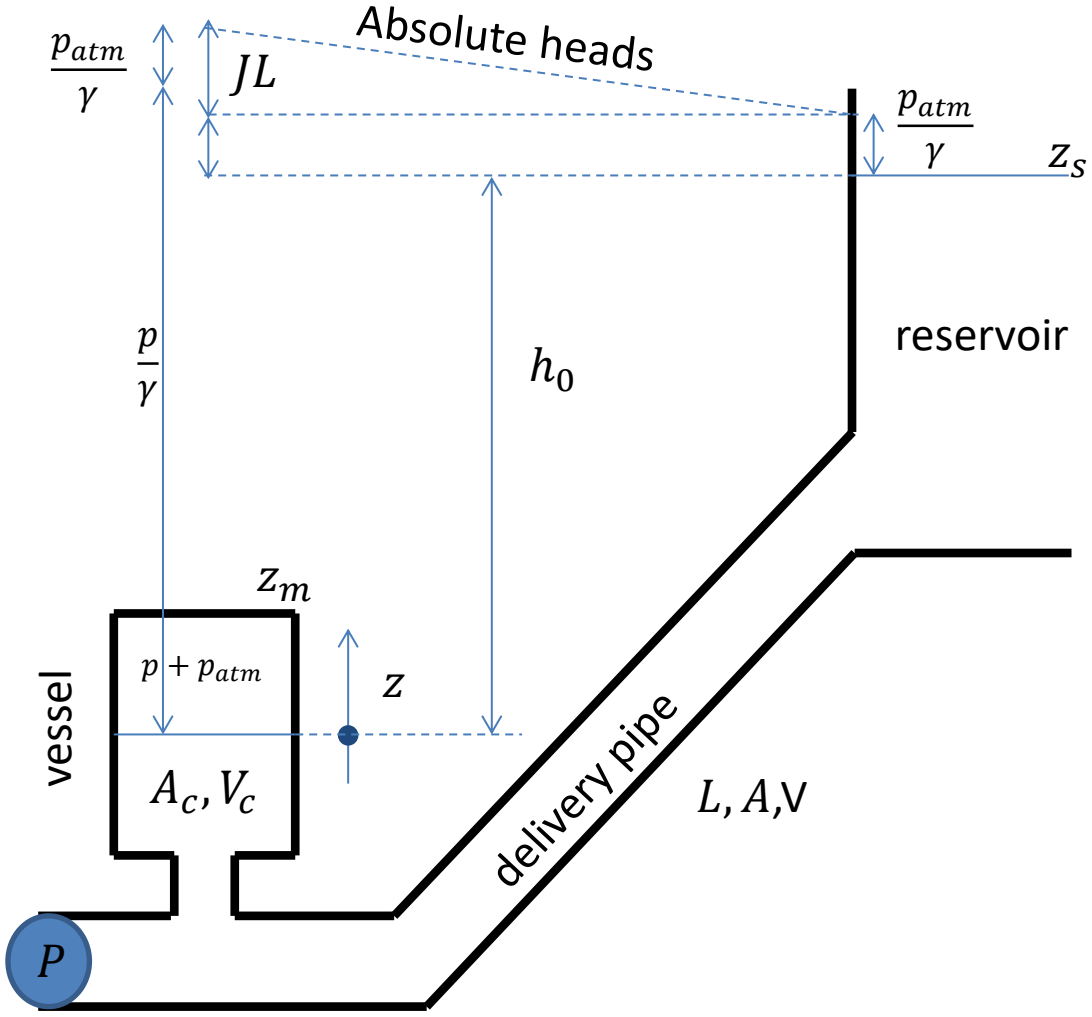
The graph below shows a comparison, for the closure maneuver, between the proposed numerical solution and the analytical one derived.
In this case we note, besides what was already said for the start-up maneuver, that the oscillation of z converges to z=0, i.e. the expected condition of equal level in the reservoir and in the tank.



# Mass oscillations – Air vessel

## Mass oscillations in an air vessel

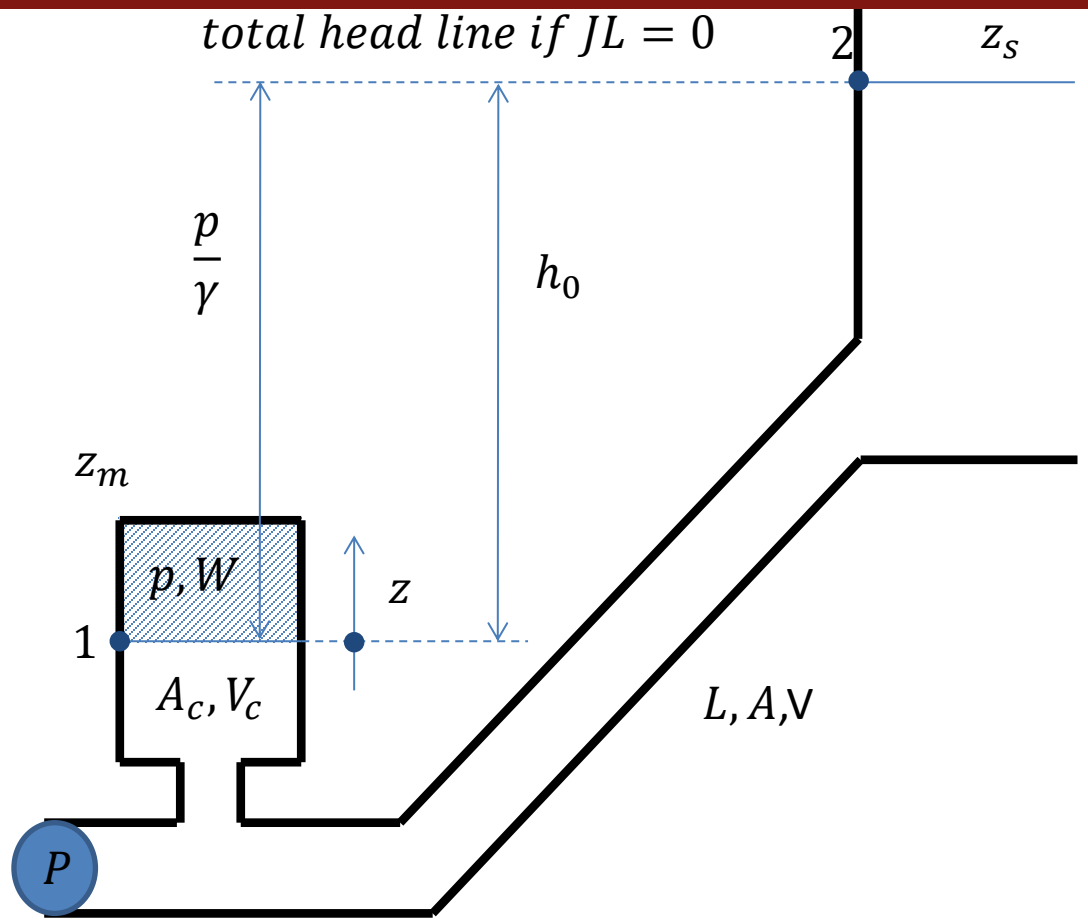
Let us now consider a pumping system, with a pump providing the head needed to carry the fluid to a receiving reservoir. In this case too a protection system similar to the surge tank is needed. Since a tank as tall as the reservoir is not feasible for obvious economic reasons, a pressurized vessel is used instead, i.e. a closed volume containing air, which absorbs, through its changes of volume and pressure, the dynamic variations occurring in the system.




Discharge variations due to pump maneuvers produce an oscillatory motion, analogous to the previous surge tank case, involving the air vessel, the delivery pipe and the receiving reservoir. It is of interest to evaluate the maximum pressure and the minimum volume inside the vessel.

# Mass oscillations – Air vessel

The reservoir is assumed to keep a constant level, and any residual discharge from the pump is neglected.
In case of sudden pump stoppage, the flow in the pipe continues, drawing from the air vessel.
→ ithe gas expands
→ the pressure decreases
→ sthe discharge decreases.
The discharge becomes zero when the pressure reaches a minimum and the gas a maximum volume. An oscillatory motion then starts between this condition and the one of maximum pressure and minimum volume in the air vessel.




The phenomenon is studied with the same equations seen previously, considering the fluid incompressible and the pipe rigid.

$$\frac{\partial h}{\partial s} + \frac{1}{g}\frac{\partial V}{\partial t} + J = 0 \qquad \frac{\partial Q}{\partial s} = 0$$

# Mass oscillations – Air vessel

Integrating between the free-surface level in the vessel and in the reservoir, assuming negligible the local losses, as well as the losses and inertial phenomena inside the air vessel, the equation of motion reduces to

$$h_2 - h_1 = -\frac{L}{g}\frac{dV}{dt} - JL$$

If I take as reference plane the one through the free-surface level in the vessel at equilibrium conditions, we can express the head at 2 in absolute terms as:

$$h_2 = h_0 + \frac{p_{atm}}{\gamma}$$

While:

$$h_1 = z + \frac{p}{\gamma} + \frac{p_{atm}}{\gamma}$$

Where we have separated the pressure inside the vessel from a component equal to the atmospheric pressure. We will then have:

$$h_2 - h_1 = h_0 - \left(z + \frac{p}{\gamma}\right) = -\frac{L}{g}\frac{dV}{dt} - JL$$

# Mass oscillations – Air vessel

The equation of motion is a function of three variables, z, V and p. Two additional equations are therefore needed to solve the problem. As in the cases analyzed before, we rely on the continuity equation:

$$Q_p = Q_c + Q_s = A_c V_c + AV$$

The discharge $Q_p$ coming from the pump will flow partly into the air vessel $Q_c$, and partly towards the reservoir, $Q_s$. The latter two are expressed as functions of the cross-section areas of the vessel, $A_c$, and of the pipe, A, characterized respectively by the velocities $V_c$ and V. From the continuity equation we can then express the velocity V, considering that $V_c$ can be expressed as a function of the variation in time of the free surface in the vessel, as:

$$V = \frac{Q_p}{A} - \frac{A_c}{A} V_c = \frac{Q_p}{A} - \frac{A_c}{A} \frac{dz}{dt}$$

Finally, we consider that for the gas the pressure p and the volume W satisfy the relation

$$pW^k = const$$

Where k depends on the type of transformation: isothermal (k=1) or adiabatic (k=1.41).

# Mass oscillations – Air vessel

Defined $p_0$ , $z_0$ and $W_0$ the equilibrium condition in the vessel in the absence of flow:

$$(p_0 + p_{atm})W_0^k = (p + p_{atm})W^k$$

Then considering $z_m$ the elevation of the crown of the air vessel:

$$(p_0 + p_{atm})(z_m - z_0)^k A_c^k = (p + p_{atm})(z_m - z)^k A_c^k$$

From which

$$p = \frac{(z_m - z_0)^k}{(z_m - z)^k}(p_0 + p_{atm}) - p_{atm}$$

With the drawback of having the pressure as a function of z at the denominator. However, considering the Taylor series expansion (truncated at the first term) for p around z=0:

$$p \cong p\Big|_{z=0} + \frac{\partial p}{\partial z}\Big|_{z=0} z$$

$$p \cong p_0 + \frac{k(p_0 + p_{atm})}{z_m} z$$

# Mass oscillations – Air vessel

In this case too we can find an analytical solution by neglecting the continuous head losses, JL=0. Combining the continuity equation and the equation of motion, similarly to the previous cases, with $Q_p$=0 we have:

$$h_0 - \left(z + \frac{p}{\gamma}\right) = \frac{A_c}{A}\frac{L}{g}\frac{d^2z}{dt^2}$$

Introducing the expression found for p

$$h_0 - \left(z + \frac{p_0}{\gamma} + \frac{k(p_0 + p_{atm})}{\gamma z_m}z\right) = \frac{A_c}{A}\frac{L}{g}\frac{d^2z}{dt^2}$$

Also noting that at equilibrium $p_0/\Upsilon$ is equal to $h_0$:

$$\frac{d^2z}{dt^2} + \left(1 + \frac{k(p_0 + p_{atm})}{\gamma z_m}\right)\frac{A}{A_c}\frac{g}{L}z = 0$$

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$

# Mass oscillations – Air vessel

Where ω is a frequency

$$\omega = \sqrt{\left(1 + \frac{k(p_0 + p_{atm})}{\gamma z_m}\right)\frac{g}{L}\frac{A}{A_c}}$$

And the equation admits a solution of the type

$$z = c_1 \sin(\omega t) + c_2 \cos(\omega t)$$

At time t=0 we have $z=z_0=0$, since it is at the level of the chosen reference plane

$$z = c_1 \sin(0) + c_2 \cos(0) = 0 \quad hence \quad c_2 = 0$$

Again at time t=0 we have $V=V_0$ , i.e. the velocity in steady-flow conditions:

$$V = V_0 = -\frac{A_c}{A}\frac{dz}{dt}$$

$$V_0 = -\frac{A_c}{A}\, c_1\, \omega \cos(\omega t) \quad hence \quad c_1 = -\frac{V_0 A}{\omega A_c}$$

# Mass oscillations – Air vessel

we obtain

$$z = -\frac{V_0 A}{\omega A_c}\sin(\omega t)$$

The complete form of the equations determined can be solved numerically, by writing the system as

$$\frac{dV}{dt} = \frac{g}{L}\left[z + \frac{p}{\gamma} - h_0 - JL\right]$$

$$\frac{dz}{dt} = \frac{Q_p - VA}{A_c}$$

$$p = (p_0 + p_{atm})\,\frac{(z_m - z_0)^k}{(z_m - z)^k} - p_{atm}$$

And solving it with an explicit approach

# Mass oscillations – Air vessel

we obtain

$$\frac{V^n - V^{n-1}}{\Delta t} = \frac{g}{L}\left[ z^{n-1} + \frac{p^{n-1}}{\gamma} - h_0 - \frac{f}{D}\frac{V^{n-1}|V^{n-1}|}{2g} L \right]$$

$$\frac{z^n - z^{n-1}}{\Delta t} = \frac{Q_p^{n-1} - V^{n-1} A}{A_c}$$

$$p^n = (p_0 + p_{atm}) \frac{(z_m - z_0)^k}{(z_m - z^n)^k} - p_{atm}$$

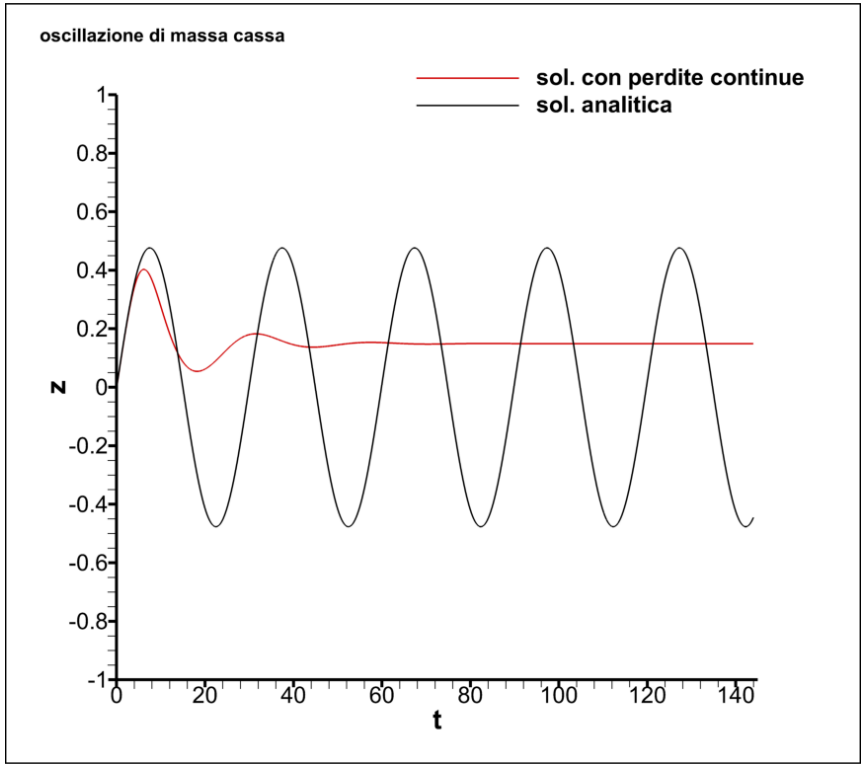
Where the variables V, z and p at time n can be obtained as a function of the previous step n-1 only.

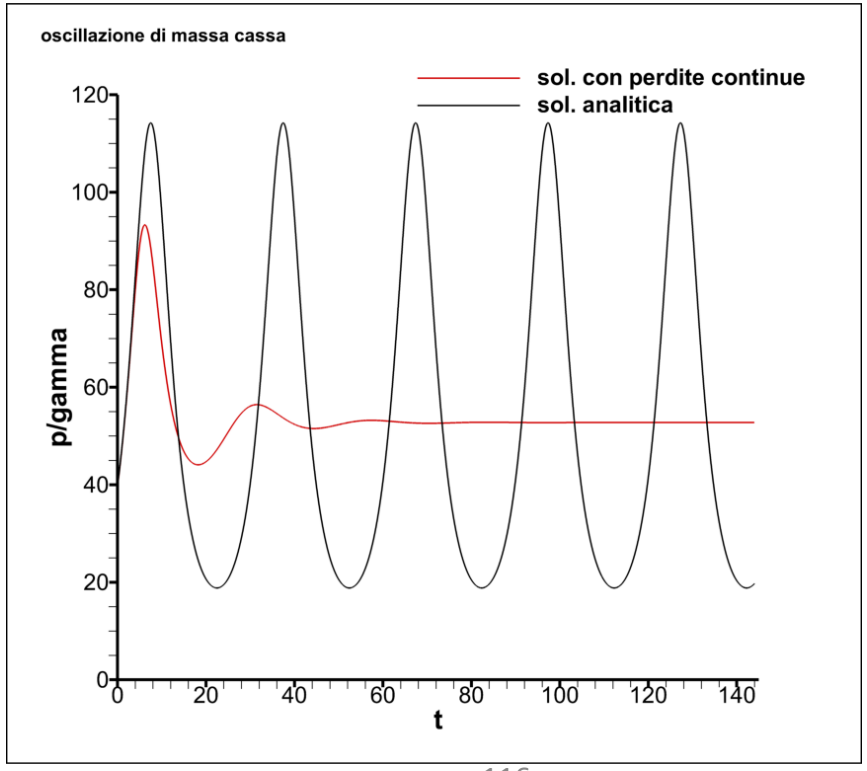
# Mass oscillations – Air vessel

Let us first consider a start-up maneuver. The free surface in the vessel is at level z=z0=0 and the pressure is p0/ϒ=h0, while the fluid is at rest. We have:

$$t = 0 \;\rightarrow\; z = 0\,;\; V = 0\,;\; Q_p^0 = 0$$

While for t>0 a condition for $Q_p=AV_0$. The graphs below show z and p inside the vessel. Note how, when z increases, the volume decreases and therefore p increases



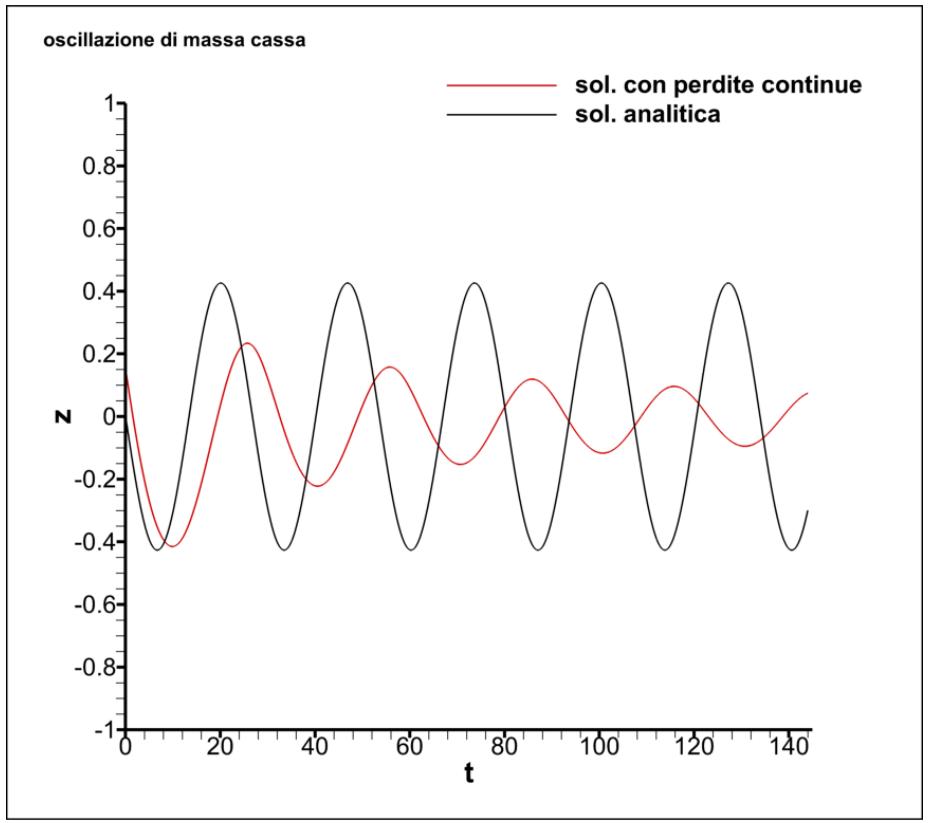



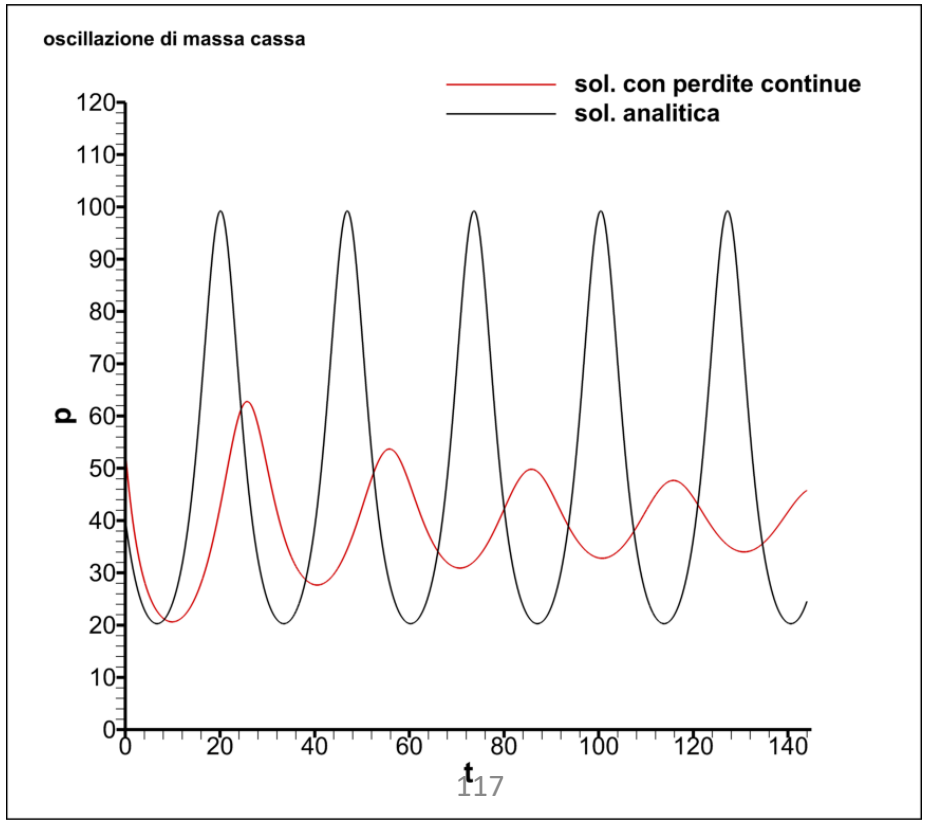
# Mass oscillations – Air vessel

For a closure maneuver. The free surface in the vessel is at the steady-flow level (see the previous graph), like the pressure, while the fluid has velocity $V_0$. We have:

$$t = 0 \rightarrow \quad z = z_{steady} \; ; \; V = V_0 \; ; Q_p^0 = AV_0$$

While for t>0 a condition for $Q_p$=0. The graphs below show z and p inside the vessel. Note how, when z increases, the volume decreases and therefore p increases, and z tends towards the 0 level of the reference plane.



# Summary – Key relations

### Joukowsky

$$\Delta p = \rho \cdot c \cdot V_0$$

Maximum overpressure for rapid closure ($T_c \leq 2L/c$)

### Celerity

$$c = c_0 / \sqrt{(1 + \varepsilon D/(Ee))}$$

$c_0 \approx 1440$ m/s in water; pipe elasticity lowers c

### Period

$$T = 4L / c$$

Time for one complete cycle of the pressure wave

### Michaud

$$\Delta p \approx 2\rho \cdot L \cdot V_0 / T_c$$

Slow closure ($T_c > 2L/c$), linear maneuver

### Surge tank

$$z_max = V_0 \sqrt{(A \cdot L / (g \cdot A_p))}$$

Frictionless maximum level oscillation

### Surge tank

$$\omega = \sqrt{(g \cdot A / (L \cdot A_p))}$$

Angular frequency; $T = 2\pi / \omega$