Lecture 5

Subgame Perfect Nash Equilibrium and Backward Induction: economic applications

Definition:
Subgame perfect Nash equilibrium

A Nash equilibrium is subgame perfect (Nash equilibrium) if the players' strategies constitute a Nash equilibrium in every subgame.

(Selten 1965)

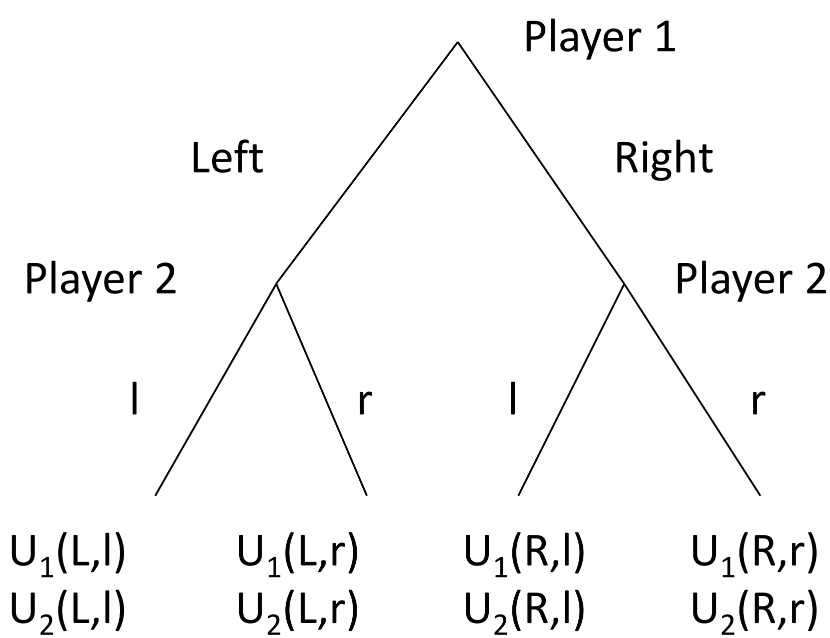
Note that every finite sequential game of complete information has at least one subgame perfect Nash equilibrium

We can find all subgame perfect NE using backward induction

Backward Induction in dynamic games of perfect information

- **Procedure:**
  - We start at the end of the trees
  - first find the optimal actions of the last player to move
  - then taking these actions as given, find the optimal actions of the second last player to move
  - continue working backwards
- If in each decision node there is only one optimal action, this procedure leads to a **Unique Subgame Perfect Nash equilibrium**

- Player 1 choose action a_1 from the set A_1={Left, Right}
- Player 2 observes a_1 and choose an action a_2 from the set A_2={l, r}
- Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$



- When Player 2 gets the move, she observes player’s 1 action a_1 and faces the following problem

$$\underset{a_2 \in \{A_2\}}{Max}\ u_2(a_1, a_2)$$

- Solving this problem, for each possible $a_1 \in A_1$, we get the best response of Player 2 to Player 1’s action.
- We denote it by $R_2(a_1)$, the reaction function of Player 2.
- Player 1 can anticipate Player 2’s reaction, then Player 1’s problem is:

$$\underset{a_1 \in \{A_1\}}{Max}\ u_1(a_1, R_2(a_1))$$

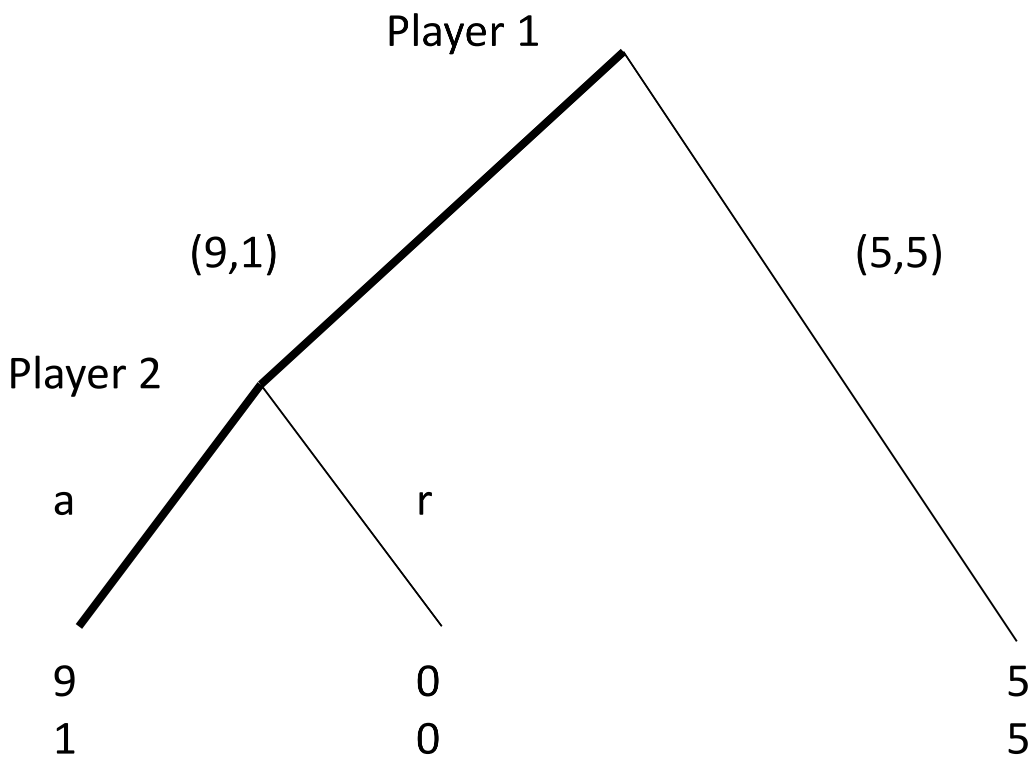
- The ***backwards induction outcome*** is denoted by

$$(a_1^*, R_2(a_1^*))$$

- It is different from the description of the equilibrium
- To describe the Nash equilibrium we need to describe the equilibrium strategies:
  - Action of Player 1 (Left or Right)
  - Action of Player 2 after Left, action of player 2 after Right

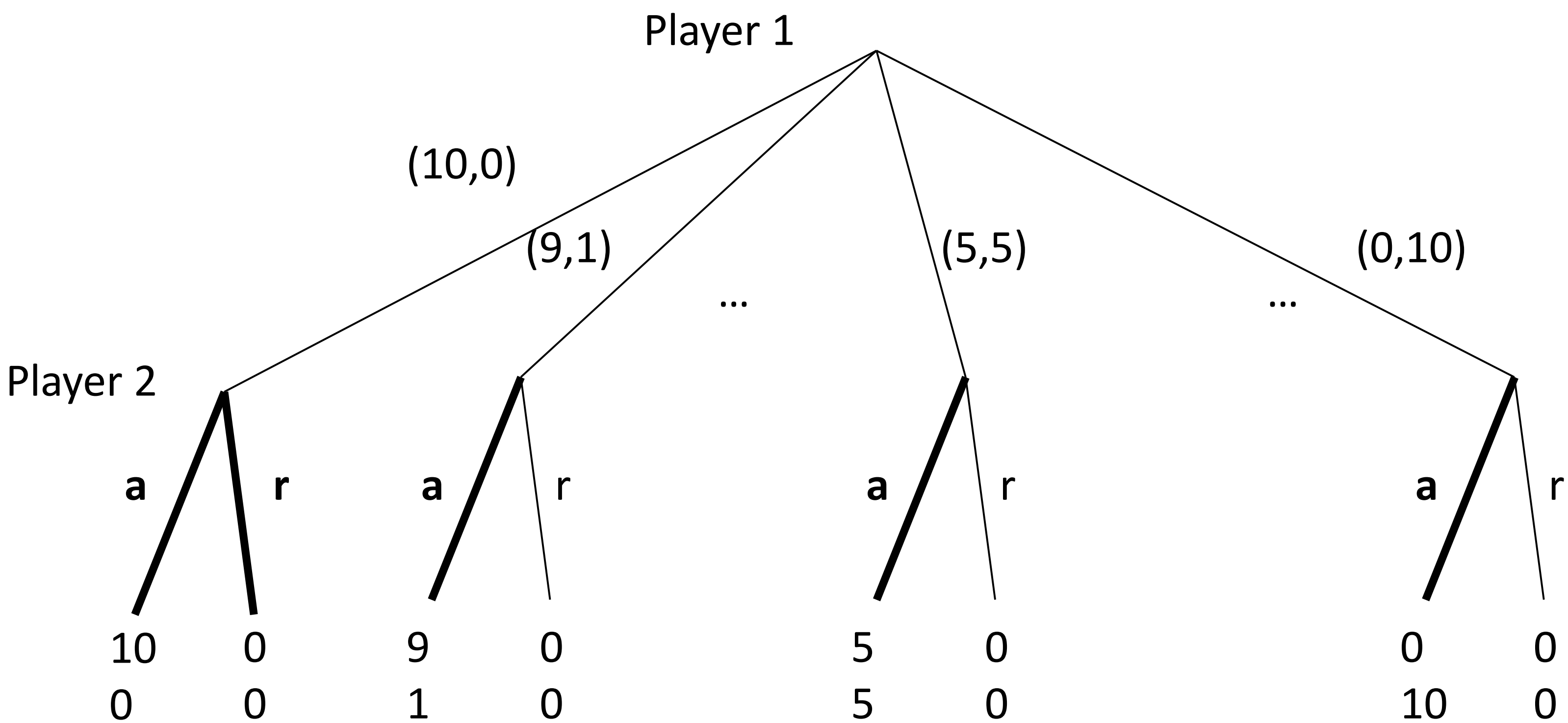
Example: Mini Ultimatum Game

- Consider Player 2, the optimal action is accept
- Taking "accept" as given, we see that (9,1) is the optimal action for player 1



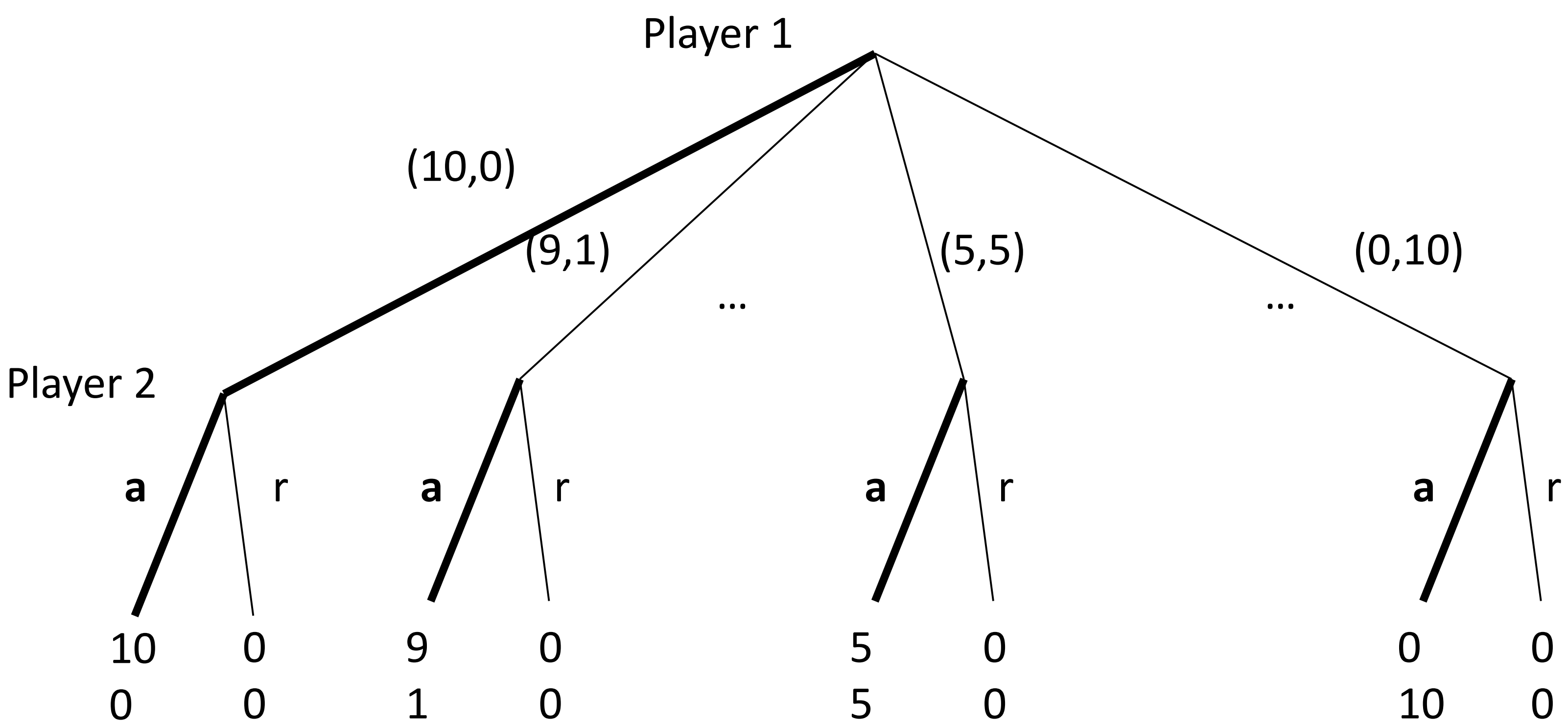
The Ultimatum Game

- Proposer (Player 1) suggest (integer) split of a fixed pie, say £10.
- Responder (Player 2) accepts the proposal or rejects (both receive 0)
- There is no unique best response following (10,0), so we have two SPNE

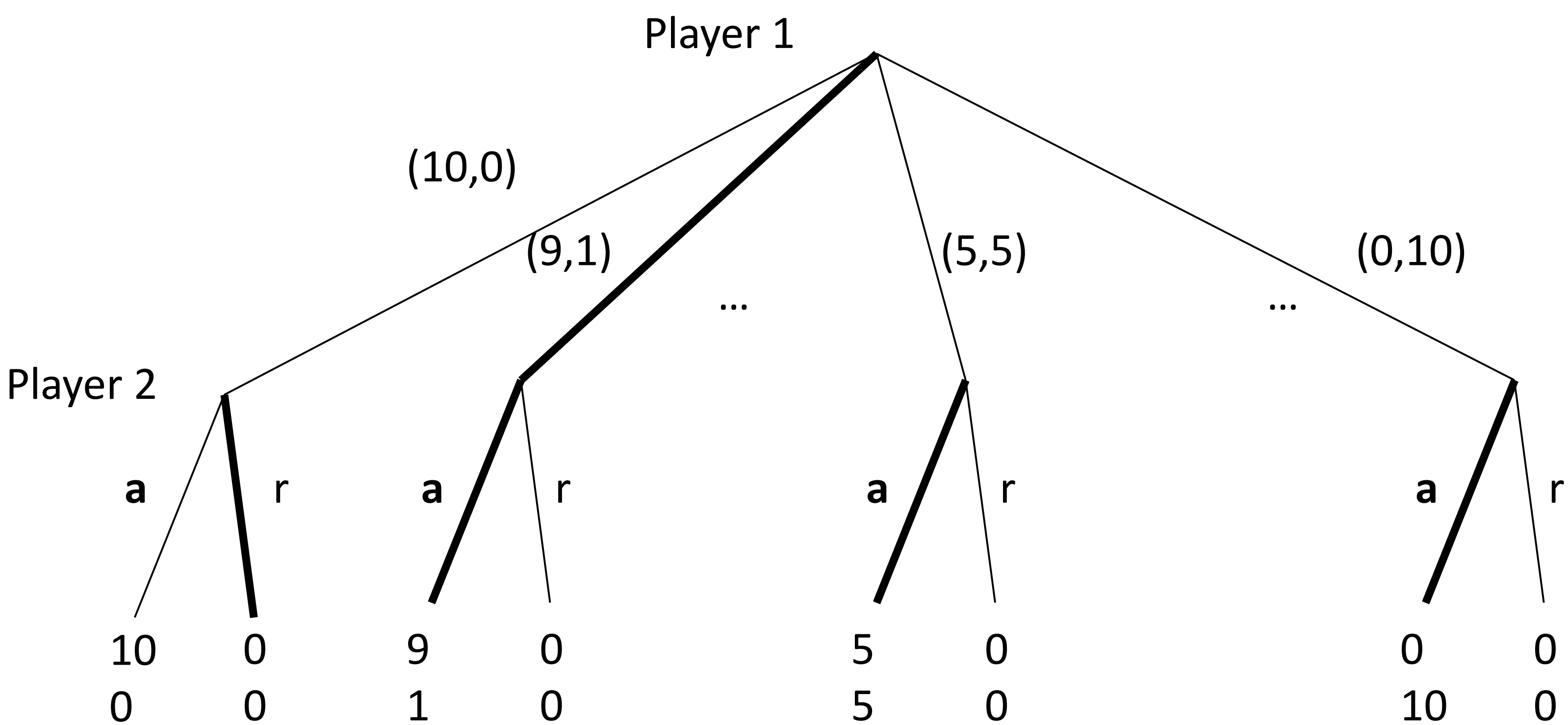


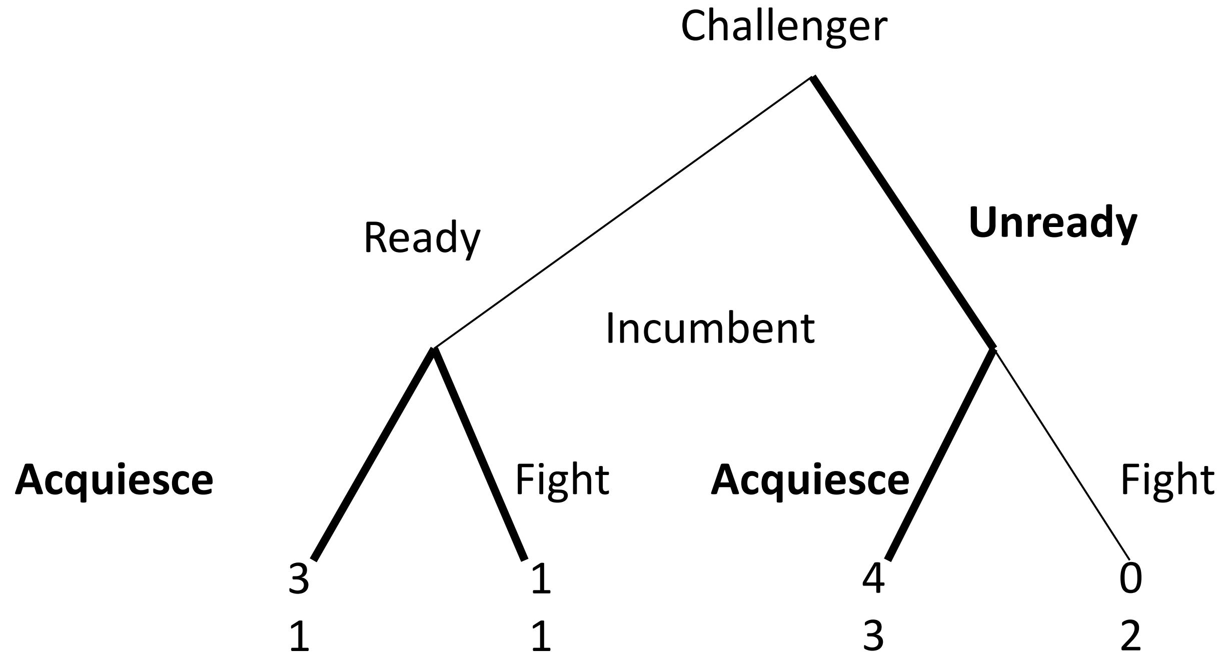
- First SPNE:
  - Player 1 proposes (10, 0)
  - Player 2 accepts in all of his decision nodes

    (a, a, a, a, a, a, a, a, a, a, a)
- Backward Induction Outcome: Player 1 proposes (10, 0), Player 2 accepts



- Second SPNE:
  - Player 1 proposes (9, 1)
  - Player 2 rejects after (10, 0) and accepts in all other decision nodes (r, a, a, a, a, a, a, a, a, a, a)
- Backward Induction Outcome: Player 1 proposes (9, 1), Player 2 accepts





		Incumbent			
		(A, A)	(A, F)	(F, A)	(F, F)
Challenger	Ready	3, 1	**3, 1**	1, 1	**1, 1**
	Unready	**4, 3**	0, 2	**4.3**	0, 2

Four Nash equilibria:

1. Ready, (A, F)
2. Ready, (F, F)
3. Unready, (A, A)
4. Unready, (F, A)

{Unready, (A, A)} and {Unready, (F, A)} are subgame perfect

Only one backward induction outcome: (Unready, Acquiesce)

Stackelberg model of Duopoly

- 2 firms, 1 and 2 (Leader and Follower)
- Firms choose quantities (as in Cournot) q_1 and q_2.
- Leader (firm 1) moves first and chooses a quantity q_1
- Followers (Firm 2) moves second, observes q_1 and then chooses a quantity q_2
- Each firm faces constant marginal cost c and no fixed cost.

The payoff of firm 1 is:

$$\pi_1(q_1, q_2) = q_1\,(P(Q) - c)$$

The payoff of firm 2 is:

$$\pi_2(q_1, q_2) = q_2\,(P(Q) - c)$$

where

$P(Q) = a - Q$ is the inverse demand function and

$Q = q_1 + q_2$

Solution by backwards-induction

- We can solve this problem by backwards-induction:

  1. We solve the problem that Firm 2 faces for a generic observed quantity q_1

  2. The solution gives us the optimal quantity q_2* as function of the observed quantity q_1:

  $q_2^* = R_2(q_1)$ where R_2 () is the reaction function.

  3. We solve the problem of Firm 1 assuming that Firm 1 knows R_2 (q_1), i.e.

  For every quantity (q_1) Firm 1 decides to produces, Firm 1 correctly anticipate the quantity (q_2) Firm 2 will decide to produce.

Firm 2's problem

$$\pi_2(q_1, q_2) = q_2\,(P(Q) - c) = q_2\,(a - q_1 - q_2 - c)$$

$$\text{Max}_{\{q2\}}\; q_2\,(a - q_1 - q_2 - c)$$

Using the FOCs

$$R_2\,(q_1) = (a - q_1 - c\,)\,/\,2$$

Note, this is the same reaction function to that we found in Cournot Oligopoly

Firm 1's problem

$$\pi_2(q_1, q_2) = q_1\,(P(Q) - c) = q_1\,(a - q_1 - q_2 - c)$$

$$\text{Max}_{\{q1\}}\; q_1\,(a - q_1 - q_2 - c)$$

Given that Firm 1 knows $R_2(q_1)$, its problem is

$$\text{Max}_{\{q1\}}\; q_1\,(a - q_1 - R_2(q_1) - c)$$

replacing $R_2(q_1)$ we get:

$$\text{Max}_{\{q1\}}\; q_1\,(a - q_1 - c) / 2$$

Using the FOCs

$$q_1^* = (a - c) / 2$$

Replacing in $R_2(q_1)$ we get:

$$R_2(q_1^*) = (a - c) / 4$$

The backward induction outcome is

$$q_1 = (a - c) / 2$$

$$q_2 = (a - c) / 4$$

The Subgame Perfect Nash Equilibrium is

$$q_1 = (a - c) / 2$$

$$q_2 = (a - q_1 - c) / 2$$

Wage and employment

- Relation between an Union and a Firm
- Union has exclusive control on the wages
- Firm has exclusive control over employment
- Union utility function is $U(w, L)$ where w is the wage the union demands and L is the employment
- $U(w, L)$ is increasing in w and L and concave
- Firm's profit function is:

$$\pi(w, L) = R(L) - wL$$

where $R(L)$ is the revenue of the firm when employment is L.

$R(L)$ is increasing and concave

Timing of the game

1. The union makes a wage demand w

2. The firm observes w and then chooses employment L

3. Firms and Union receive their payoffs, $\pi(w, L)$ and $U(w, L)$

Solution by backwards-induction

1. We analyze (and solve) the Firm problem for a generic observed wage w.

2. The solution gives us the optimal level of employment for any salary level.

3. Then we solve the problem of the Union assuming that Union knows the reaction of the firm to any wage demand w.

Firm's problem

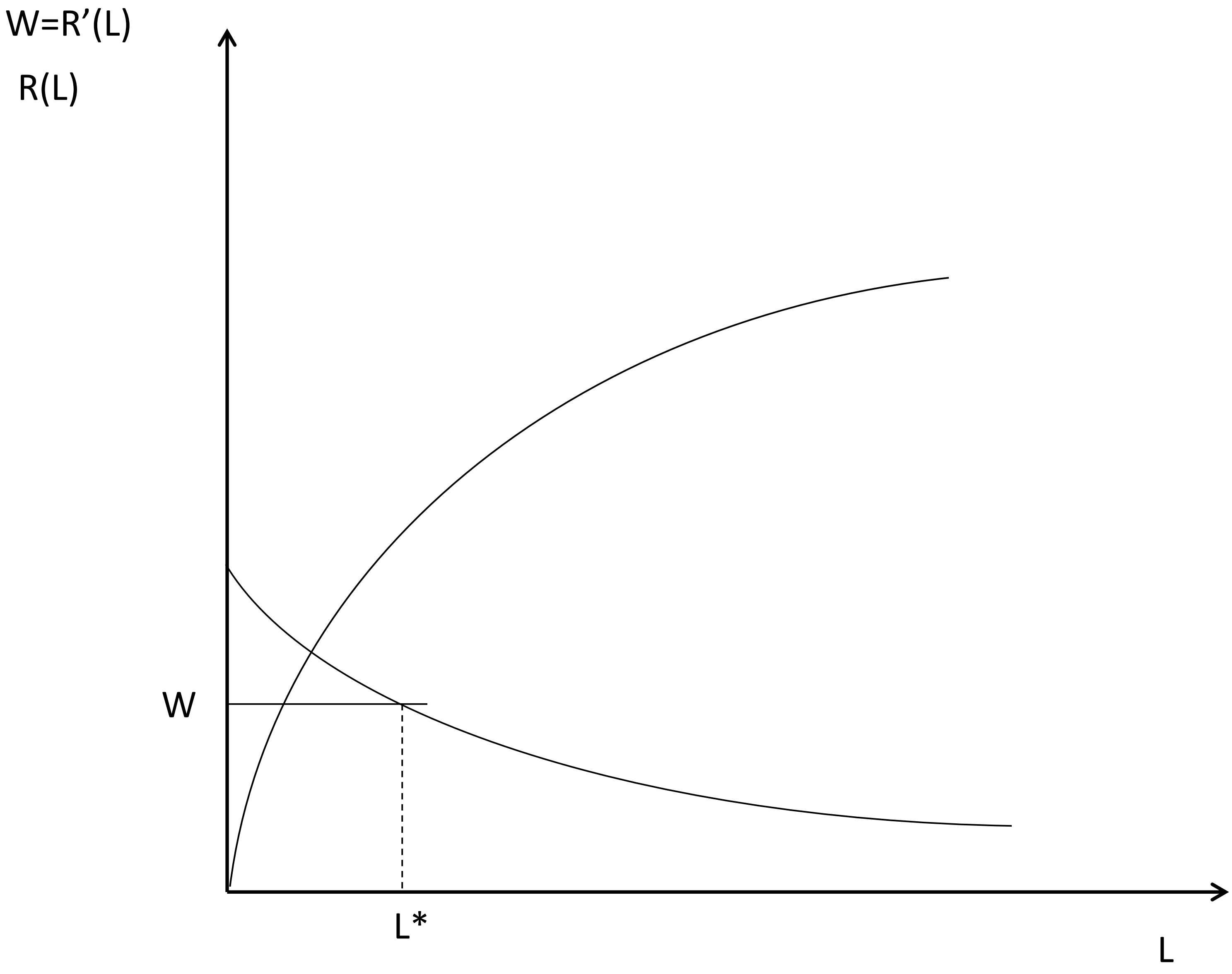
$$\text{Max}_{\{L\}}\ R(L) - w\,L$$

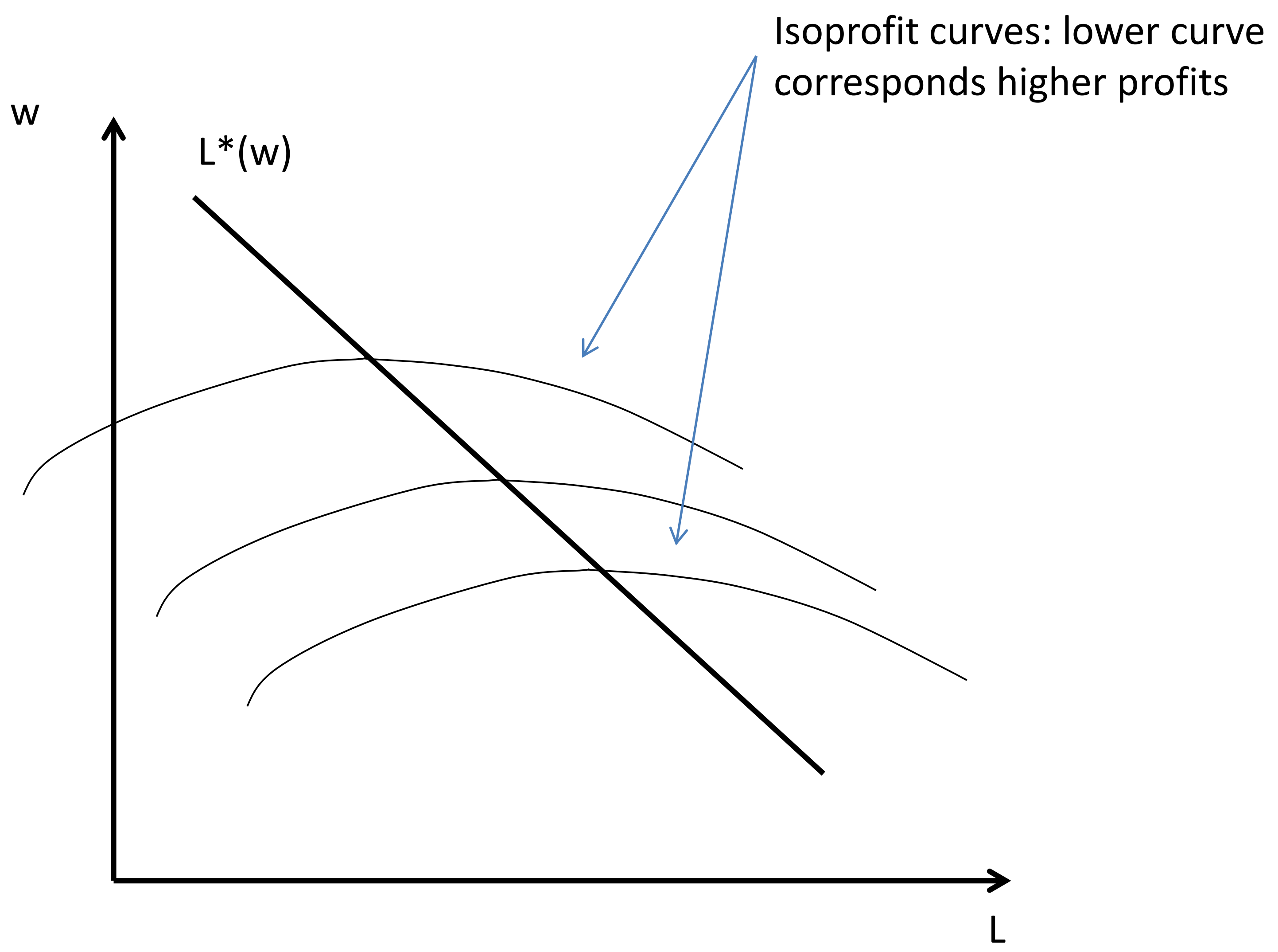
FOCs are:

$$R'(L) - w = 0 \ \rightarrow\ w = R'(L)$$

The solution will give us the reaction (best response) of the firm to a salary demand w, i.e. $L^*(w)$

Given that R(L) is increasing and concave, it follows that R'(L) is decreasing respect to L , $L^*(w)$ will be decreasing respect to w


W=R’(L)
R(L)
W
L*
L


Isoprofit curves: lower curve corresponds higher profits
w
L*(w)
L

Union's Problem

$$\max_{\{w\}} U(w, L)$$

Note that Union can anticipate the firm reaction to a wage demand w

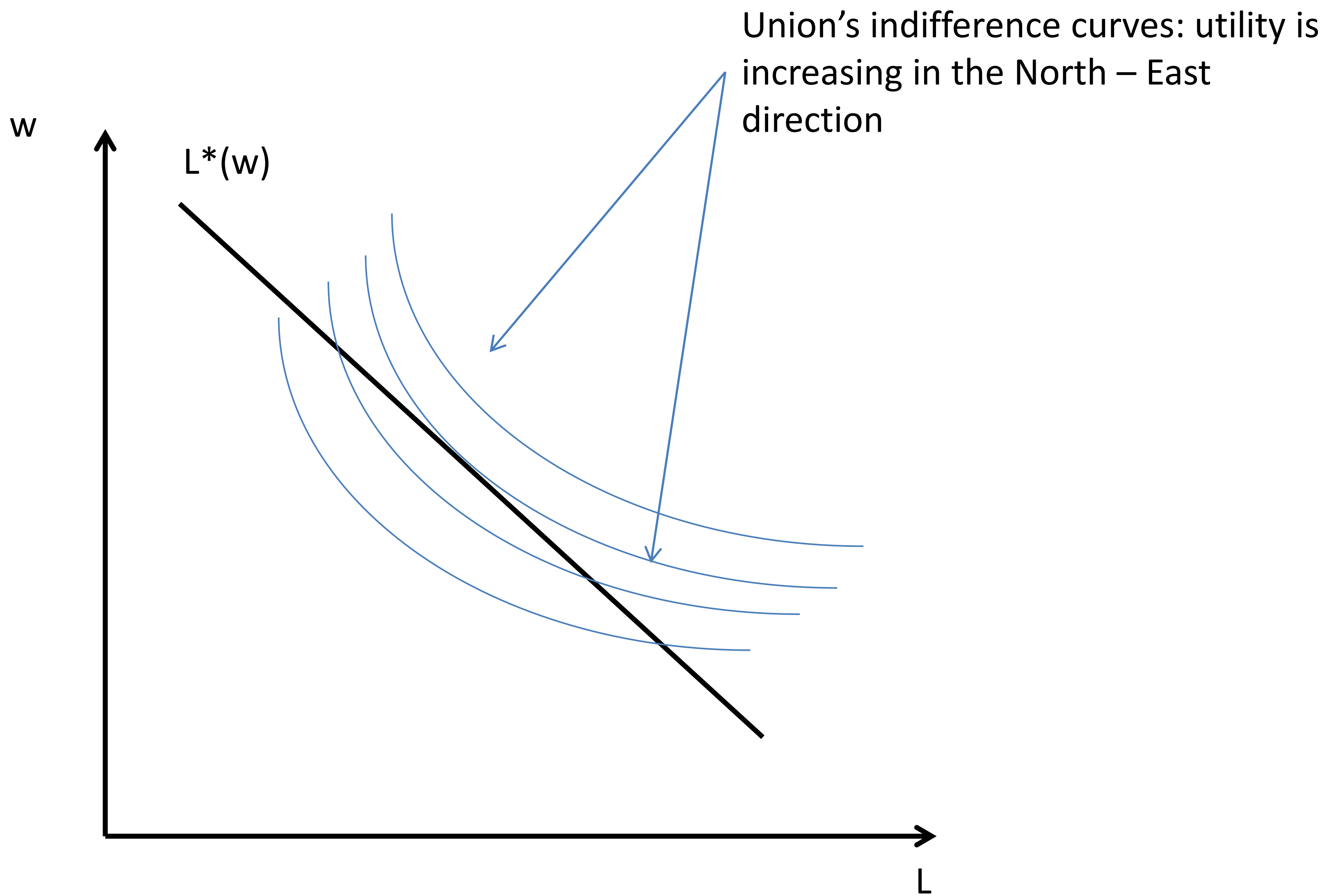
(Union can solve the firm's problem as well as the firm can solve it)

Then its problem is:

$$\max_{\{w\}} U(w, L^*(w))$$

FOCs are:

$$\frac{d\text{U}(\text{w}, L^*(\text{w}))}{dw} = U'_1 + U'_2 L^{*\prime} = 0$$



The union's problem is to choose a w such that (w, L*(w)) is on the highest possible indifference curve

The solution will be:

$(w^*, L^*(w^*))$

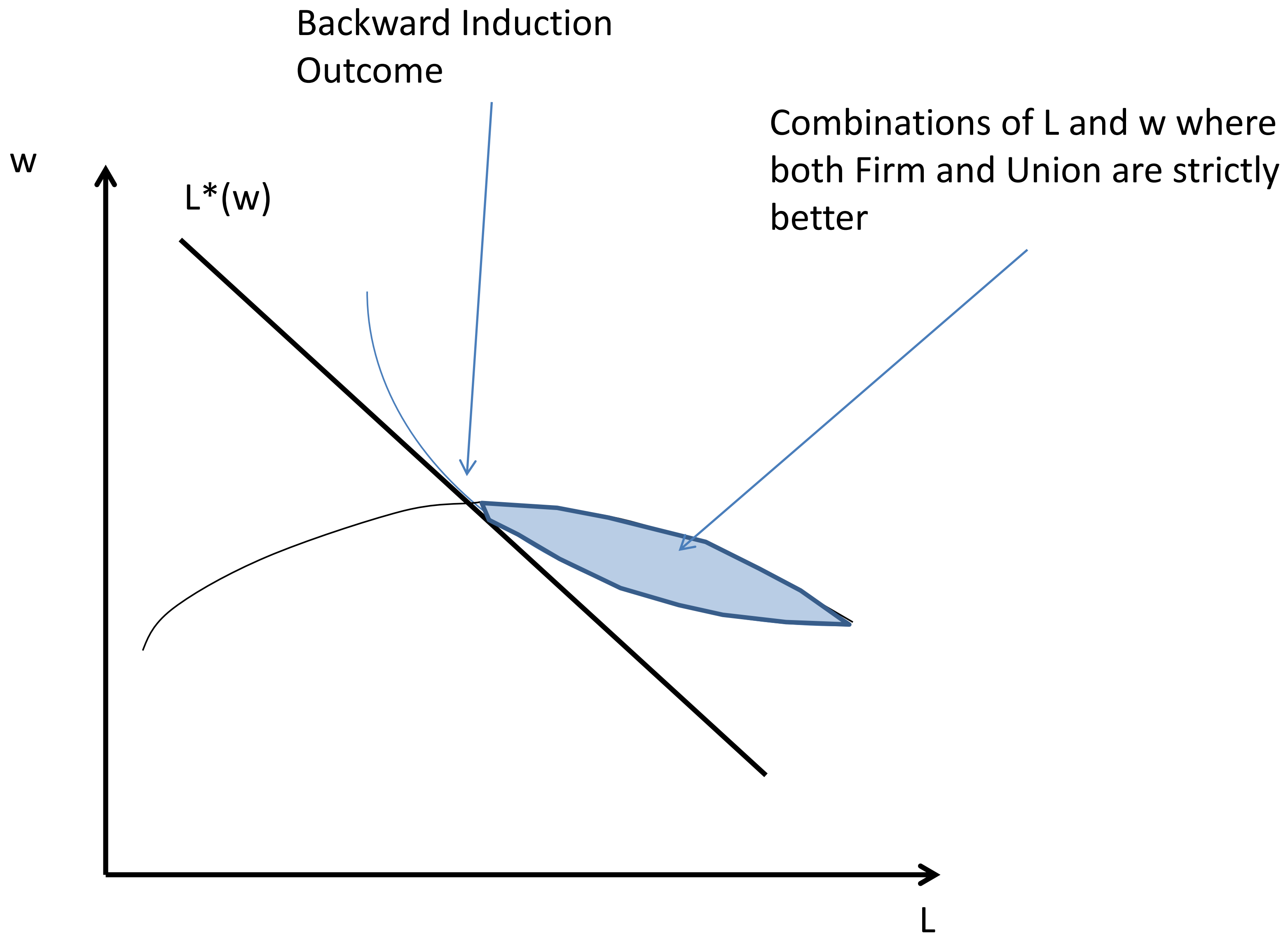
where the union indifference curve through the point $(w^*, L^*(w^*))$ will be tangent to $L^*(w)$ at that point

This solution is not efficient, in the sense that exist other combinations of L and w where Union and Firm are strictly better.

Note:

$(w^*, L^*(w^*))$ is the backward induction outcome

$(w^*, L^*(w))$ is the Subgame Perfect Nash Equilibrium


Backward Induction Outcome
Combinations of L and w where both Firm and Union are strictly better
w
L*(w)
L

Example

$$\pi(\mathrm{w}, \mathrm{L}) = 10\,L - L^2 - wL$$

$$U(w, L) = \mathrm{L} + \ln w$$

Sequential bargaining

- Two players, 1 and 2, are bargaining over \$1
- Bargaining procedure, alternating offers:
  - Player 1 makes a proposal that player 2 accepts or rejects
  - If player 2 rejects then player 2 makes a proposal that player 1 accepts or rejects
  - If player 1 rejects then player 1 makes a proposal that player 2 accepts or rejects
  - ......
- Each offer takes one period
- Players discount future payoffs by factor δ per period, $0 < \delta < 1$.

Three periods bargaining

(1a). Player 1 proposes to take a share $\boldsymbol{s_1}$ of the dollar, leaving $\mathbf{1} - \boldsymbol{s_1}$ for player 2

(1b). Player 2 either accepts (game ends) or rejects (Play goes to period 2)

(2a). Player 2 proposes a share $\boldsymbol{s_2}$ of the dollar for player 1, leaving $\mathbf{1} - \boldsymbol{s_2}$ for player 2

(2b). Player 1 either accepts (game ends) or rejects (Play goes to period 3)

(3). Player 1 receives a share $\boldsymbol{s}$ of the dollar, player 2 receives $\mathbf{1} - \boldsymbol{s}$.

Solution by backwards-induction

- The problem of player 1 in period 2 is a choice between
  - to have $\boldsymbol{s_2}$ immediately or
  - $\boldsymbol{s}$ one period later.

  The best response of Player 1 is to accept $\boldsymbol{s_2}$
  if $\boldsymbol{s_2} \geq \boldsymbol{\delta s}$, otherwise reject ($\boldsymbol{s_2} < \boldsymbol{\delta s}$)

- The problem of Player 2 in period 2 is a choice between:
  - to offer $\boldsymbol{s_2 = \delta s}$ (player 1 accepts) and receive immediately $\boldsymbol{1 - \delta s}$ or
  - to offer less (player 1 rejects) and receive $1 - s$ one period later

The best response of Player 2 is to propose

$\boldsymbol{s_2 = \delta s}$, because $\boldsymbol{1 - \delta s > \delta (1 - s)}$

- The problem of player 2 in period 1 is a choice between:
  - To accept $\boldsymbol{s_1}$ and receive $\boldsymbol{1} - \boldsymbol{s_1}$ immediately
  - To reject and receive $(\boldsymbol{1} - \boldsymbol{\delta s})$ one period later

  The best response of Player 2 in period 1 is to accept $\boldsymbol{s_1}$ if and only if $\boldsymbol{1} - \boldsymbol{s_1} \geq \boldsymbol{\delta}(\boldsymbol{1} - \boldsymbol{\delta s})$,

  i.e. $\boldsymbol{s_1} \leq \boldsymbol{1} - \boldsymbol{\delta}(\boldsymbol{1} - \boldsymbol{\delta s})$

- The problem of Player 1 in period 1 is a choice between:
  - To offer $\boldsymbol{s_1 = 1 - \delta(1 - \delta s)}$ (player 2 accepts) and receive $\boldsymbol{1 - \delta(1 - \delta s)}$ immediately
  - To offer less (player 2 rejects) and receive δs one period later

The best response of Player 1 in period 1 is to propose $\boldsymbol{s_1 = 1 - \delta(1 - \delta s)}$ because

$\boldsymbol{1 - \delta(1 - \delta s) > \delta^2 s}$