Eq. of motion of a photon travelling toward us: $\boxed{a_0 r(z)}$ 1

$$\frac{a\,dr}{\sqrt{1-kr^2}} = -c\,dt = -c\frac{da}{\dot a} = -c\frac{da}{aH}$$

$$H = \frac{\dot a}{a}$$

$$a = a_0/1+z \rightarrow da = -\frac{a_0}{(1+z)^2}dz$$

$$\rightarrow \frac{a_0}{1+z}\cdot\frac{dr}{\sqrt{1-kr^2}} = -\frac{c}{H}\cdot\frac{1+z}{a_0}\cdot\left[-\frac{a_0}{(1+z)^2}\right]dz$$

$$\boxed{\frac{a_0\,dr}{\sqrt{1-kr^2}} = \frac{c}{H(z)}dz} \qquad H(z) \equiv H_0\cdot E(z)$$

$$E(z) \equiv \left[\Omega_R(1+z)^4 + \Omega_M(1+z)^3 + \Omega_\Lambda + (1-\Omega_0)(1+z)^2\right]^{1/2}$$

But

$$f_k(r) = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \frac{c}{a_0 H_0}\int_0^z \frac{dz'}{E(z')} = \begin{cases} \arcsin(r) & k=+1 \\ r & k=0 \\ \operatorname{arcsinh}(r) & k=-1 \end{cases}$$

Remember $\frac{kc^2}{a_0^2} = H_0^2(\Omega_0-1) \rightarrow \frac{c}{a_0 H_0} = \sqrt{|\Omega_0-1|}$

$$a_0 = \frac{c}{H_0\sqrt{|\Omega_0-1|}}$$

For instance, for $k=+1$,

$$\frac{c}{a_0 H_0}\int_0^z \frac{dz'}{E(z')} = \arcsin(r) \rightarrow r = \sin\left(\frac{c}{a_0 H_0}\int\frac{dz}{E}\right) = \underbrace{\frac{c}{a_0 H_0\sqrt{|\Omega_0-1|}}}_{\equiv 1}\cdot\sin\left[\sqrt{|\Omega_0-1|}\cdot\int\frac{dz}{E}\right] \quad \cancel{a_0}$$

and

$$\boxed{a_0 r(z) = \frac{c}{H_0\sqrt{|\Omega_0-1|}}\cdot\sin\left[\sqrt{|\Omega_0-1|}\cdot\int_0^z\frac{dz'}{E(z')}\right]} \quad \boxed{k=+1}$$

For $k=-1$ we get the same, with sin → sinh

For $k=0$ we get simply: $\boxed{a_0 r(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}}$

For $z < z_{eq}$ we neglect Ω_R and

$$E(z) = \left[(1+z)^2 (1+\Omega_M z) - z(2+z)\Omega_\Lambda\right]^{1/2}$$

{ When Ω_Λ is also negligible, $E(z) \simeq (1+z)\sqrt{1+\Omega_M z}$ }

When $\Omega_0 = 1$ and $\Omega_M + \Omega_\Lambda = 1$

$$E(z) = \left[1 - \Omega_M + \Omega_M (1+z)^3\right]^{1/2}$$

There are no general, analytical expressions for $a_0 r(z)$.

If $\Omega_\Lambda = 0$, Mattig Formula:

$$\boxed{a_0 r(z) = \frac{2c}{H_0} \cdot \frac{\Omega_M z + (\Omega_M - 2)\left[(1+\Omega_M z)^{1/2} - 1\right]}{\Omega_M^2 (1+z)}}$$

which holds for both $\Omega_M > 1$ and $\Omega_M < 1$

For $z \gg 1$ ($z \to \infty$) Mattig formula gives ($\Omega_\Lambda \equiv 0$)

$$\boxed{a_0 r(z) \cong \frac{2c}{H_0 \Omega_M}} \quad \boxed{\Omega_\Lambda = 0}$$

When $\Omega_M + \Omega_\Lambda \equiv 1$ a useful approximation is

$$\boxed{a_0 r(z) \cong \frac{2c}{H_0 \Omega_M^{0.4}}} \quad \boxed{\text{flat } \Omega_M + \Omega_\Lambda = 1}$$

$\boxed{\text{! } a_0 r(z) \text{ is very important for observational cosmology !}}$

We have seen that

$$a(t) \simeq a_0 \left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots\right]$$

If we use the redshift z: $1+z = \frac{a_0}{a(t)}$:

$$1+z = \left[\dots\right]^{-1}$$

Remember the $(1+x)^\alpha \simeq 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$

$H_0 t_0 \sim 1$ remember EdS model

so

$H_0(t-t_0) \ll 1$ if

$|t-t_0| \ll t_0$

So

$$\cancel{1}+z \simeq \cancel{1} - H_0 (t-t_0) + \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots + \frac{-1(-1-1)}{2}\left[H_0^2 (t-t_0)^2 + \dots\right]$$

$$\boxed{z \simeq H_0 (t_0-t) + \left(1+\frac{q_0}{2}\right) H_0^2 (t_0-t)^2 + \dots}$$

t_0-t = look-back time

We want t_0-t as a function of z: we have to invert the power series.

Power series inversion

given: $y = a_1 x + a_2 x^2 + a_3 x^3 + \dots$

we want $x = A_1 y + A_2 y^2 + A_3 y^3 + \dots$

and

$$y = a_1 \left(A_1 y + A_2 y^2 + A_3 y^3 + \dots\right) + a_2 \left(A_1^2 y^2 + A_2^2 y^4 + \dots + 2 A_1 A_2 y^3 + 2 A_1 A_3 y^4 + \dots\right) +$$
$$+ a_3 \left(A_1^3 y^3 + \dots\right) + \dots$$

Equating the coefficients of the powers of y:

$a_1 A_1 = 1 \to \boxed{A_1 = 1/a_1}$

$a_1 A_2 + a_2 A_1^2 = 0 \to \boxed{A_2 = -\frac{a_2}{a_1} \cdot A_1^2 = -\frac{a_2}{a_1^3}}$

$a_1 A_3 + 2 a_2 A_1 A_2 + a_3 A_1^3 = 0 \to A_3 = \frac{1}{a_1^5}\left[2 a_2 - a_3 a_1\right]$

...

$$x(y) = \frac{1}{a_1} y - \frac{a_2}{a_1^3} y^2 + \frac{1}{a_1^5}\left[2 a_2 - a_3 a_1\right] y^3 + \dots$$

Coming back we then have

$$\boxed{t_0 - t \simeq \frac{1}{H_0} \cdot z - \frac{(1+q_0/2) H_0^2}{H_0^3} \cdot z^2 + \dots \simeq \frac{z}{H_0}\left[1 - \left(1+\frac{q_0}{2}\right) \cdot z + \dots\right]}$$

Remember that

$$\underbrace{\int_t^{t_0} \frac{c\,dt'}{a(t')}} = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} \cong r + \frac{k}{6} r^3 \approx r \quad \left[\text{neglecting terms } O(r^3)\right]$$

$$\int_t^{t_0}\frac{c\,dt'}{a(t')}\frac{a_0}{a_0} = \frac{c}{a_0}\int_t^{t_0}\frac{dt'}{a(t')/a_0} \cong \frac{c}{a_0}\int_{t-t_0}^{t_0-t_0}\left[1 - H_0(t'-t_0) + \frac{1}{2} q_0 H_0^2 (t'-t_0)^2 + \dots\right] d(t'-t_0) \cong$$

$$\cong \frac{c}{a_0}\left[0-(t-t_0) - \frac{H_0}{2}\left[0-(t-t_0)^2\right] + \dots\right] \cong \frac{c}{a_0}\left[(t_0-t)+\frac{1}{2} H_0 (t_0-t)^2 + \dots\right]$$

and

$$r \cong \frac{c}{a_0}\left[(t_0-t)+\frac{1}{2} H_0(t_0-t)^2+\dots\right] \leftarrow t_0-t \text{ vs } z \quad \text{(see above)}$$

$$r \cong \frac{c}{a_0}\left\{\frac{z}{H_0} - \frac{z^2}{H_0}\left(1+\frac{q_0}{2}\right) + \dots + \frac{1}{2} H_0 \frac{z^2}{H_0^2}\left[1-\left(1+\frac{q_0}{2}\right)z + \dots\right]^2 + \dots\right\}$$

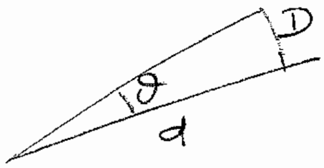
and

$$a_0 r(z) \cong \frac{c}{H_0}\left[z - z^2\left(1+\frac{q_0}{2}-\frac{1}{2}\right)+\dots\right]$$ and finally

$$\boxed{a_0 r(z) \cong \frac{cz}{H_0}\left[1-\frac{1+q_0}{2}\cdot z+\dots\right]}$$

Distances

In euclidean space we write



$$\frac{D}{d} = \vartheta \quad\downarrow\quad d = \frac{D}{\vartheta}$$

Flux F

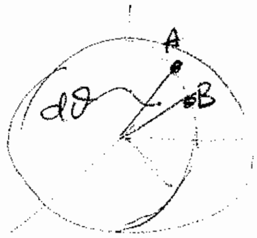
$$F = \frac{L}{4\pi d^2} \rightarrow d = \sqrt{\frac{L}{4\pi F}}$$

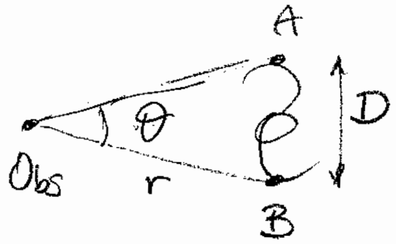
How can we do in curved space?

From R&W metric, $d\ell^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$

and, if $dr = 0$, $d\ell^2 = a^2(t)\, r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$

Remember also that the radial coordinate r was chosen in such a way that the surface of a sphere $r = \text{const}$ is $4\pi r^2$ and, taking into account scale factor, $\underline{\text{Area} = 4\pi a^2(t)\, r^2}$





$$d\ell = a(t_e)\cdot r \cdot d\theta$$

$$\downarrow \quad (d\varphi \equiv 0)$$

$$D = a(t_e)\cdot r \cdot \theta$$

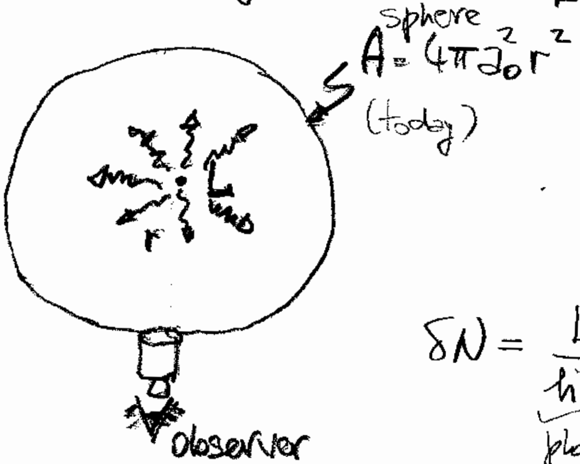
angular diameter distance d_A

$$\boxed{d_A = \frac{D}{\theta} = a(t_e)\cdot r = \frac{a_0\, r(z)}{1+z}}$$

$$\lambda_0 = \lambda_e(1+z) \qquad \nu = \frac{c}{\lambda}$$

$$\nu_e = \nu_0(1+z) \rightarrow \frac{\nu_0}{\nu_e} = \frac{1}{1+z}$$

luminosity distance d_L



Isotropic source with luminosity (bolometric) L

Suppose: monochromatic (no absorption)

$$L^{(\nu)} = L\cdot\delta(\nu - \nu_e) \qquad \nu_e = \nu_{\text{emission}}$$

$$\delta N = \underbrace{\frac{L}{h\nu_e}\cdot\delta t_e}_{\text{photons emitted in } \delta t_e} = \text{number photons crossing sphere today in } \delta t_0 = \frac{F}{h\nu_0}\cdot 4\pi a_0^2 r^2\, \delta t_0$$

$$F = \frac{L}{4\pi a_0^2 r^2}\cdot\frac{\nu_0}{\nu_e}\cdot\frac{\overset{\sim \frac{1}{\nu_e}}{\delta t_e}}{\underset{\sim \frac{1}{\nu_0}}{\delta t_0}} = \frac{L}{4\pi a_0^2 r^2}\left(\frac{\nu_0}{\nu_e}\right)^2 = \frac{L}{4\pi\left[a_0 r(z)(1+z)\right]^2} = \frac{L}{4\pi d_L^2}$$

$$\boxed{d_L = a_0\, r(z)\cdot(1+z)} \rightarrow \boxed{d_A = \frac{d_L}{(1+z)^2}}$$

Remember $a_0 r(z) \cong \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}\cdot z + \dots\right]$

$$d_L \cong \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}\cdot z + \dots\right]\cdot\left[1+z\right] \cong \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}\cdot z + \dots + z + \dots\right]$$

$$-\frac{1+q_0}{2} + 1 = \frac{-1-q_0+2}{2} = \frac{1-q_0}{2}$$

$$\boxed{d_L \cong \frac{cz}{H_0}\left[1 + \frac{1-q_0}{2}\cdot z + \dots\right]}$$

$$d_A \simeq \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2} z + \dots\right]\cdot\underbrace{\left[1+z\right]^{-1}}_{\sim 1-z} \simeq \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2}\cdot z + \dots - z + \dots\right]$$

$$-\frac{1+q_0}{2} - 1 = \frac{-1-q_0-2}{2} = -\frac{3+q_0}{2}$$

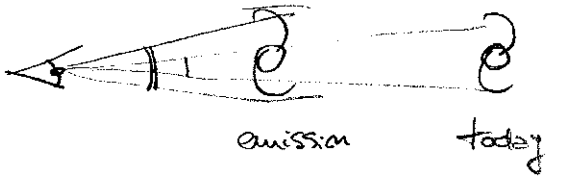
$$\boxed{d_A \cong \frac{cz}{H_0}\left[1 - \frac{3+q_0}{2} z + \dots\right]}$$

$$d_A = \frac{D}{\theta} \rightarrow \theta = \frac{D}{d_A} \approx \frac{H_0 D}{cz}\left[1 + \frac{3+q_0}{2} z + \dots\right]$$

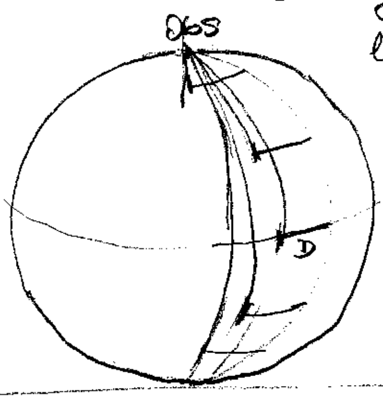
- "Naive" point of view: $\frac{\Delta\lambda}{\lambda} = z = \frac{v}{c} \rightarrow cz = v = H_0 d \rightarrow d = \frac{cz}{H_0}$

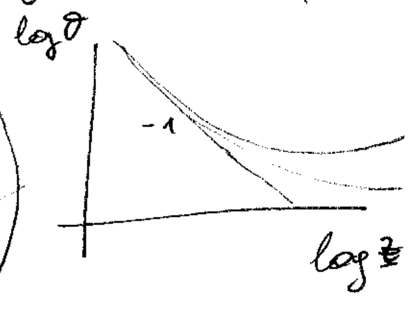
$$\theta_N = \frac{D}{d} = \frac{DH_0}{cz} \sim \frac{1}{z}$$ smaller than observed!

Because when photons emitted, source was nearer and seen under a larger angle.



In addition also geometrical effects

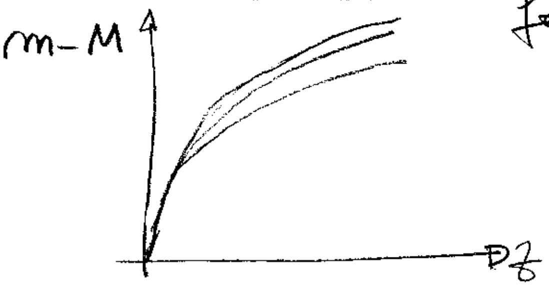




- Luminosity distance

$$F = \frac{L}{4\pi d_L^2} \rightarrow \underbrace{m - M}_{\text{distance modulus}} = 5 \log d_L(pc) + 5$$

Hubble diagram



for SN Ia ⇒ models without Λ do not fit observations!

⇒ $\Omega_\Lambda \neq 0$ accelerated expansion (Nobel Prize 2011) — Perlmutter, Schmidt, Riess

Do we have a standard length to use for cosmological tests?

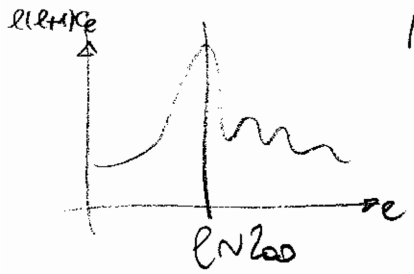
In the early univers matter is ionized and protons + electrons are tightly coupled to radiation: they form a unique fluid with sound speed $c_s \sim \frac{c}{\sqrt{3}}$. This situation ends at the "Recombination" epoch, at $z \sim 1000$, $t \sim 400.000$ yrs.
Then photons and baryons are no more coupled, and the sound speed is almost zero if compared to $c/\sqrt{3}$.
So a density perturbation propagates at the speed of sound (see the figure for more details; CDM is the dark matter component). The relevant scale is the "Sound horizon", $d_s(t) = a(t) \int_0^t \frac{c_s\, dt'}{a(t')}$

This, if $c_s \sim$ const. $\sim \frac{c}{\sqrt{3}}$, is $\sim \frac{1}{\sqrt{3}} d_H(t)$ (← particle horizon)

Recombination occurs in the matter dominate epoch, when $d_H(t) = 3ct$, when also $R_H(t) \simeq \frac{1}{2} d_H(t)$.
So, at recombination,

$$d_s(t_{rec}) \sim \frac{1}{\sqrt{3}} d_H(t_{rec}) \sim \frac{1}{\sqrt{3}} \cdot 2\, R_H(t_{rec}) \sim R_H(t_{rec})$$

Sound horizon gives a signature in the angular power spectrum of CMB, corresponding to the position of the first peack.



Multipole $\ell \sim \frac{\pi}{\theta_{rad}}$

Remember that, for $z \gg 1$, $a\, r(z) \sim \begin{cases} \frac{2c}{H_0 \Omega_M} & \Omega_\Lambda = 0 \\ \frac{2c}{H_0 \Omega_M^{0.4}} & \text{flat} \end{cases}$

Recombination corresponds also to the last scattering of CMB photons, at $z_{ls} \sim z_{rec} \sim 1000$

$$d_A(z_{ls}) \simeq \frac{a_0 r(z_{ls})}{1+z_{ls}} \approx \frac{2c}{H_0 \Omega_M^{\alpha} (1+z_{ls})} \quad \text{with } \alpha = \begin{cases} 1 & \Omega_\Lambda = 0 \\ 0.4 & \text{flat} \end{cases}$$

$$R_H(z_{ls}) \simeq \frac{c}{H(z_{ls})} \overset{EdS}{\simeq} \frac{c}{H_0 \sqrt{\Omega_M} (1+z_{ls})^{3/2}}$$

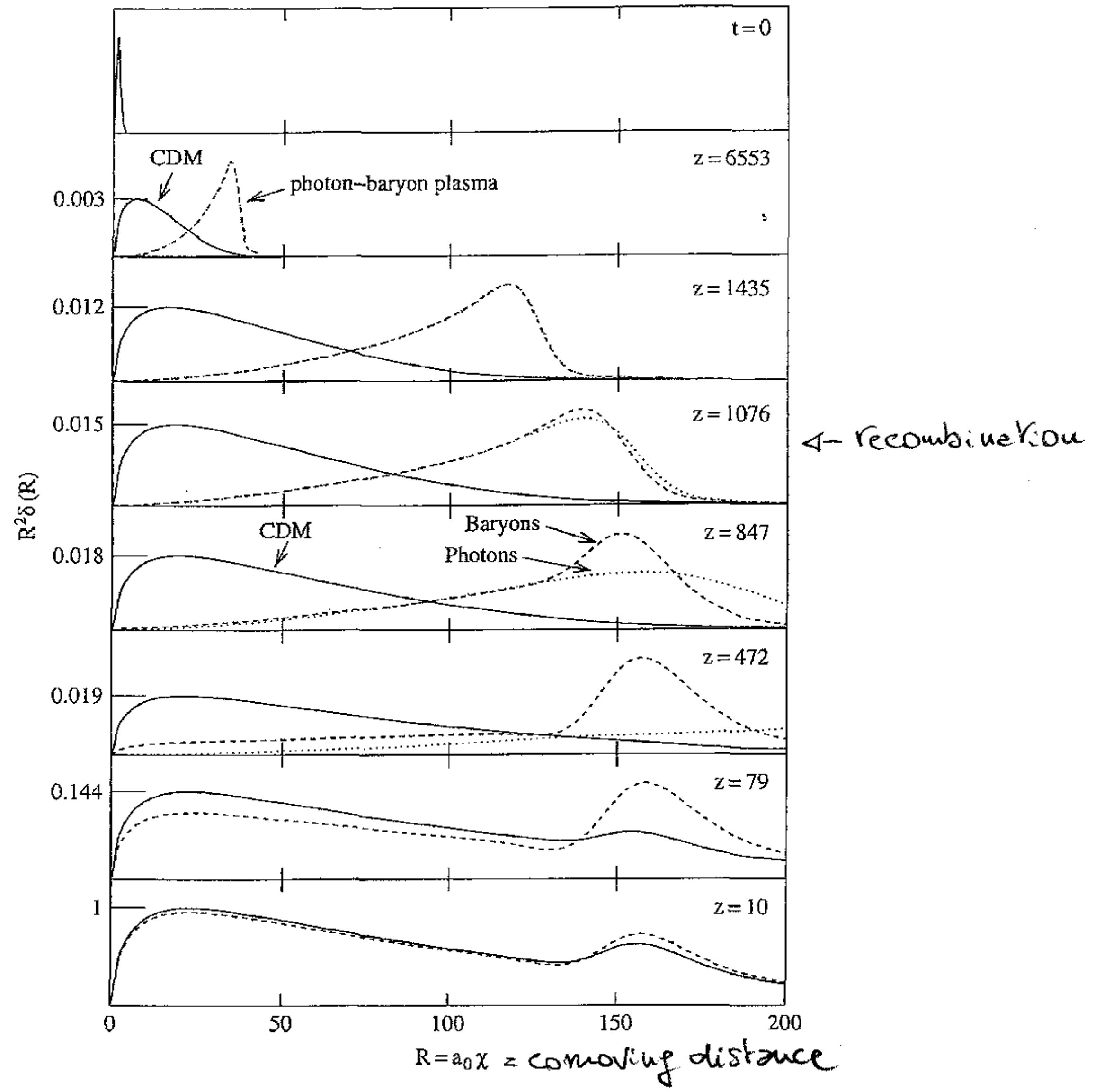


Fig. 5.5 The time development of an initial adiabatic over-density in a universe with CDM, neutrinos, baryons, and photons [152, 123]. At $t = 0$, the over-density of all components are superimposed but the pressure of the baryon–photon plasma causes it to propagate away from the origin at the speed of sound. Light neutrinos (not shown) free stream away with the speed of light. The baryons stop shortly after recombination when the baryons and photons decouple, allowing the photons to free stream away. CDM and baryons from the homogeneous reservoir will then be gravitationally attracted into the potential wells formed by the CDM at the origin and the shell of baryons. This infall of homogeneous matter will generate CDM–baryon over-densities where galaxies will preferentially form. This results in the galaxy–galaxy correlation function seen in Fig. 5.7.

So sound horizon correspond to an angle

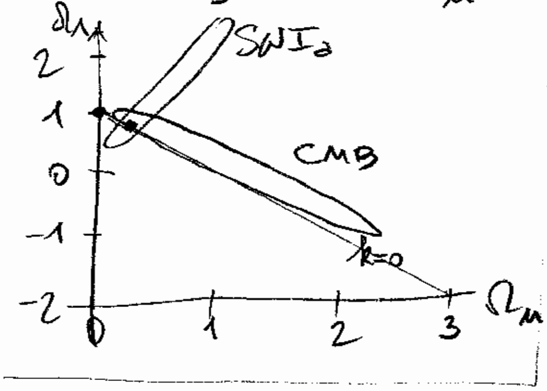
$$\theta_S \simeq \frac{R_H(z_{es})}{d_A(z_{es})} \cong \frac{c}{H_0\sqrt{\Omega_M}\,(1+z_{es})^{3/2}} \cdot \frac{H_0\,\Omega_M^{\alpha}\,(1+z_{es})}{2c} \simeq \frac{\Omega^{\alpha-1/2}}{2\,(1+z_{es})^{1/2}}$$

$\theta_S \sim 0.9^\circ\,\Omega_M^{\alpha-1/2}$ ← Does not depend on H_0!

Going to the multipole $\ell \simeq \pi/\theta$

$$\ell_S \simeq \frac{\pi}{\theta_S} \simeq \frac{2\pi\,(1+z_{es})^{1/2}}{\Omega_M^{\alpha-1/2}} \sim \frac{200}{\Omega_M^{\alpha-1/2}}$$

← Very weak dependence on Ω_M if universe flat: $\alpha \cong 0.4 \to \ell_S \sim 200 \cdot \Omega_M^{0.1}$



$\theta_S \sim 1^\circ \sim d_H(z_{es})$

The particle horizon at $z \sim 1000$ subtends an angle of $\sim 1^\circ$

But $(\Delta T/T)_{CMB} \sim 10^{-5}$ all over the sky

<u>Question</u>: Why region which were never in causal contact have so similar temperature?

This is the so-called "<u>horizon problem</u>" of the Standard Cosmological model.

It is solved by inflation, which makes $d_H(z_{es}) \gg R_H(z_{es})$

Source counts

Other observational constraints come from source counts

Number of objects with proper density n in the range $r \mapsto r+dr$ and within the solid angle $d\Omega$:

$$dN = n\,dV = n\,a^3 r^2 d\Omega\,\frac{dr}{\sqrt{1-kr^2}} = n\,a^3\,\frac{r^2\,dr\,d\Omega}{\sqrt{1-kr^2}}$$

But $\frac{dr}{\sqrt{1-kr^2}} = \frac{c}{a_0 H(z)}\,dz$

so

$$dN = n\,a^3\,\frac{r^2 c}{\underset{\uparrow\,H_0 E(z)}{a_0 H(z)}}\,dz\,d\Omega \to \boxed{\frac{dN}{d\Omega\,dz} = \frac{c}{H_0}\,\frac{n\,a^3\,a_0^2 r^2}{a_0\,E(z)\,a_0^2} = \frac{c\,n(z)\,[a_0 r(z)]^2}{H_0\,E(z)\,(1+z)^3}}$$

$\frac{a^3}{a_0^3} = \frac{1}{(1+z)^3}$

$$\boxed{\frac{dN}{dz\,d\Omega} = \frac{c}{H_0} \cdot \frac{n(z)\left[a_0 r(z)\right]^2}{E(z)\,(1+z)^3}}$$

If sources are not created nor distroyed, $n(z) = n_0\,(1+z)^3$
const.

and $$\frac{dN}{dz} = \frac{c}{H_0}\,\frac{n_0\left[a_0 r(z)\right]^2 d\Omega}{E(z)}$$

from the expension for $a\,r(z)$:

$$\frac{dN}{dz} \approx \frac{c^3 n_0\, d\Omega}{H_0^3}\left[z^2 - 2(1+q_0)z^3 + \dots\right]$$

and, by integration,

$$\boxed{N(<z) \approx \frac{n_0 c^3\, d\Omega}{3H_0^3}\, z^3\left[1 - \frac{3}{2}(1+q_0)z + \dots\right]}$$ sources conserved!

Usually astronomers observe all the objects, within a solid angle, having a flux larger then a given value F.
So moving from z to d_L, and from d_L to L, we can write

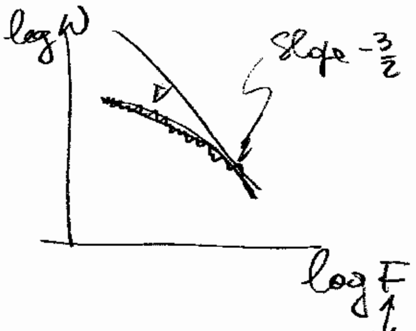
$$N(>F) \simeq \frac{4\pi n_0}{3}\left(\frac{L}{4\pi F}\right)^{3/2}\left[1 - \frac{3H_0}{c}\left(\frac{L}{4\pi F}\right)^{1/2} + \dots\right]$$ Sources conserved!

(here we have assumed that all objects have the same intrinsic luminosity L, but the relation can be generalized to a spectrum of values of L)

$\left[\log_{10}(x) = \underbrace{\log_{10} e}_{0.4343} \cdot \ln(x)\ ;\ \ln(1+x) \sim x\right]$

So

$$\log N(>F) \cong -\frac{3}{2}\log F - 0.4343\,\frac{3H_0}{c}\left(\frac{L}{4\pi}\right)^{1/2} \cdot \frac{1}{\sqrt{F}} + \dots$$



$\log F \leftrightarrow \log S$ for radioastronomers

This is correct if sources are conserved and if luminosity L does not change in time
$\downarrow$
Check of evolution of Sources

- $\frac{dN}{dz} = \frac{c\, n_0\, d\Omega\, [a_0 r(z)]^2}{H_0\, E(z)}$ (in the case of conservation of source number) | Dist 5, brs

- $a_0 r(z) \simeq \frac{cz}{H_0}\left[1 - \frac{1+q_0}{2} z + \cdots\right]$

- We need the expansion of $E(z)^{-1}$

$$E(z) = \frac{H(z)}{H_0} = \frac{1}{H_0}\frac{\dot a}{a} \rightarrow E(z)^{-1} = H_0 \frac{a}{\dot a}$$

from $a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \cdots\right]$ we have

$$\dot a = a_0\left[H_0 - \frac{1}{\cancel{2}} q_0 H_0^2 \cdot \cancel{2}(t-t_0) + \cdots\right] \simeq a_0 H_0\left[1 - q_0 H_0 (t-t_0) + \cdots\right]$$

$$E(z)^{-1} = H_0 \frac{a_0\left[1 + H_0(t-t_0) + \cdots\right]}{a_0 H_0\left[1 - q_0 H_0(t-t_0)\right]} \simeq \left[1 + H_0(t-t_0) + \cdots\right] \cdot \left[1 + q_0 H_0 (t-t_0) + \cdots\right]$$

$$\simeq 1 + H_0(t-t_0) + \cdots + q_0 H_0 (t-t_0) + \cdots \simeq 1 + (1+q_0) H_0 (t-t_0) + \cdots$$

$$E(z)^{-1} \simeq 1 - (1+q_0) H_0 (\overbrace{t_0 - t}^{t_{lb}})$$

But $H_0(t_0 - t) \simeq z + O(z^2)$, so

$$E(z)^{-1} \simeq 1 - (1+q_0) \cdot z + \cdots$$

- $$\frac{dN}{dz} \simeq \frac{c\, n_0\, d\Omega}{H_0} \underbrace{\frac{c^2 z^2}{H_0^2}\left[1 - (1+q_0)\cdot z + \cdots\right]}_{[a_0 r(z)]^2} \cdot \underbrace{\left[1 - (1+q_0)\cdot z + \cdots\right]}_{E(z)^{-1}}$$

$$\simeq \left(\frac{c}{H_0}\right)^3 n_0\, d\Omega \cdot z^2 \left[1 - 2(1+q_0) z + \cdots\right]$$

$$\simeq \left(\frac{c}{H_0}\right)^3 n_0\, d\Omega \left[z^2 - 2(1+q_0) z^3 + \cdots\right]$$

- $$N(<z) = \int_0^z \frac{dN}{dz} dz = \left(\frac{c}{H_0}\right)^3 n_0\, d\Omega \left[\frac{z^3}{3} - \frac{2(1+q_0)}{\cancel{4}\ 2} z^4 + \cdots\right]$$

$$N(<z) \simeq \left(\frac{c}{H_0}\right)^3 \frac{n_0\, d\Omega}{3} z^3 \left[1 - \frac{3}{2}(1+q_0) z + \cdots\right]$$

Dest 5, ter

- $N(<z) = \left(\frac{c}{H_0}\right)^3 \frac{n_0 d\Omega}{3} z^3 \left[1 - \frac{3}{2}(1+q_0) z + \dots\right]$

we know that:

$$d_L \cong \frac{cz}{H_0}\left[1 + \frac{1-q_0}{2} z + \dots\right] \rightarrow \frac{H_0 d_L}{c} \simeq z + \frac{1-q_0}{2} z^2 + \dots$$

inverting:

$$z \simeq \left(\frac{H_0 d_L}{c}\right) - \left(\frac{1-q_0}{2}\right)\left(\frac{H_0 d_L}{c}\right)^2 + \dots \simeq \left(\frac{H_0 d_L}{c}\right)\left[1 - \left(\frac{1-q_0}{2}\right)\left(\frac{H_0 d_L}{c}\right) + \dots\right]$$

- $N(<z)$ corresponds to the number of sources within a given d_L, so $N(<z) \rightarrow N(<d_L)$

$$N(<d_L) \cong \cancel{\left(\frac{c}{H_0}\right)^3} \frac{n_0 d\Omega}{3} \cdot \underbrace{\cancel{\left(\frac{H_0}{c}\right)^3} d_L^3 \left[1 - \frac{3}{2}(1-q_0)\left(\frac{H_0 d_L}{c}\right) + \dots\right]}_{z^3} \left[1 - \frac{3}{2}(1+q_0) \cdot \left(\frac{H_0 d_L}{c}\right) \dots\right]$$

$$\cong \frac{n_0 d\Omega}{3} d_L^3 \underbrace{\left[1 - \frac{3}{2}(1-q_0)\left(\frac{H_0 d_L}{c}\right) + \dots - \frac{3}{2}(1+q_0)\left(\frac{H_0 d_L}{c}\right) + \dots\right]}_{1 - \frac{3}{2}\left(\frac{H_0 d_L}{c}\right)\left[1 \cancel{- q_0} + 1 \cancel{+ q_0}\right] = 1 - 3\left(\frac{H_0 d_L}{c}\right)}$$

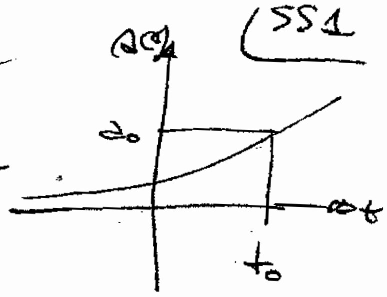
But $d_L = \left(\frac{L}{4\pi F}\right)^{1/2}$

Sources within d_L have flux larger than F, so

$$N(>F) \simeq \frac{n_0 d\Omega}{3} \left(\frac{L}{4\pi F}\right)^{3/2} \left[1 - 3 \frac{H_0}{c} \left(\frac{L}{4\pi F}\right)^{1/2} + \dots\right]$$

Stato Stazionario - Steady State

(SS1



• $a(t)$? $H = \text{cost.}$ $\frac{da}{a} = H\,dt$

$\ln a = Ht + \text{cost}$ $\boxed{a = Ae^{Ht}}$

• $k = ?$ Scalare di Ricci o curvatura K

$K = \frac{6k}{a^2(t)}$ → K does not depend on time is S.S.! ma K non può dipendere dal tempo in S.S. $\Rightarrow \boxed{k=0}$

• $q_0 = -\frac{\ddot{a}\,a}{\dot{a}^2} = ?$ $\dot{a} = Ha$, $\ddot{a} = H\dot{a}$ → $q = -\frac{H\dot{a}\cdot a}{H^2a^2} = -\frac{H^2a^2}{H^2a^2} = -1$

• $d_H = ?$ $d_H = a(t)\int_{t_i}^{t} \frac{c\,dt'}{a(t')} = \cancel{A}e^{Ht}\int_{t_i}^{t}\frac{c\,dt'}{\cancel{A}e^{Ht'}} = e^{Ht}\cdot\frac{c}{H}\int_{Ht_i}^{Ht} e^{-Ht'}\,d(t'H)$

$d_H(t) = \frac{c}{H}e^{Ht}\cdot\left[e^{-Ht_i} - e^{-Ht}\right] = \frac{c}{H}\left[e^{H(t-t_i)} - 1\right]$ se $t_i \to -\infty$

$\boxed{\int_{t_i}^{t} e^{-x}dx = -e^{-x}\Big|_{t_i}^{t}}$

$d_H(t) \to \infty$ $\tilde{\exists}\, d_H$

• $d_E = ?$

$d_E(t) = a(t)\int_{t}^{\infty}\frac{c\,dt'}{a(t')} = \cancel{A}e^{Ht}\int_{t}^{\infty}\frac{c\,dt'}{\cancel{A}e^{Ht'}} = \frac{c}{H}e^{Ht}\left[e^{-Ht} - 0\right] = \frac{c}{H}$ finito

$\exists\, d_E$

• $r(z)$: $k=0$ $H = H_0E(z)$ con $E(z) \equiv 1$

$a_0 r(z) = \frac{c}{H}\int_0^z dz' \to \boxed{a_0 r(z) = \frac{cz}{H}}$

$\boxed{k=0 \quad a_0 r(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}}$

- $\underline{d_L(z)} = a_0\, r(1+z) = \dfrac{c\, z(1+z)}{H}$

- $\underline{d_A(z)} = \dfrac{d_L}{(1+z)^2} = \dfrac{c\, z}{(1+z)\, H}$

- $\underline{d\theta} = \dfrac{D}{d_A} = \dfrac{D\,(1+z)\, H}{c\, z}$

- $\underline{\dfrac{dN}{dz}} = \dfrac{c}{H_0}\, d\Omega \cdot \dfrac{n(z)\left[a_0\, r(z)\right]^2}{(1+z)^3\, E(z)}$ $n(z) = \tilde{n}$ costante, $E(z) \equiv 1$

$$= \frac{c}{H_0}\, \frac{\tilde{n}\, d\Omega}{(1+z)^3} \cdot \frac{c^2 z^2}{H_0^2} = \left(\frac{c}{H_0}\right)^3 \frac{\tilde{n}\, z^2}{(1+z)^3}\, d\Omega$$

---

no creation or destruction of ~~matter~~

Se non c'è creazione o distruzione di materia.

$$\rho a^3 = \text{cost.} = K \rightarrow \rho(t) = \frac{K}{a^3(t)} = K\, a^{-3}(t)$$

$$\dot{\rho} = -3 a^{-4} \cdot K\, \dot{a} = -3\, \frac{K}{a^3} \cdot \frac{\dot{a}}{a} = -3 H \rho$$

$$\boxed{\dot{\rho} = -3 H \rho}$$

If there's creation

Se c'è creazione: $\dot{\rho} = -3H\rho + \dot{\rho}_{cr}$

In

Nello S.S. $\dot{\rho} = 0 \rightarrow \dot{\rho}_{cr} - 3H\rho = 0 \rightarrow \dot{\rho}_{cr} = 3H\rho$

$$\dot{\rho}_{cr} = 3H\, \rho_{cr} = 3 \cdot \frac{h}{3 \times 10^{17}\, s} \cdot 2 \times 10^{-29}\, h^2\, g/cm^3 =$$

$$= 2 \times 10^{-46}\, h^3\, g\, cm^{-3}\, s^{-1}$$

$n_H = \rho / m_H$ $m_H = 1.6 \times 10^{-24}\, g$

$\downarrow$

$\sim 4\, h^3\, \dfrac{\text{atomi di H}}{\text{km}^3\ \text{anno}}$ (atoms of H / year) $\rightarrow (h \sim 0.7) \sim 1\, \dfrac{\text{atomi di H}}{\text{km}^3 \cdot \text{anno}}$ (year)

Flux

Assume now $L(\nu) \neq L\,\delta(\nu-\nu_e)$, but $L(\nu) = L \cdot \varphi(\nu)$

such that $\int_0^\nu \varphi(\nu)\,d\nu \equiv 1$ [extended spectrum of source]

We receive radiation in a spectral range $\Delta\nu_0$ $[\nu_0 \le \nu \le \nu_0 + \Delta\nu_0]$

which was emitted $[\nu_e = \nu_0(1+z)]$ in a range $\Delta\nu_e = \Delta\nu_0(1+z) > \Delta\nu_0$

So what we receive in $\Delta\nu_0$, $F(\nu_0)\cdot\Delta\nu_0$ was emitted at $\nu_e = \nu_0(1+z)$

and in a larger spectral range $\Delta\nu_e$

$$F(\nu_0)\Delta\nu_0 = \frac{L\,\varphi[\nu_0(1+z)]}{4\pi\, a_0^2 r_{(z)}^2 (1+z)^2} \cdot \underbrace{\Delta\nu_e}_{\Delta\nu_0(1+z)}$$

and then the observed flux per unit frequency range (Hz^{-1}) is

$$\boxed{F(\nu_0) = \frac{L \cdot \varphi[\nu_0(1+z)]}{4\pi\, a_0^2 r_{(z)}^2 (1+z)}}$$

How to shift from $F(\nu_0)$ to $F(\lambda_0)$?

$\lambda\nu = c \to \nu = \frac{c}{\lambda} \to d\nu = -\frac{c}{\lambda^2}d\lambda \to |d\nu| = \frac{c}{\lambda^2}|d\lambda|$

So $\varphi[\nu_0(1+z)] = \varphi(\nu_e)$

$$\varphi(\nu_e)\,d\nu_e = \tilde{\varphi}(\lambda_e)\,d\lambda_e \to \varphi(\nu_e)\cdot\frac{c}{\lambda_e^2}\,d\lambda_e = \tilde{\varphi}(\lambda_e)\,d\lambda_e$$

$\boxed{\tilde{\varphi}(\lambda_e) = \frac{c}{\lambda_e^2}\varphi(\nu_e) = \frac{\nu_e^2}{c}\varphi(\nu_e)}$ and the same for all spectral functions

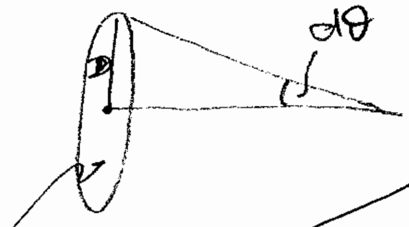
- $F(\nu_0)\,d\nu_0 = F(\lambda_0)\,d\lambda_0 \to F(\nu_0)\cdot\frac{c}{\lambda_0^2} = F(\lambda_0)$ — $\lambda_0 = \lambda_e(1+z)$

$$F(\nu_0)\cdot\frac{c}{\lambda_e^2(1+z)^2} = F(\lambda_0)$$

$\tilde{\varphi}(\lambda_e) = \tilde{\varphi}[\lambda_0/(1+z)]$

$$F(\lambda_0) = \frac{L\,\varphi[\nu_e]\cdot\frac{c}{\lambda_e^2}}{4\pi\, a_0^2 r^2(z)\cdot(1+z)}\cdot\frac{1}{(1+z)^2}$$

$$\boxed{F(\lambda_0) = \frac{L \cdot \tilde{\varphi}\left[\frac{\lambda_0}{1+z}\right]}{4\pi\, a_0^2 r^2(z)(1+z)^3}}$$



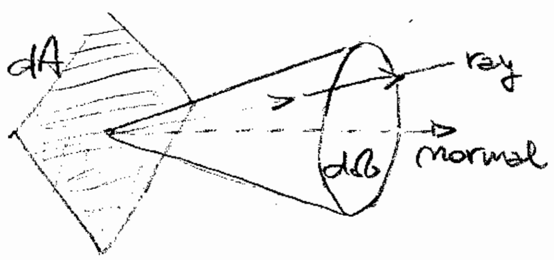
$d_A = \frac{D}{d\theta}$ $\quad d\theta = \frac{D}{d_A} = \frac{D(1+z)}{a_0 r}$

$$\to d\Omega = \frac{dA}{d_A^2} = \frac{\pi D^2}{4}\cdot\frac{(d\theta)^2}{D^2} = \frac{\pi}{4}(d\theta)^2$$

$A = \pi\left(\frac{D}{2}\right)^2$ at the source

Specific intensity or brightness

Area dA normal to the direction of a light ray, consider all rays passing through dA, whose direction is within a solid angle $d\Omega$ of the given ray.

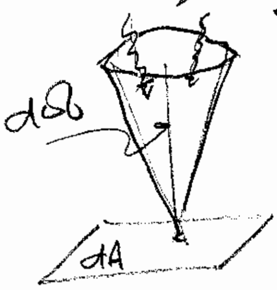


The energy crossing dA in time dt, in frequency range $d\nu$, within the solid angle $d\Omega$ is defined by the relation

$$dE = I_\nu \, dA \, dt \, d\nu \, d\Omega$$

I_ν : brightness or specific intensity

$$I_\nu = \frac{dE}{dt\, d\nu\, dA\, d\Omega} \quad (\text{erg s}^{-1}\,\text{Hz}^{-1}\,\text{cm}^{-2}\,\text{sterad}^{-1})$$



If the Flux $F(\nu)$ [erg s^{-1} Hz^{-1} cm^{-2}] arriving around the normal is divided by the solid angle, we get the brightness

$$I_\nu = \frac{F(\nu)}{d\Omega}$$

$$\left\{ F(\nu_0) = \frac{L\,\varphi[\nu_0(1+z)]}{4\pi a_0^2 r^2 (1+z)} \right.$$

Then

$$I(\nu_0) = \frac{F(\nu_0)}{d\Omega} = \frac{L\,\varphi[\nu_0(1+z)]}{(1+z)\,4\pi a_0^2 r^2}\cdot\frac{4}{\pi}\,\frac{a_0^2 r^2}{D^2(1+z)^2} = \frac{L\,\varphi[\nu_0(1+z)]}{\pi^2 D^2 (1+z)^3}$$

$$\pi^2 D^2 = 4\pi\cdot\pi\left(\frac{D}{2}\right)^2$$

At the source

$$\frac{L\,\varphi[\overbrace{\nu_0(1+z)}^{\nu_e}]}{\underbrace{4\pi}_{\text{Solid angle}}\cdot\underbrace{\pi\left(\frac{D}{2}\right)^2}_{\text{Area of source}}} = \text{erg s}^{-1}\,\text{Hz}^{-1}\,\text{cm}^{-2}\,\text{sterad}^{-1} = I(\nu_e) \text{ at the source}$$

So

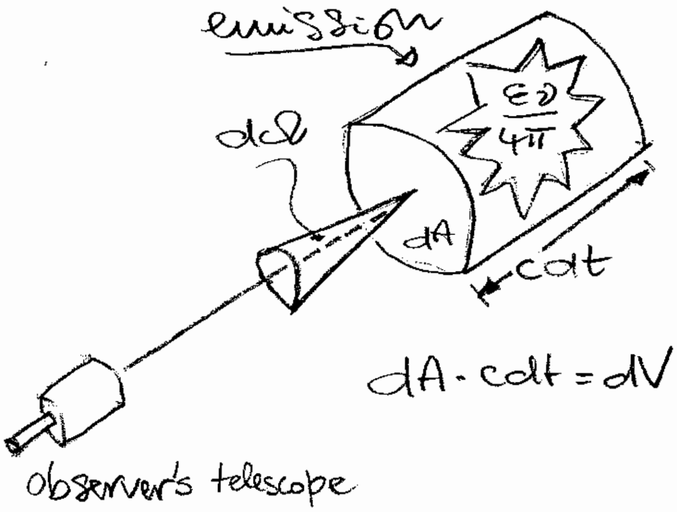
$$I(\nu_0) = \frac{I(\nu_e)}{(1+z)^3} \Big/ \cdot \frac{1}{\nu_0^3} \rightarrow \frac{I(\nu_0)}{\nu_0^3} = \frac{I(\nu_e)}{\underbrace{\nu_0^3(1+z)^3}_{\nu_e^3}} \rightarrow \boxed{\frac{I(\nu)}{\nu^3} = \text{const}}$$

along the ray

[If the space is not expending, no redshift, $1+z \equiv 1$ and $I(\nu_0) = I(\nu_e)$ along the ray]

This result is independent of the cosmological model!

- Suppose now there is a diffuse source of radiation, emitting isotropically ε_ν erg s^{-1} cm^{-3} Hz^{-1}, so for unit solid angle: $\frac{\varepsilon_\nu}{4\pi}$ erg s^{-1} cm^{-3} Hz^{-1} $sterad^{-1}$



$dA \cdot cdt = dV$

The energy crossing dA in dt, within $d\Omega$, in $d\nu$ is

$$dE \overset{def}{\equiv} dI_\nu \, dA \, dt \, d\Omega \, d\nu$$

$$= \underbrace{\frac{\varepsilon_\nu}{4\pi} \cdot (\underbrace{dA \cdot cdt}_{dV})}_{\text{energy crossing dA in time dt, erg } s^{-1} Hz^{-1} sterad^{-1}} dt \, d\nu \, d\Omega$$

From the relation we get

$$dI(\nu_{em}) = \frac{c\,\varepsilon(\nu_{em})}{4\pi} dt$$

At the arrival

$$dI(\nu_0) = \left(\frac{\nu_0}{\nu_{em}}\right)^3 dI(\nu_{em}) = \frac{1}{(1+z)^3} \cdot \frac{c\,\varepsilon[\nu_0(1+z)]}{4\pi} dt$$

since $\frac{dt}{dz} = -\frac{1}{(1+z)H_0 E(z)}$

the observed brightness of the sky in the given direction, for sources between z_1 and z_2, is then

$$\boxed{I(\nu_0) = \frac{c}{4\pi H_0} \int_{z_1}^{z_2} \frac{\varepsilon[\nu_0(1+z), z]}{(1+z)^4 E(z)} dz}$$

where ε depends also on z.
This relation allows estimation, for any cosmological model, the contribution of a kind of source to the cosmic background.

In the "classical", newtonian, static universe, I_ν is constant along the path, ε does not depend on time, so the brigthness of the sky produced between time t_i and t_0 is

$$I_{cl}(\nu_0) = \frac{c\,\varepsilon(\nu_0)}{4\pi}(t_0 - t_i)$$

If the universe in infinite in time, $t_i \to -\infty$, $I_{cl}(\nu) \to \infty$.
This is the so-called Olbers paradox: Why is the sky dark in the night? The paradox is solved if the universe is finite in time or in space (if R is the size of the universe, $t_i = R/c$), or if one takes in to accont the finite life of stars. Expansion helps to solve the problem, but is not the key point.

Absorption of radiation

Absorption of radiation is usually measured by a quantity named optical depth τ_ν. If σ_ν is the absorption cross section, n the proper number density of absorbers, $\tau_\nu \sim n\,\sigma_\nu \cdot \ell$ [ℓ = length of the path, neglecting cosmology]
To use cosmology, remember that a photon received with frequency ν_0 has, at redshift z, a freq. $\nu_0(1+z)$. In $t \mapsto t+dt$, corresponding to $z \mapsto z+dz$, optical depth [which is measured starting from the observer] grows by

$$d\tau(\nu_0) = \sigma[\nu_0(1+z)] \cdot n(z) \cdot c\,dt =$$
$$= \sigma[\nu_0(1+z)] \cdot n(z) \cdot c \cdot \frac{dt}{dz} \cdot dz$$

$$\frac{dt}{dz} = -\frac{1}{(1+z)\,H_0 E(z)}$$

$$\tau(\nu_0) = \int_t^{t_0} \sigma n c\, dt = -\int_z^0 \frac{\sigma n c}{(1+z) H_0 E(z)} dz$$

So

$$\boxed{\tau_{\nu_0}(z) = \frac{c}{H_0} \int_0^z \frac{\sigma[\nu_0(1+z')]\, n(z')}{(1+z')\, E(z')} dz'}$$

We apply this to absorption by neutral hydrogen (HI) in the Lyman-α line (1216 Å), $\nu_\alpha = 2.46 \times 10^{15}$ Hz

$$\sigma_\nu \cong \frac{\pi e^2}{m_e c} f \cdot \delta(\nu - \nu_\alpha)$$

$f \simeq 0.416$ oscillator strength (quantum correction to classical treatment of transitions), m_e electron mass, line profile $\sim$ Dirac δ

$$\tau_{\nu_0}(z) = \frac{c}{H_0} \frac{\pi e^2 f}{m_e c} \int_0^z \frac{\delta[\nu_0(1+z') - \nu_\alpha]\, n_{HI}(z')}{(1+z')\, E(z')} dz' \qquad z' \to (1+z')\nu_0$$

$$= \frac{\pi e^2 f}{H_0 m_e} \int_{\nu_0}^{\nu_0(1+z)} \frac{\delta[\nu_0(1+z') - \nu_\alpha]\, n_{HI}(z')}{E(z') \cdot (1+z') \cdot \nu_0} d[\nu_0(1+z')]$$

[ν_α must be in the range $\nu_0 \mapsto \nu_0(1+z)$]

Remember $\int \delta(x - x_0) f(x) dx = f(x_0)$

The contribution from the integrand comes only (given ν_0) from the redshift $\tilde{z}$ such that $\nu_0(1+\tilde{z}) = \nu_\alpha$

$$\tau\left(\nu_0 = \frac{\nu_\alpha}{1+\tilde{z}}\right) = \frac{\pi e^2 f}{H_0 m_e} \frac{n_{HI}(\tilde{z})}{E(\tilde{z})\, \underbrace{\nu_0(1+\tilde{z})}_{\nu_\alpha}} \simeq 4 \times 10^{10} h^{-1} \frac{n_{HI}(\tilde{z})}{E(\tilde{z})} \quad (n_{HI} \text{ in } cm^{-3})$$

or

$$n_{HI}(\tilde{z}) \simeq 2.5 \times 10^{-11} h\, E(\tilde{z})\, \tau\left(\nu_0 = \frac{\nu_\alpha}{1+\tilde{z}}\right) \quad cm^{-3}$$

Note: changing ν_0 we can test absorption at all redshifts

Density of baryons (supposed to be all hydrogen)

$$n_b(z) = (1+z)^3 \cdot \frac{\Omega_b \rho_{0c}}{m_H} = (1+z)^3 \frac{0.02 h^{-2} \cdot 2 \times 10^{-29} h^2}{1.6 \times 10^{-24}} \ cm^{-3}$$

$$\cong 2.5 \times 10^{-7} (1+z)^3$$

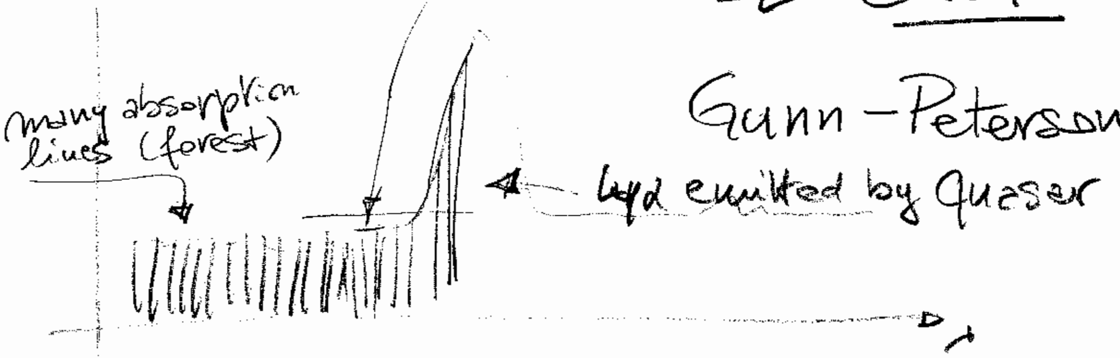
If $z > 1$ we use EdS: $E(z) \sim (1+z)\sqrt{1+\Omega_M z}$ | Flux | 6

$$\frac{n_{HI}(\tilde{z})}{n_b(\tilde{z})} \simeq 10^{-4} h \frac{\sqrt{1+\Omega_M \tilde{z}}}{(1+\tilde{z})^2} \cdot \tau \left[\nu_0 = \frac{\nu_0}{1+\tilde{z}}\right]$$

For $z \sim 3$ $h \sim 0.7$ $\Omega_M \sim 0.3$ $\left.\frac{n_{HI}}{n_b}\right|_{z=3} \sim 6 \times 10^{-6} \cdot \tau$

$\ll 1$ from observations

Ly-α forest of quasars



Gunn-Peterson test

The intergalactic medium (IGM) is highly ionized for $z \lesssim 6$. But the universe became neutral at $z \sim 1000$ (recombination); this period (Dark Ages) ends when the first stars form and ionize IGM at $z \sim 10$.

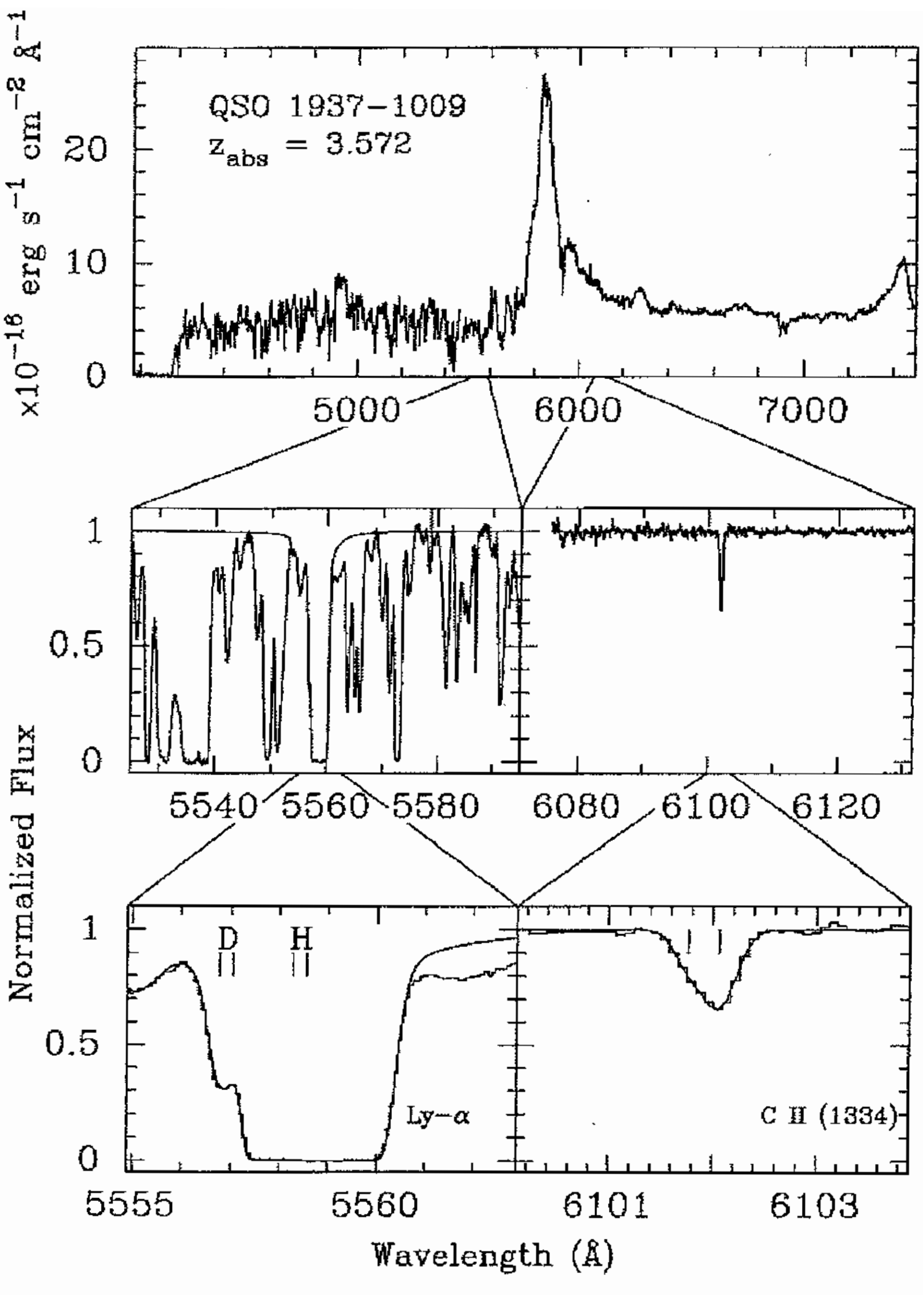


Fig. 6.8 The spectrum of a quasar at $z \sim 3.79$ showing Ly-α emission at 580 nm and, blueward of this line, the "forest" of Ly-α absorption lines by intervening atomic hydrogen [133]. The zoom on the left shows Ly-α hydrogen and deuterium absorption by a cloud at $z = 3.572$. The deuterium line is shifted with respect to the hydrogen line because the atomic energy levels are proportional to the reduced electron–nucleus mass. The ratio between the hydrogen and deuterium absorption can be used to determine the two abundances within the cloud. Courtesy of D. Tytler

Djorgouski et al., 2001

$z = 5.73$

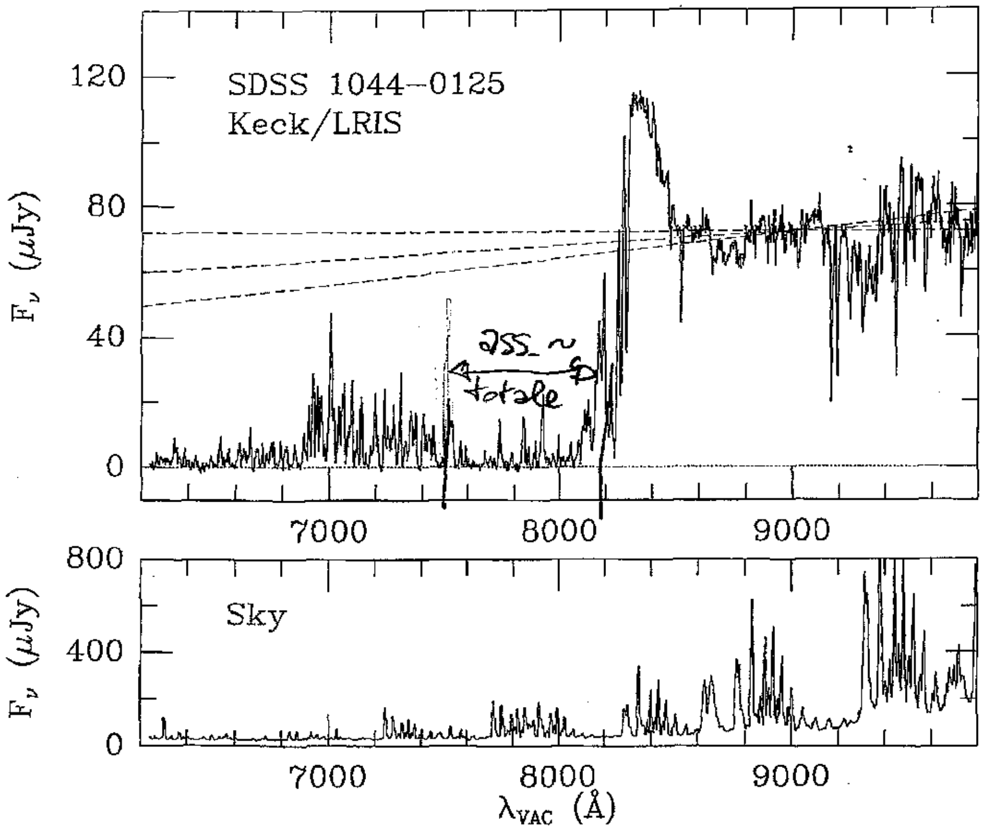


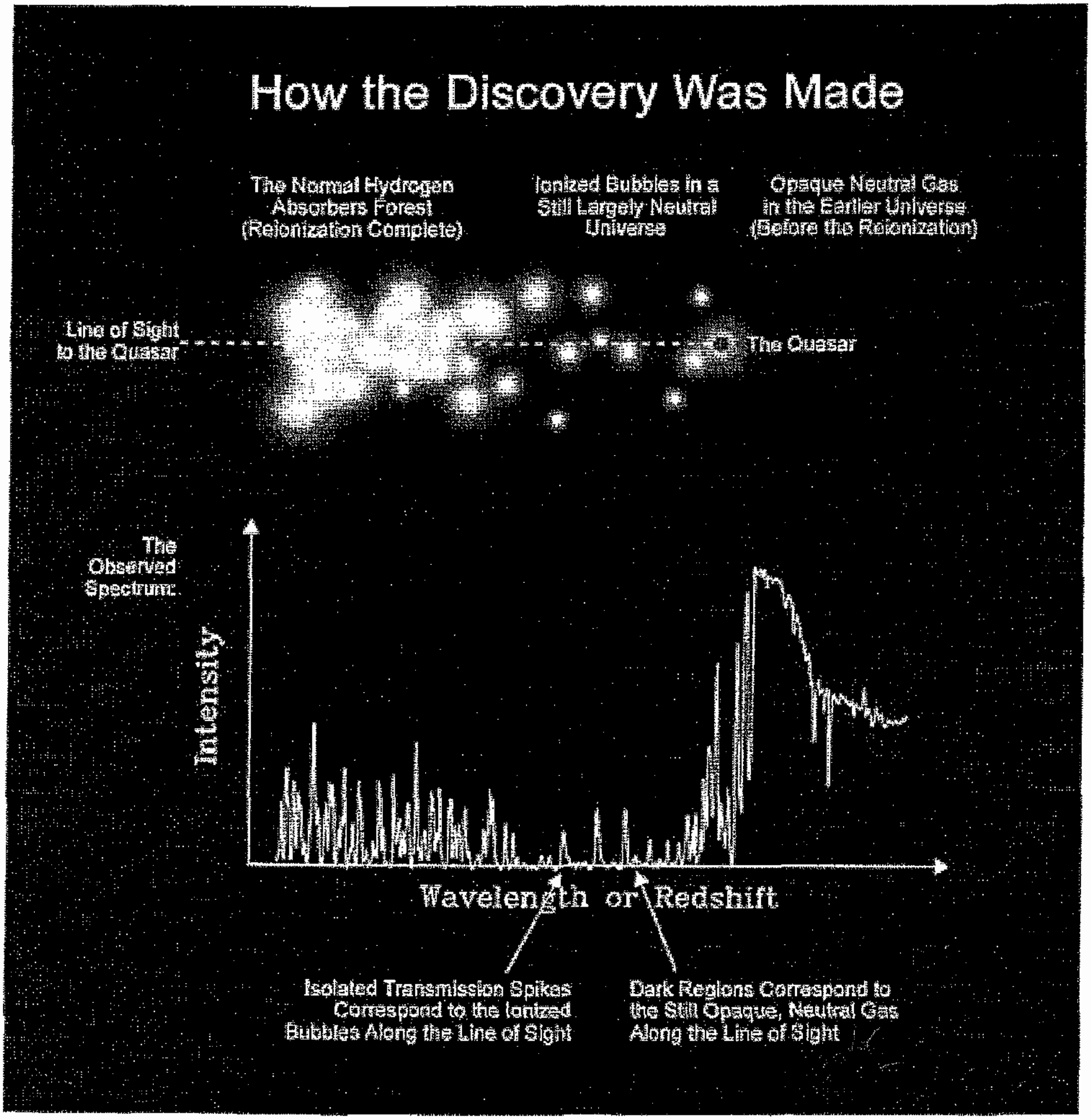
FIG. 1.— Spectrum of SDSS 1044–0125 obtained with LRIS (top), and the corresponding night sky (bottom). The three dashed lines represent a plausible range of the unabsorbed quasar power-law continua.

Steidel, C., Adelberger, K., Giavalisco, M., Dickinson, M., & Pettini, M. 1999, ApJ, 519, 1
Stern, D. & Spinrad, H. 1999, PASP, 111, 1475
Stern, D., Spinrad, H., Eisenhardt, P., Bunker, A., Dawson, S., Stanford, A., & Elston, R. 2000, ApJ, 533, L75
Zheng, W., *et al.* (the SDSS Collaboration) 2000, AJ, 120, 1607
Umemura, M., Nakamoto, T., & Susa, H. 2001, preprint (astro-ph/0108176)

$$7500\,\text{Å} = (1+z)\cdot \underbrace{1216\,\text{Å}}_{\text{Ly}\alpha} \rightarrow z \sim 5.2$$

Qualcosa succede a $z \sim 5.5-6$
A $z \sim 6$ fase finale della reionizzazione

Something happens at $z \sim 5.5-6$
At $z \sim 6$ final phase of reionization


How the Discovery Was Made
The Normal Hydrogen Absorbers Forest (Reionization Complete)
Ionized Bubbles in a Still Largely Neutral Universe
Opaque Neutral Gas in the Earlier Universe (Before the Reionization)
Line of Sight to the Quasar
The Quasar
The Observed Spectrum:
Intensity
Wavelength or Redshift
Isolated Transmission Spikes Correspond to the Ionized Bubbles Along the Line of Sight
Dark Regions Correspond to the Still Opaque, Neutral Gas Along the Line of Sight