Lecture 10

Dynamic Games of Incomplete Information: signalling games

Guessing Game

Two persons play the following game.

Person i receives a signal s_i that can be either 0 or 1, $i \in \{1, 2\}$.

Signals are independently distributed and each signal's probability of being 1 is p>0.5.

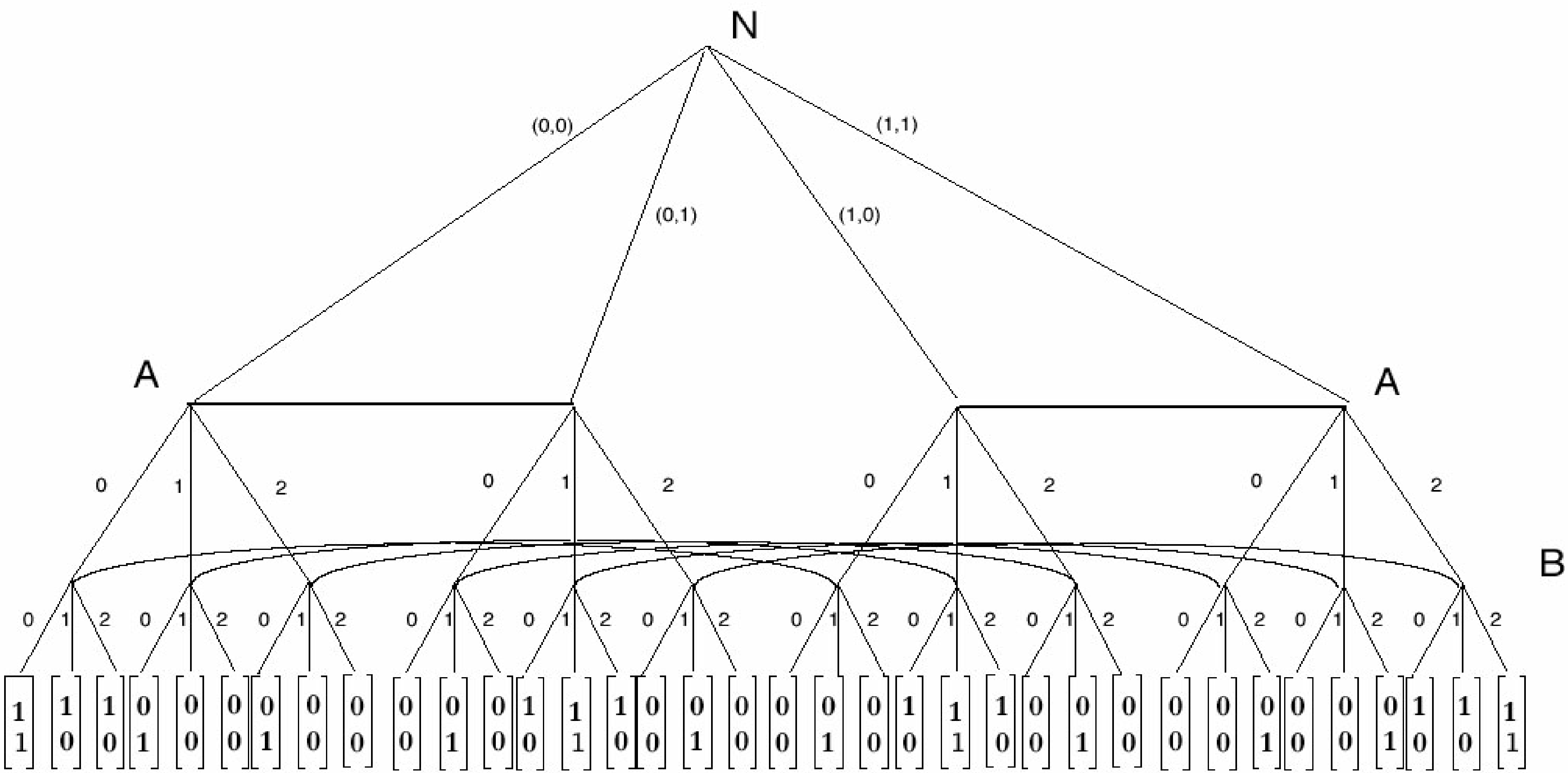
Each person observes only his signal but not the signal of the other persons.

The probability distributions of the signals are common knowledge.

Each person has to guess the sum of the signals.

If the guess is correct, the player gets £1, otherwise she receives 0

Person 1 guesses first, person 2 observes the guess of player 1 before making his guess.



N
(0,0)
(0,1)
(1,0)
(1,1)
A
A
B

Payoff of Person 1 does not depends on the player 2's strategies

"Requirement 2": Person 1 has to choose the strategy with the higher expected value.

Person 1 has two information sets: $s_1 = 0$ and $s_1 = 1$.

Suppose $s_1 = 0$,the expected value of each possible guess are:

$E(g_1 = 0) = Pr(s_2 = 0) = 1 - p$

$E(g_1 = 1) = Pr(s_2 = 1) = p$

$E(g_1 = 2) = Pr(s_2 = 2) = 0$

Suppose $s_1 = 1$, the expected value of each possible guess are:

$E(g_1 = 0) = Pr(s_2 = -1) = 0$

$E(g_1 = 1) = Pr(s_2 = 0) = 1 - p$

$E(g_1 = 2) = Pr(s_2 = 1) = p$

Then in PBE person 1 plays the strategy $\{1, 2\}$ i.e $g_1 = 1$ when $s_1 = 0$ and $g_1 = 2$ when $s_1 = 1$.

Person 2 has 6 information sets: all possible combination between g_1 and s_2. Let (g_1, s_2) denotes an information set, then the person 2's information sets are:

$(0,0), (0,1), (1,0), (1,1), (2,0), (2,1),$

Only the following four information sets are on the equilibrium path $(1,0), (1,1), (2,0), (2,1)$, (because in equilibrium person 1 plays either 1 or 2).

In these information sets beliefs are determined by Bayes rule, in the others information sets beliefs can be arbitrary.

Let be $b_2(g_1, s_2)$ the belief of person 2 about the signal s_1, i.e. the probability person 2 assigns to the event $s_1 = 1$, in the information set (g_1, s_2).

In the information sets on the equilibrium path the player 2's beliefs are:

$$b_2(1,0) = b_2(1,1) = 0$$
$$b_2(2,0) = b_2(2,1) = 1$$

Consider information set $(1,0)$(i.e. $\boldsymbol{g_1 = 1}$ **and** $\boldsymbol{s_2 = 0}$)

the expected value of each possible guess are

$E(g_2 = 0) = Pr(s_1 = 0) = \left(1 - b_2(1,0)\right) = 1$

$E(g_2 = 1) = Pr(s_1 = 1) = b_2(1,0) = 0$

$E(g_2 = 2) = Pr(s_1 = 2) = 0$

<u>Then $g_2 = 0$ has the higher expected value</u>

Consider information set $(1,1)$(i.e. $\boldsymbol{g_1 = 1}$ **and** $\boldsymbol{s_2 = 1}$)

the expected value of each possible guess are:

$E(g_2 = 0) = Pr(s_1 = -1) = 0$

$E(g_2 = 1) = Pr(s_1 = 0) = \left(1 - b_2(1,1)\right) = 1$

$E(g_2 = 2) = Pr(s_1 = 1) = b_2(1,1) = 0$

<u>Then $g_2 = 1$ has the higher expected value</u>

Consider information set $(2,0)$(i.e. $\boldsymbol{g_1} = \mathbf{2}$ **and** $\boldsymbol{s_2} = \mathbf{0}$)

the expected value of each possible guess are

$E(g_2 = 0) = Pr(s_1 = 0) = \left(1 - b_2(2,0)\right) = 0$

$E(g_2 = 1) = Pr(s_1 = 1) = b_2(2,0) = 1$

$E(g_2 = 2) = Pr(s_1 = 2) = 0$

<u>Then $g_2 = 1$ has the higher expected value</u>

Consider information set $(2,1)$(i.e. $\boldsymbol{g_1} = \mathbf{2}$ **and** $\boldsymbol{s_2} = \mathbf{1}$)

the expected value of each possible guess are:

$E(g_2 = 0) = Pr(s_1 = -1) = 0$

$E(g_2 = 1) = Pr(s_1 = 0) = \left(1 - b_2(2,1)\right) = 0$

$E(g_2 = 2) = Pr(s_1 = 1) = b_2(2,1) = 1$

<u>Then $g_2 = 2$ has the higher expected value</u>

Then in a BPE,

Person 2 plays $g_2 = s_2$ after observing $g_1 = 1$

Person 2 plays $g_2 = s_2 + 1$ after observing $g_1 = 2$

Out of equilibrium path there are arbitrary beliefs and the strategy must be optimal given the (arbitrary) beliefs.

For example $b_2(0,0) = b_2(0,1) < 0.5$ (for example 0)

In this case for player 2 is optimal to play $g_2 = s_2$

PBE

Person 1 plays $s_1 + 1$

Person 2 plays:

$g_2 = s_2$ when $g_1 \leq 1$

$g_2 = s_2 + 1$ when $g_1 = 2$

$b_2(0,0) = b_2(0,1) < 0.5$

$b_2(1,0) = b_2(1,1) = 0$

$b_2(2,0) = b_2(2,1) = 1$

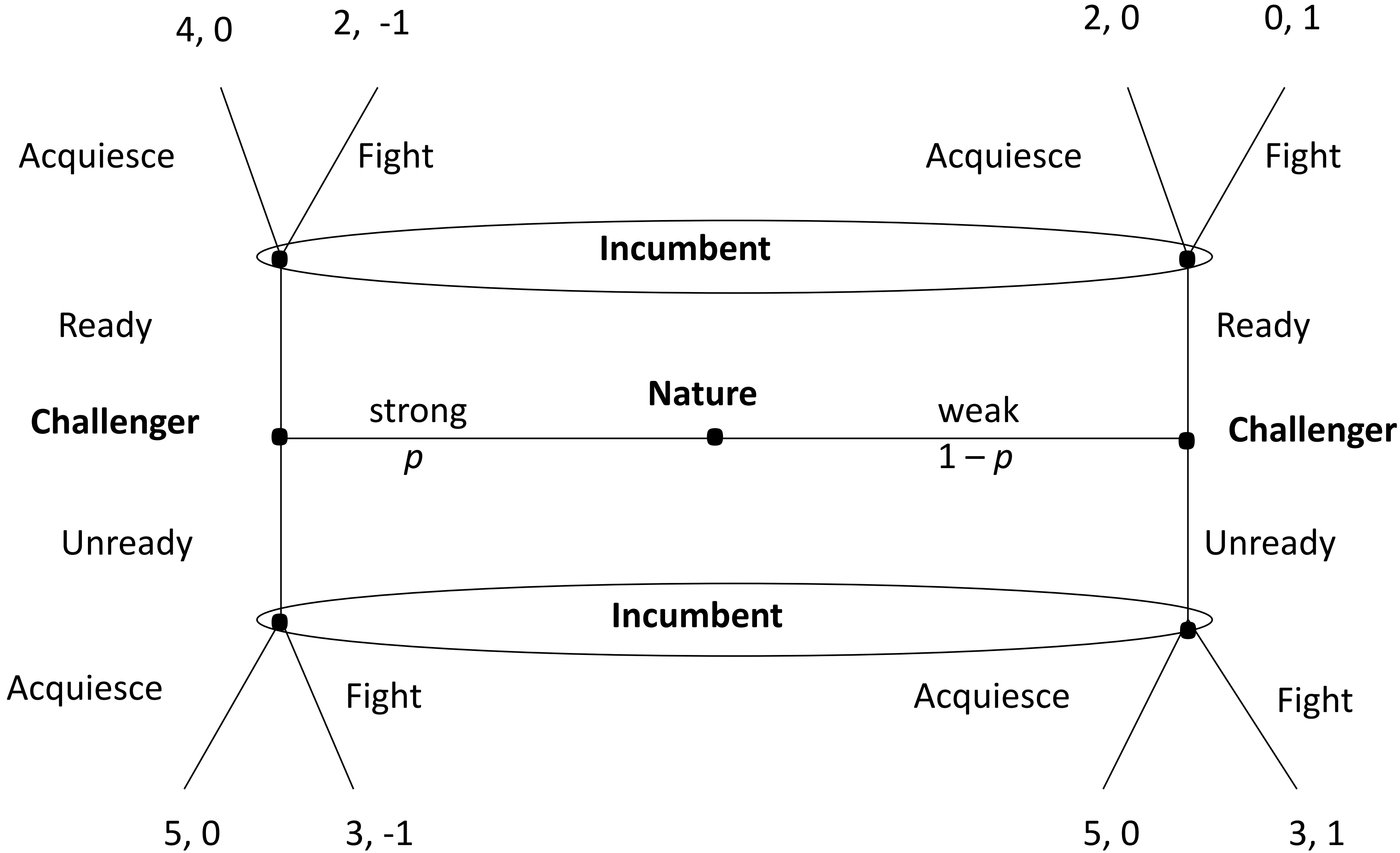
Two-players Signalling Games

- Sequential game where player 1 is the first mover.
- Player 1 (the sender) is informed about a variable relevant to both her and player 2 (that is uninformed)
- Player 1 takes an action that is observed by player 2 (receiver)
- Observing the action of player 1, player 2 receives some information about the relevant (unobserved) variable
- Player 2 takes an action that affect the payoff of both players.
- Player 1 can play

  A) an action in according to the observed variable (**separating strategy**)

  B) an action that is independent from the observed variable (**pooling strategy**)

  In the case A player's 1 action conveys some info to player 2, no in case B.



Entry as signalling game

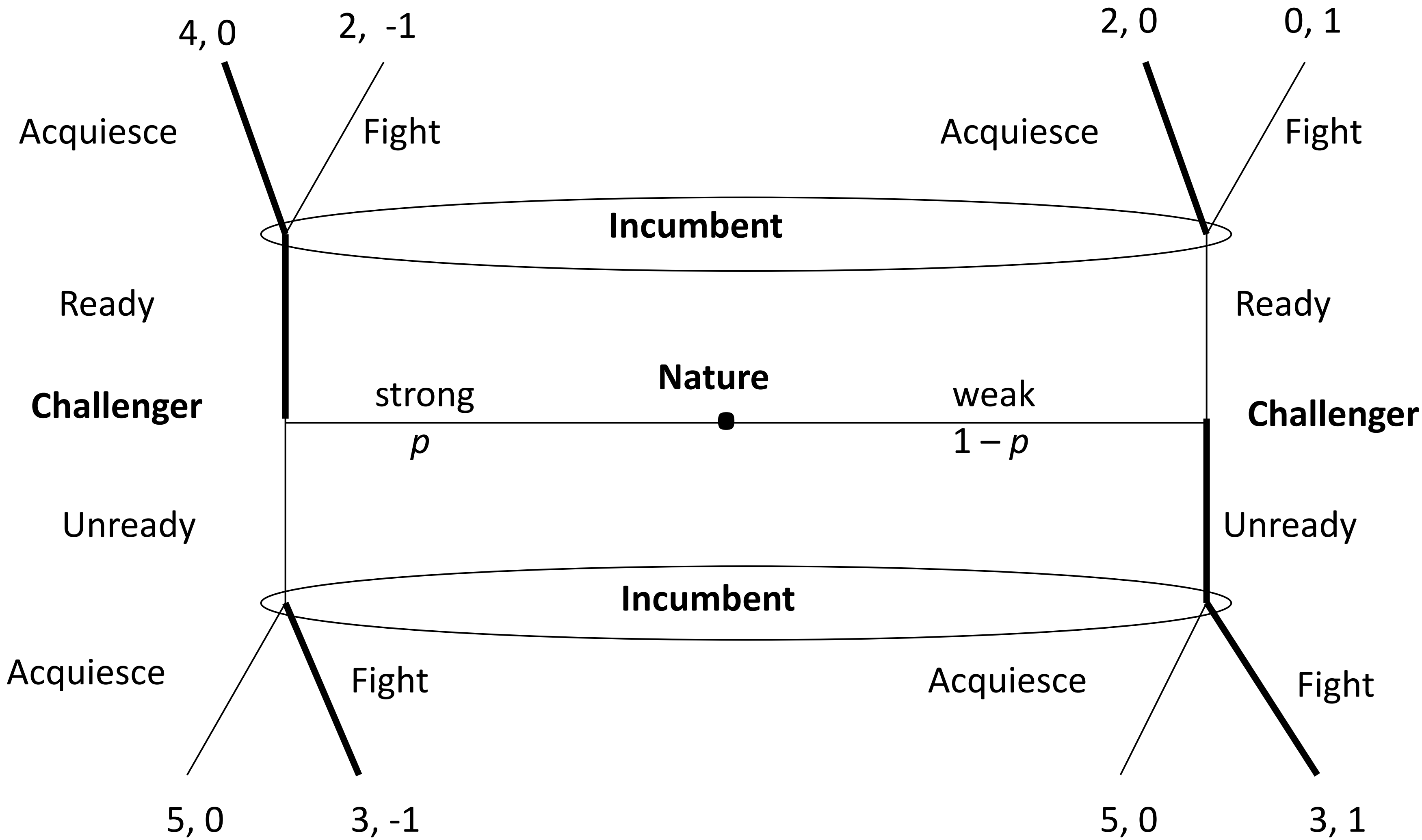
Note first that the weak challenger prefers *unready* whatever action the incumbent takes, so in any PBE the weak challenger must choose *unready*

That leaves two possibilities for equilibria:

1. Strong challenger chooses *ready,* weak challenger chooses *unready (**separating** strategy)*
2. Challenger chooses *unready* in both cases *(**pooling** strategy)*

<u>Consider 1: (*ready, unready*)</u>

- both information sets of incumbent are reached.
- beliefs are: Pr(*strong* | *ready*) =1   Pr(*weak* | *unready*) = 1
- incumbent chooses *acquiesce* after *ready* and *fight* after *unready*
- no type of player 1 has an incentive to deviate
- so this is a (**separating**) PBE (for any *p*)



PBE: Challenger : *ready* if strong; *unready* if weak

Incumbent *: acquiesce* after *ready, fight* after *unready*

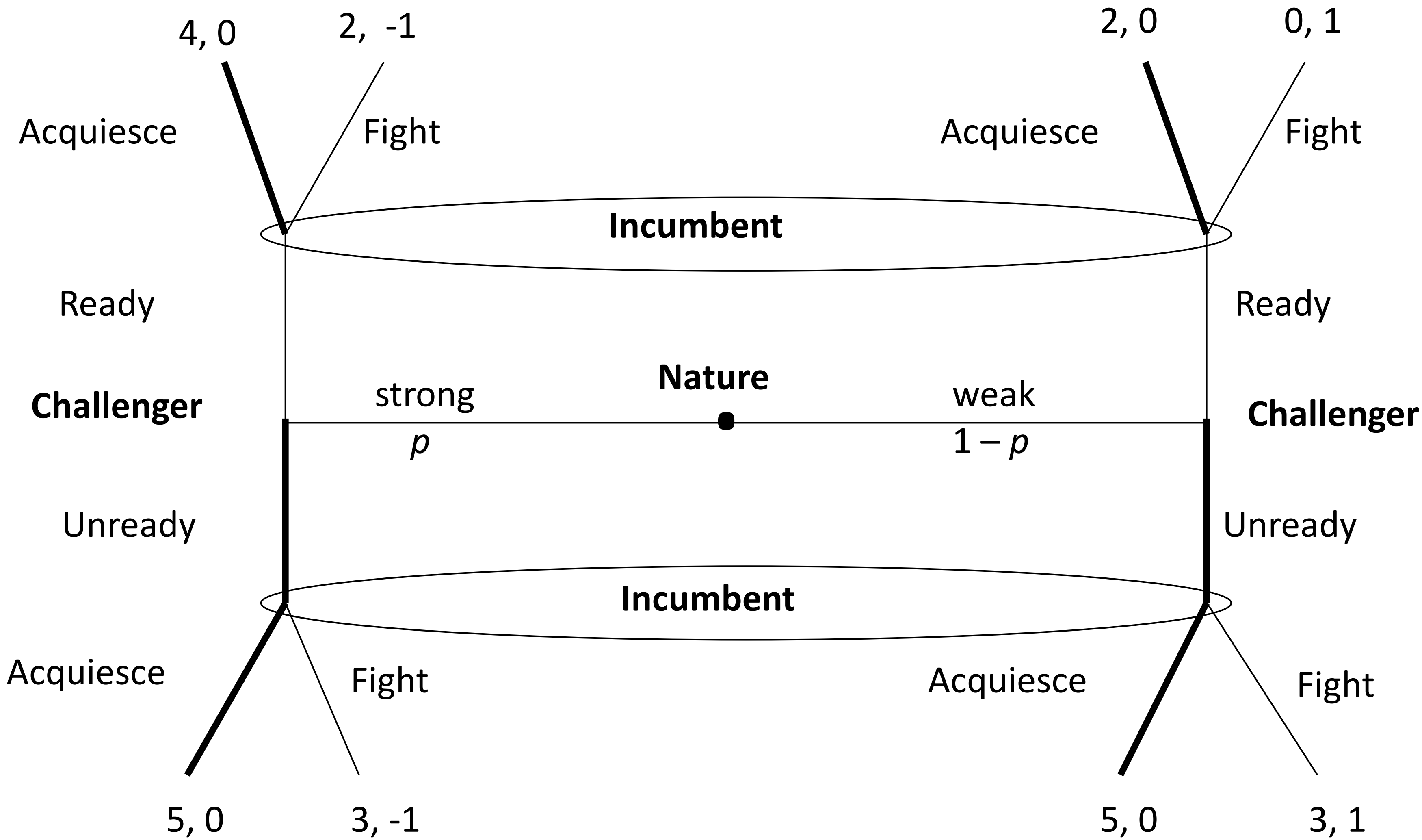
Incumbent' s beliefs:

Pr(*strong* | *ready*) =1   Pr(*weak* | *unready*) = 1

Consider 2: (*unready, unready*)

- only the information set (*unready*) of incumbent is reached
- beliefs are:
  - $\Pr(strong \mid unready) = p$;
  - $\Pr(weak \mid unready) = 1 - p$
- $E(A \mid unready) = 0$;
- $E(F \mid unready) = -p + 1 - p = 1 - 2p$
- $E(A \mid unready) \geq E(F \mid unready) \Leftrightarrow 0 \geq 1 - 2p \Leftrightarrow p \geq ½$
- Then for the incumbent is optimal to play *acquiesce* after *unready* if and only if $p \geq ½$
- Now assume $p \geq ½$

- We need to specify strategy given *ready*, although it's never reached, to check if challenger would want to deviate
- since probability to reach *ready* is 0, beliefs are not restricted
- If incumbent chooses *acquiesce* after *ready,* no type of challenger would want to deviate.
- This is optimal for the incumbent by a belief that challenger is strong if he play ready.
- Note challenger does not deviate even if incumbent chooses fight after ready.



For $p \geq 1/2$, there are (pooling) **PBE** where:

challenger: both types choose *unready*

incumbent: *acquiesce* after *unready* and *acquiesce* after *ready*

Incumbent' s beliefs: Pr(*strong* | *ready*) =1

Pr(*strong* | *unready*) = p

Note: there are other PBE for $p \geq ½$

- The same strategy profile but different beliefs Pr(*strong* | *ready*).
- For the incumbent is optimal to play *fight* after *ready* if Pr(*strong* | *ready*) ≤ ½

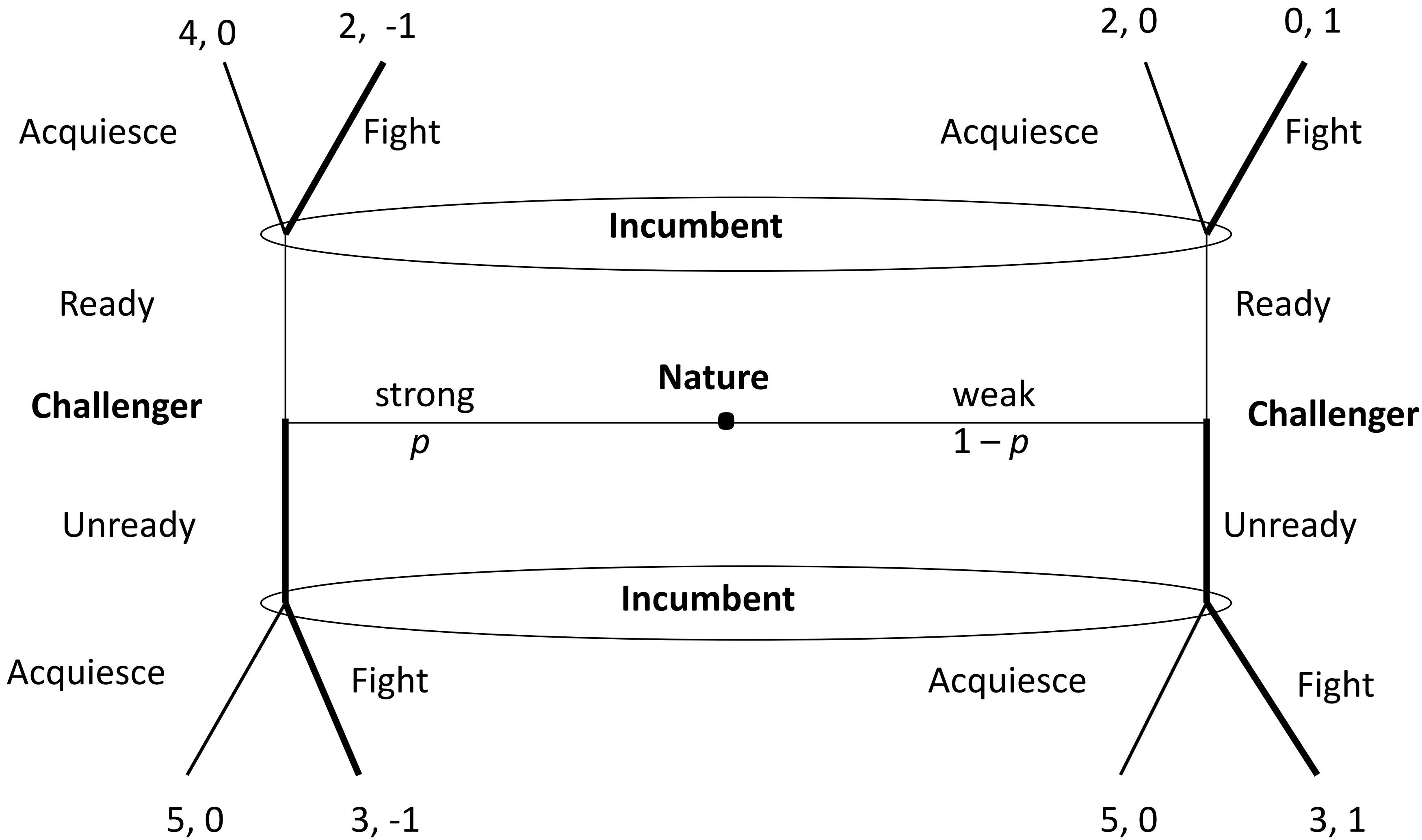
- challenger: both types choose *unready*

  Incumbent plays *fight* after *ready* and

  Pr(*strong* | *ready*) ≤ ½

  Pr(*strong* | *unready*) = p

What happen if challenger chooses (*unready*, *unready*) but $p \leq ½$? ?



For $p \leq 1/2$, there are (pooling) **PBE** where:

challenger: both types choose *unready*

incumbent: *fight* after *unready* and *fight* after *ready*

Incumbent' s beliefs: Pr(*strong* | *ready*) =0

Pr(*strong* | *unready*) = p

Cheap Talk

Two players, 1 and 2, could buy an object.

The object can be either white or black with equal probability

Each person evaluates the object in according his preferred colour:

100 if the object is of his preferred colour, 0 otherwise.

The preferred colour of each person can be either white or black with equal probability

The realizations of the object colour and those of the preferred colours are independent.

If both buy the object its price is 60, if only one buy the object its price is 30

Object black
Player 1 white
Player 2 white

Object white
Player 1 black
Player 2 black

Object black
Player 1 white
Player 2 black

Object white
Player 1 black
Player 2 white

$v_1 = 0, v_2 = 0$			
		Player 2	
		Ask	No Ask
Player 1	Ask	-60, -60	-30, 0
	No Ask	0, -30	0, 0

$v_1 = 0, v_2 = 100$			
		Player 2	
		Ask	No Ask
Player 1	Ask	-60, 40	-30, 0
	No Ask	0, 70	0, 0

Object black
Player 1 black
Player 2 white

Object white
Player 1 white
Player 2 black

Object black
Player 1 black
Player 2 black

Object white
Player 1 white
Player 2 white

$v_1 = 100, v_2 = 0$			
		Player 2	
		Ask	No Ask
Player 1	Ask	40, -60	70, 0
	No Ask	0, -30	0, 0

$v_1 = 100, v_2 = 100$			
		Player 2	
		Ask	No Ask
Player 1	Ask	40, 40	70, 0
	No Ask	0, 70	0, 0

Before to take simultaneously their actions:

1. Player 1 is informed of
   - its preferred colour,
   - player 2's colour
   - object's colour
2. Player 2 is informed about his preferred colour.
3. Player 1 sends a message to player 2 about the object's colour (" the colour of the object is …) (Note that the message can be false or true)

We explore three possible equilibria:

No informative, partial informative and full informative equilibria

No informative equilibrium

The player 1’s message does contain any information: the message is uncorrelated with object’s colour and player 1’s colour

Partial informative equilibrium

There is some level of correlation between the message and the colour of the object

Full informative equilibrium

The message is always true

We check if these equilibria exist in our game

For player 1 ***ask*** is dominant strategy when $v_1 = 100$, ***no ask*** is dominant strategy when $v_1 = 0$.

<u>No informative equilibrium</u>

Player 2's beliefs are the priors, so he believes that all four combinations (object's colour/player 1's colour) are equally probable

The expected payoffs are

$E(ask) = \frac{70+40-30-60}{4} = 5 \qquad E(no\ ask) = 0$

Then $s_2 = ask$

The strategy profile where:

a. player 1 sends messages randomly chosen, plays ***ask*** when $v_1 = 100$, ***no ask*** otherwise
b. player 2 assigns equal probability to each combination (object's colour/player 1's colour) and plays ***ask***

is a ***PBE***

<u>Full informative equilibrium</u>

The colour of the object is fully revealed to player 2, then he knows v_2 and believes that v_1 is either 0 or 100 with equal probability.

Player 1 plays ***ask*** when $v_1 = 100$, ***no ask*** otherwise

For player 2 ***no ask*** is dominant when $v_2 = 0$, ***ask*** is dominant when $v_2 = 100$.

<u>This is not an equilibrium</u>, because when $v_1 = 100$ and $v_2 = 100$ player 1 can improve his payoff (from 40 to 70) sending a false message

In this case player 2 believes (incorrectly) that $v_2 = 0$ and plays ***no ask***.

Partial informative equilibrium

when either $v_1 = 0$ or ($v_1 = 100$ and $v_2 = 0$)

message = object's colour

when $v_1 = 100$ and when $v_2 = 100$

message ≠ object's colour

Player 1 plays ***ask*** when $v_1 = 100$, ***no ask*** otherwise

$s_2 = ask$ if message = player 2's colour

$s_2 = no\ ask$ if message ≠ player 2's colour

Player 2's beliefs

when message = player 2's colour

$$\Pr(v_1 = 0, v_2 = 100) = 1$$

when message ≠ player 2's colour

$$\Pr(v_1 = 0, v_2 = 0) = \Pr(v_1 = 1, v_2 = 0) = \Pr(v_1 = 1, v_2 = 1) = \frac{1}{3}$$

message = player 2's colour

$E(ask) = 70$ and $E(no\ ask) = 0$ → ***ask*** is the best response

message ≠ player 2's colour

$E(ask) = \frac{-30-60+40}{3} < 0$
$E(no\ ask) = 0$ → ***no ask*** is the best response

<u>Incentives of player 1 to send a different message</u>

when $v_1 = 0$ a different message does not change his payoff

when $v_1 = 100$ a different message induces player 2 playing ***ask*** reducing player 1's payoff.

The strategy profile and player 2's beliefs are a PBE