The HOT BIG BANG

The ***Hot Big Bang*** model, i.e. the standard (cosmological) model, and its time evolution rests on **3 pillars**:

1. **the expansion of the Universe**

2. **the microwave background radiation** showing a **Black Body spectrum** at *2.73 K* (*CMB*), which reveals the existence of a phase in the life of the universe during which the there was **thermodynamic equilibrium**

3. The prediction of the abundances of the light elements (D, ^3He, ^4He, ^7Li), in particular helium; this **cosmological nucleosynthesis** requires also that there was an era in which $T \approx 10^9 K$

   To these facts it may be added that the predicted **age** for the universe is comparable to the age estimated directly for some types of cosmic objects (globular clusters, …), and that it is possible to give a reasonable theoretical explanation for the **formation of cosmic structures** through their gravitational collapse, starting from the perturbations in the microwave background (CMB).

   We also mention the **problems** of **flatness** and **horizon** (+ the **monopoles** problem, see below) which we have already mentioned, and whose solution is not found in the standard model of cosmic evolution, but which are solved through the mechanism of inflation.

The Standard Model of Particle Physics and beyond

We describe here some aspects of Particle Physics which are connected to cosmology and to particular epochs in the evolution of the Universe.

In the Standard Model (*SM*) of particle physiscs, described by Quantum Field Theory (*QFT*), only three interactions are considered: electromagnetism, weak and strong interactions. Gravitation is much weaker and is not considered, at least at the energy scales involved in present experimental projects. But, as we imagine to go back in time, the temperature and the energy of particles increases and new aspects have to be taken into account. As we shall see, cosmology can set useful constraints to Particle Physics, beyond the *SM*.

In *QFT* it is useful to use dimensionless quantities to estimate the strength of these interactions, the dimensionless couplings, like the fine structure constant

$$\alpha_{EM} = \frac{e^2}{\hbar c} \cong \frac{1}{137}$$

for electromagnetism. For weak interactions one can use the Fermi weak coupling constant G_F [$G_F/(\hbar c)^3 = 1.166\times10^{-5}\ GeV^{-2}$] and the dimensionless coupling for a typical hadronic mass, the proton mass m_p, is given by

$$\frac{G_F m_p^2 c^4}{(\hbar c)^3} \approx 1.03 \times 10^{-5},$$

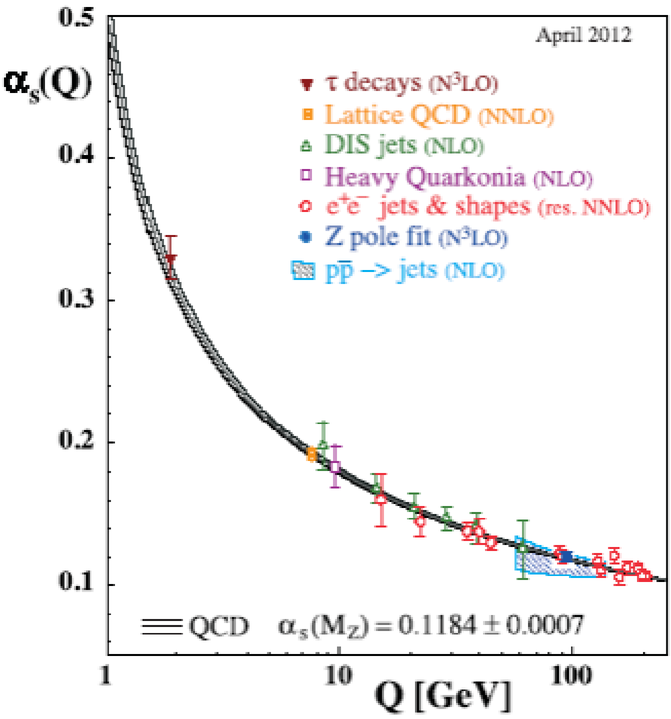
The weakness of weak interactions is due to the improbability of the emission of the very massive bosons W^+, W^-, Z^0. The dimensionless coupling, according to Weinberg-Salam theory, is linked to G_F by the relation

$$\frac{G_F}{(\hbar c)^3} = \frac{\pi}{\sqrt{2}} \frac{\alpha_W}{{M_W}^2 c^4}$$

where $M_W \sim 80\ GeV/c^2$. For strong interactions (Quantum Chromo Dynamics, *QCD*) a dimensionless coupling α_S can be defined.

In *QFT* these couplings are not constant, but are "running", i.e. change their values with the energy scale, linked to a distance scale $r \sim \hbar c/E \sim \hbar/mc$ ($E=mc^2$). For instance, in *QCD*,

$$\alpha_s(E) \sim \frac{0.73}{\ln\left(\frac{E}{\Lambda_{QCD}}\right)} \sim \frac{0.73}{\ln\left(\frac{\hbar c}{\Lambda_{QCD} r}\right)} \qquad \Lambda_{QCD} \equiv \Lambda_C$$



Hence $\alpha_s \to 0$ as $r \to 0$, which gives rise to the so-called "asymptotic freedom," i.e., the fact that quarks and gluons inside a hadron behave like free particles when very close together. On the other hand, α_s apparently diverges as $r \to R_C \equiv \hbar c/\Lambda_C$, where Λ_C is the hadronic energy scale. This divergence simply heralds the breakdown of perturbation theory, of course, but nonetheless it leads us to expect that the strength of the force between quarks will increase if they are pulled apart. As a result, quarks and gluons are "confined" inside hadrons whose size is of order R_C.

The crucial parameter characterizing the strong interaction is the energy scale Λ_C that appears in (1.12). It turns out to be

$$\Lambda_C \simeq 200 \text{ MeV} \tag{1.13}$$

and $R_C \sim 10^{-13}$ *cm* (1 *fm*), the size of hadrons.

The interesting point, as shown in the following figure, is that the couplings tend to converge to one single value at energies on the order of 10^{15} *GeV*, or higher. From this comes the idea that at high energy there is only one interaction, whose symmetry is broken at lower energies, as the electroweak interaction splits into weak interaction and electromagnetism at energies below ~ *100 GeV*. One speaks of Grand Unified Theories (*GUTs*).

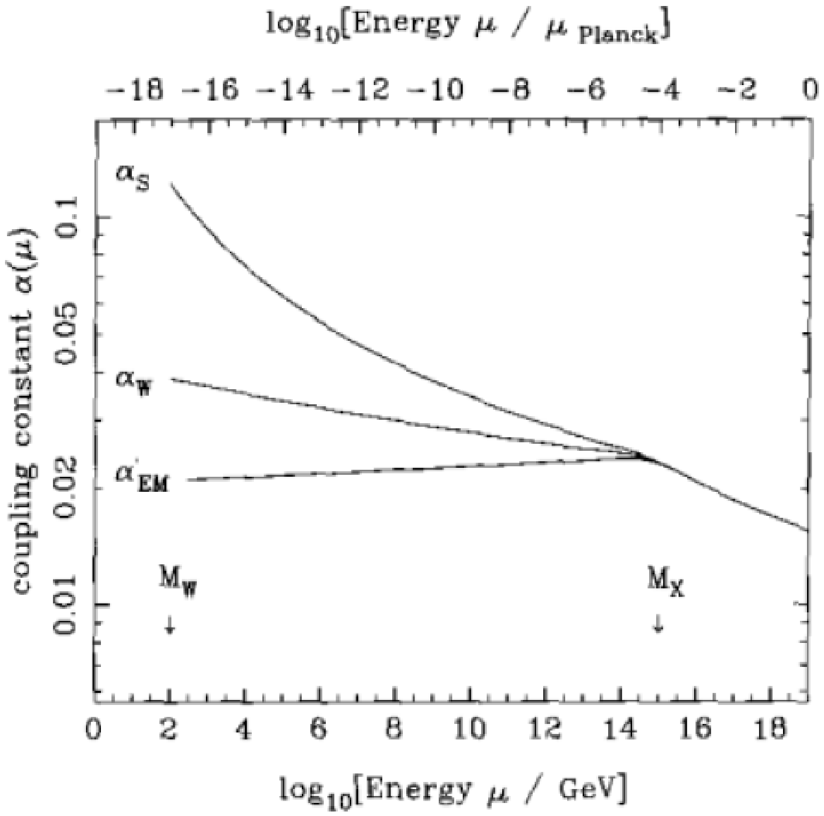
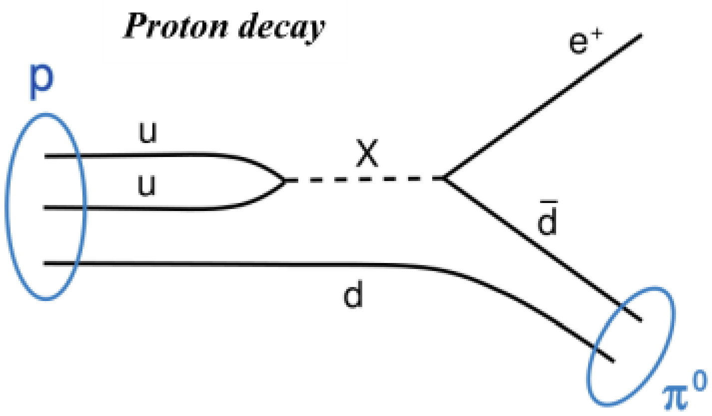


Figure 8.6. The predicted variation of the coupling constants α_i with energy scale (here denoted by μ), according to the $SU(5)$ GUT. All three interactions reach comparable strength at a scale around $M_X \sim 10^{15}$ GeV (adapted from figure 2 of Iliopoulos 1990).

In GUTs there are new gauge bosons X which link quarks and leptons and mediate interactions that violate Baryon number B and Lepton number L. These new interactions must be wery weak since they have eluded detection so far, which means that the X bosons must be very massive ($M_X c^2 \sim E_{GUT} \sim 10^{15}$-$10^{16}$ GeV). Even if B and L conservation are violated, in some GUTs B-L is conserved.

The B-violating interactions would make the proton unstable. Since no proton decay has been observed so far, there are lower limits on proton lifetime $\tau_P > 10^{31}$-10^{32} years.



The Planck era

What about **gravity**? A natural chice for a dimensionless, gravitational coupling is given by

$$\alpha_G = Gm_p^2 / \hbar c \approx 6\times10^{-39}$$

which is extremely small. But $m=E/c^2$ and

$$\alpha_G = GE^2/\hbar c^5 \approx 1 \text{ if } E = E_{Pl} \sim \left(\frac{\hbar c^5}{G}\right)^{1/2} \sim M_{Pl}c^2$$

$E_{Pl} \approx 2\times10^{16}$ erg $\approx 1.2\times10^{19} GeV$; $M_{Pl} \approx 2\times10^{-5} g$. From the relations $\Delta E\times\Delta t\sim\hbar$ and $\Delta t\sim l/c$, **Planck Energy** E_{Pl} corrisponds to a scale (**Planck lenghth**)

$$l_{Pl} \sim \frac{\hbar c}{E_{Pl}} \sim \left(\frac{\hbar G}{c^3}\right)^{1/2} \sim 1.6\times10^{-33}\ cm$$

and to a **Planck time**

$$t_{Pl} \sim \frac{l_{Pl}}{c} \sim \left(\frac{\hbar G}{c^5}\right)^{1/2} \sim 5\times10^{-44}\ s$$

So at energies of the order or above E_{Pl} gravitation becomes strong, and cannot be neglected in comparison to the other interactions. We need to link *QFT* and *GR*, but such a theory is not available at the moment (String theory could be such a theory). This means that all our extrapolations of the known and experimentally tested Physics have to stop at the Planck scale.

At E_{Pl} the age of the Universe was $t \sim t_{Pl}$, the particle horizon was $\sim l_{Pl}$, the density was

$$\rho_{Pl} \sim \frac{1}{G t_{Pl}^2} \sim \frac{c^5}{\hbar G^2} \sim 5\times10^{93}\ g\,cm^{-3}$$

and the mass within the horizon was $M_H \sim \rho_{Pl}\, l_{Pl}^3 \sim M_{Pl}$.

Moreover, E_{Pl} , l_{Pl} and t_{Pl} are the only possible results if one combines $\hbar$ (Quantum Mechanics), c (Special Relativity) and G (Gravitation) to obtain an energy-mass, a length and a time, and they are the most *natura*l choice.

SUPERSYMMETRY (SUSY)

Supersymmetry (SUSY) is a generalizetion of the space-time symmetries of quantum field theory that transforms fermions into bosons and vice versa. In particular, it is possible that supersymmetry will ultimately explain the origin of the large hierarchy of energy scales from the W and Z masses to the GUTs and Planck scales.

If supersymmetry were an exact symmetry of nature, then particles and their superpartners (which differ in spin by half a unit) would be degenerate in mass. Since superpartners have not (yet) been observed, supersymmetry must be a broken symmetry. Nevertheless, the stability of the gauge hierarchy can still be maintained if the supersymmetry breaking is soft, and the corresponding supersymmetry-breaking mass parameters are no larger than a few TeV.

In the Minimal Supersymmetric extension of the Standard Model (MSSM) *B*-*L* is conserved. As a consequence of $B-L$ invariance, the MSSM possesses a multiplicative ***R-parity*** invariance, where $R = (-1)^{3(B-L)+2S}$ for a particle of spin S. Note that this implies that all the ordinary Standard Model particles have even R parity, whereas the corresponding supersymmetric partners have odd R parity[1].

The conservation of R parity in scattering and decay processes has a crucial impact

[1] In the SM: for leptons L=1, B=0, S=1/2; for quarks L=0, B=1/3, S=1/2; for bosons B=L=0 and S is an integer. So R turns out to be always +1. For the superpartners B and L are the same, but S=0 for fermionic partners and S=1 for bosonic partners, so R is always -1.

on supersymmetric phenomenology. For example, starting from an initial state involving ordinary (R-even) particles, it follows that supersymmetric particles must be produced in pairs. In general, these particles are highly unstable and decay into lighter states. However, R-parity invariance also implies that the lightest supersymmetric particle (LSP) is absolutely stable, and must eventually be produced at the end of a decay chain initiated by the decay of a heavy unstable supersymmetric particle. In order to be consistent with cosmological constraints, a stable LSP is almost certainly electrically and color neutral. Consequently, the LSP in an R-parity-conserving theory is weakly interacting with ordinary matter, i.e., it behaves like a stable heavy neutrino and will escape collider detectors without being directly observed. So **the LSP is a promising candidate for dark matter**.

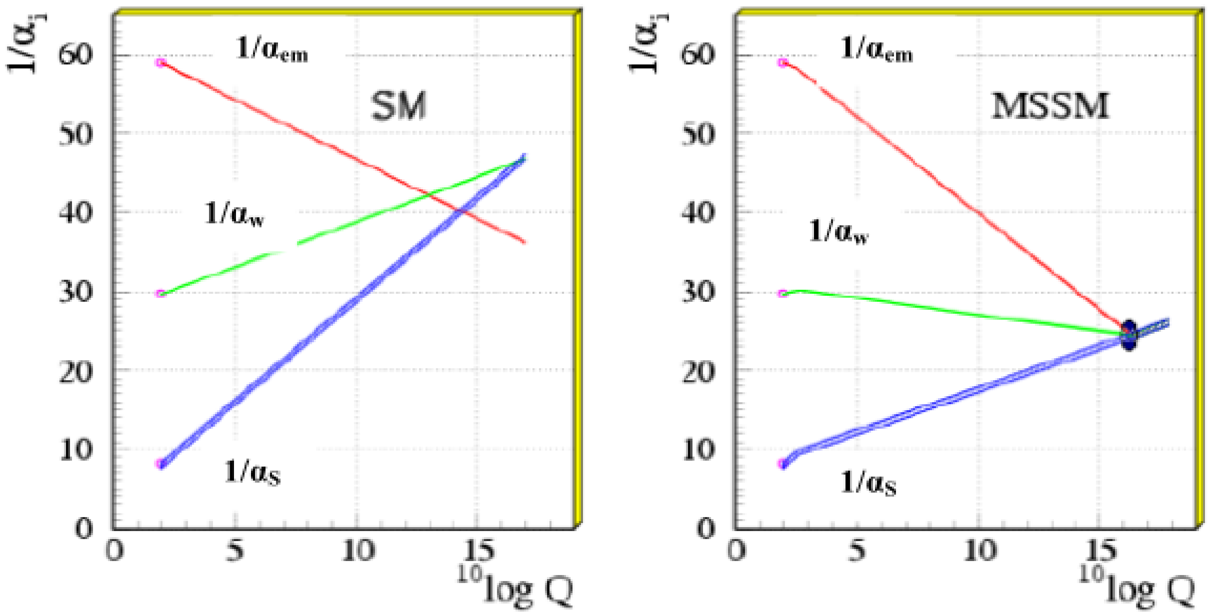


Figure 15.1: Gauge coupling unification in non-SUSY GUTs on the left vs. SUSY GUTs on the right using the LEP data as of 1991. Note, the difference in the running for SUSY is the inclusion of supersymmetric partners of standard model particles at scales of order a TeV (Fig. taken from Ref. 24). Given the present accurate measurements of the three low energy couplings, in particular $\alpha_s(M_Z)$, GUT scale threshold corrections are now needed to precisely fit the low energy data. The dark blob in the plot on the right represents these model dependent corrections.

Axions

In *QCD* the vacuum is a superposition of degenerate states. This introduces a new arbitrary parameter Θ in the theory which leads to an additional term in the *QCD* Lagrangian. However, the existence of this term violates CP, T and P and leads to a neutron electric dipole moment of $d_n/e \sim 5\times10^{-16}\ \Theta\ cm$. Observations give an upper limit $d_n/e \sim 10^{-25}\ cm$, so $\Theta \le 10^{-10}$. Why is Θ so small? This is the ***strong CP problem*** of *QCD*.

In 1977 Peccei and Quinn showed that Θ could be driven to zero by introducing in the Lagrangian a new symmetry which is spontaneously broken at an energy scale f_{PQ}. This induces the existence of a new boson, the ***axion***, which is not massless, but has a mass of the order

$$m_A c^2 \sim 0.6\ \frac{10^7\ GeV}{f_{PQ}(GeV)}\, eV$$

In their original paper Peccei and Quinn assumed that f_{PQ} was on the order of the vacuum expectation value $\boldsymbol{v}$ of the Electroweak phase transition ($\boldsymbol{v} \sim 250\ GeV$). In this case m_A would be $\sim 100\ keV$, excluded by experiments. But the value of f_{PQ} can be anywhere between $250\ GeV$ and $10^{19}\ GeV$, and m_A spans a huge range of values.

Limits on m_A are given also by stellar evolution. Detection techniques to find out evidence of the existence of axions are based on the conversion of axions into microwave photons in the presence of a very strong magnetic field. The contribution of axions to the **dark matter** is given, if they exist, by

$$\Omega_A h^2 \approx 0.3\ \left(\frac{f_{PQ}(GeV)}{10^{12} GeV}\right)^{7/6}$$

which means that, in order to represent a major contribution to dark matter, the mass of the axions must be $m_A \approx 10\ \mu eV$.

Thermodynamics of the Early Universe

Going back in time temperature T and density ρ grow and it is expected that the particles reach the thermodynamic equilibrium through rapid interactions. The rate of interaction $\Gamma = n\sigma v$ (n = number density, σ = cross section, v = particle velocity) grows more rapidly, with the temperature, than the rate of expansion H, so Γ » H at high T. This means that, with regard to the interactions, the expansion is quasi-static and there is enough time for the universe to continuously restore thermodynamic equilibrium.

This allows a very simple treatment of the distribution functions of the particles. In thermodynamic equilibrium, the number density n of particles of a given species, with momentum between P and $P + dP$ is

$$dn = \frac{g}{2\pi^2\hbar^3}\frac{P^2 dP}{e^{\frac{E-\mu}{kT}} \pm 1}$$

where $E^2=P^2c^2+m^2c^4$, μ is the chemical potential, and g is the spin-degeneracy factor , which counts the number of degrees of freedom, taking into account the spins and colors of particles (for spin states $g=1$ if $m=0,s=0$; $g=2$ if $m=0, s\neq 0$; $g=2s+1$ if $m\neq 0$; $g_\gamma=2$, $g_e=2$, *but* $g_\nu=1$ since neutrinos are only *left handed*; for each quark flavour $g=6$, a factor *2* for the spin and a factor *3* for the colors) The + or - sign corresponds to fermions (f) and bosons (b).

For photons μ is naturally zero since they have a planckian distribution with temperature $T_\gamma(t)$; if a species A is in thermal equilibrium with photons ($\Gamma_{A\gamma} \gg H$), $T_A=T_\gamma$ and the same holds for all species in equilibrium. So we use the photon temperature as reference: $T_\gamma \equiv T_{\mathrm{Universe}}=T$.

In thermodynamic equilibrium the number density n_i and the energy density $\rho_i c^2$ of "*i*" particles are given by

$$n_i = \int_0^\infty \frac{\mathrm{d}n}{\mathrm{d}P}\mathrm{d}P = \frac{g_i}{2\pi^2\hbar^3}\int_0^\infty \frac{P^2}{e^{\frac{E-\mu}{kT}} \pm 1}\mathrm{d}P$$

$$\rho_i c^2 = \int_0^\infty E\frac{\mathrm{d}n}{\mathrm{d}P}\mathrm{d}P = \frac{g_i}{2\pi^2\hbar^3}\int_0^\infty \frac{EP^2}{e^{\frac{E-\mu}{kT}} \pm 1}\mathrm{d}P$$

For the pressure p, from $p = \frac{n}{3} < \vec{P}\cdot\vec{v} >$, and

$$P = \gamma m\vec{v} \Longrightarrow \vec{P}\cdot\vec{v} = \vec{P}\cdot\vec{P}/\gamma m = P^2c^2/\gamma mc^2 = P^2c^2/E$$

$$p_i = \int_0^\infty \frac{1}{3}\left(\vec{P}\cdot\vec{v}\right)\frac{\mathrm{d}n}{\mathrm{d}P}\mathrm{d}P = \frac{g_i}{2\pi^2\hbar^3}\int_0^\infty \frac{1}{3}\frac{P^2c^2}{E}\frac{P^2}{e^{\frac{E-\mu}{kT}} \pm 1}\mathrm{d}P$$

$$p_i = \frac{g_i c^2}{6\pi^2\hbar^3}\int_0^\infty \frac{1}{E}\frac{P^4}{e^{\frac{E-\mu}{kT}} \pm 1}\mathrm{d}P$$

In the Early Universe, for various reasons, the chemical potentials are negligible (fermions are non-degenerate, bosons do not form a Bose condensate). The main argument comes from the fact that the net chemical potential in the early universe can be set to zero, because the asymmetry between particles and antiparticles is very small. From chemical thermodynamics, for a reaction $1 + 2 \leftrightarrow 3 + 4$, the relation among chemical potentials is $\mu_1 + \mu_2 = \mu_3 + \mu_4$. From a reaction like ($\gamma+\gamma \leftrightarrow A + \bar{A}$), since $\mu_\gamma=0$, then $\mu_A = -\mu_{\bar{A}}$. So, for number densities, $n_A - n_{\bar{A}} \neq 0$ and this gives a nonzero value for the quantum numbers (electric charge, baryon number, color charge, …) associated to particle *A*. But electric charge, color charge, …, of the Universe seem to be consistent with zero; moreover, the number density of baryons is much smaller than that of photons: $(n_B - n_{\bar{B}})/n_\gamma \leq 10^{-9}$. So, in the Early Universe, it is usually assumed that $n_B \cong n_{\bar{B}}$ and chemical potentials are set to zero.

The above relations for number density, energy density, and pressure are general. It is easy to evaluate these integrals in two extreme cases: ***ultrarelativistic, non-degenerate particles*** and ***non-relativistic particles***.

- **Ultrarelativistic case:** $kT \gg m_i c^2$; $E^2 \sim P^2c^2$, $|\mu| << kT$ (use $Pc/kT=u$)

$$n_i \cong \frac{g_i}{2\pi^2\hbar^3}\int_0^\infty \frac{P^2dP}{e^{Pc/kT}\pm 1} \cong \frac{g_i}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty \frac{u^2du}{e^u \pm 1}$$

(+)fermions =3/2·ζ(3) ***(-)bosons =2·ζ(3)***

ζ(x): Riemann Zeta function
ζ(3)=1.202
ζ(4)=π⁴/90=1.082

For photons ($g_\gamma=2$)

$$n_\gamma = \frac{2}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3$$

and for bosons and fermions:

$$n_{b,i} = \frac{g_i}{\pi^2}\zeta(3)\left(\frac{kT_i}{\hbar c}\right)^3 = \frac{g_i}{2}\left(\frac{T_i}{T}\right)^3 n_\gamma(T)$$

$$n_{f,i} = \frac{3}{4}\cdot\frac{g_i}{\pi^2}\zeta(3)\left(\frac{kT_i}{\hbar c}\right)^3 = \frac{3}{8}\cdot g_i\left(\frac{T_i}{T}\right)^3 n_\gamma(T)$$

For the energy density

$$\rho_i c^2 \cong \frac{g_i c}{2\pi^2\hbar^3}\int_0^\infty \frac{P^3dP}{e^{Pc/kT}\pm 1} \cong \frac{g_i (kT)^4}{2\pi^2\hbar^3c^3}\int_0^\infty \frac{u^3du}{e^u \pm 1}$$

(+)fermions =7/8·6·ζ(4) ***(-)bosons =6·ζ(4)***

Stefan-Boltzmann constant:

$$a_B = \frac{\pi^2k^4}{15\hbar^3c^3} = 7.5659\times 10^{-15}\, erg\, cm^{-3}K^{-4}$$

Remember that $\zeta(4)=\pi^4/90$ and so, using *Stefan-Boltzmann* constant,

$$\rho_{f,i}\,c^2=\frac{7}{8}6\frac{\pi^4}{90}\frac{g_i k^4 T^4}{2\pi^2\hbar^3 c^3}=\frac{7}{16}\frac{\pi^2 k^4}{15\hbar^3 c^3}g_i T_i^4=\frac{7}{16}a_B\,g_i T_i^4$$

$$\rho_{b,i}\,c^2=\frac{1}{2}g_i a_B T_i^4$$

The mean energy per particle is $< E_i >= \rho_i c^2/n_i$:

$$\langle E\rangle_b=\frac{\pi^4}{30\zeta(3)}kT=2.70\,kT \qquad \langle E\rangle_f=\frac{7\pi^4}{180\zeta(3)}kT=3.15\,kT$$

which can be approximated by $< E_i >\approx 3kT$.

For the pressure p it is easy to realize that, if $E\sim Pc$,

$$p_i=1/3\;\rho_i\,c^2.$$

- **Non relativistic case: $kT<<m_ic^2$** ***(****$Pc<< m_ic^2$****)*;**

$E^2=P^2c^2+m_i^2c^4= m_i^2c^4[1+P^2/m_i^2c^2] \;\Rightarrow\; E\sim m_i\,c^2[1+ P^2/2m_i^2c^2]$; $E\sim m_ic^2+P^2/2m$

$e^{E/kT}$ **» 1** $\Rightarrow$ *no difference between fermions and bosons***.**(use $P/\sqrt{mkT}\equiv u$)

$$n_i\cong\frac{g_i}{2\pi^2\hbar^3}\int_0^\infty P^2 e^{-\frac{m_ic^2-\mu_i}{kT}}e^{-\frac{P^2}{2m_ikT}}dP\cong$$

$$\cong\frac{g_i}{2\pi^2\hbar^3}(m_ikT)^{3/2}e^{-\frac{m_ic^2-\mu_i}{kT}}\underbrace{\int_0^\infty u^2e^{-u^2/2}du}_{=\frac{\Gamma(3/2)}{2\left(\frac{1}{2}\right)^{7/2}}=\frac{\sqrt{2\pi}}{2}}$$

and we have:

$$n_i\cong\frac{g_i}{\hbar^3}\left(\frac{m_ikT}{2\pi}\right)^{3/2}e^{-\frac{m_ic^2-\mu_i}{kT}}$$

Note the ***strong exponential cut***, since $kT << m_ic^2$. This cut is due to ***annihilation*** of particles with their antiparticles. When particles are ultrarelativistic (***kT » m****$_i$****c****^{2}*), annihilation is balanced by pair production, but for ***kT<<m****$_i$****c****^{2}* pair production is ineffective and annihilation prevails.

In a similar way we get

$$\rho_i c^2 \cong n_i m_i c^2 + \tfrac{3}{2} n_i kT \sim n_i m_i c^2$$
$$p_i \cong n_i kT << \rho_i c^2$$

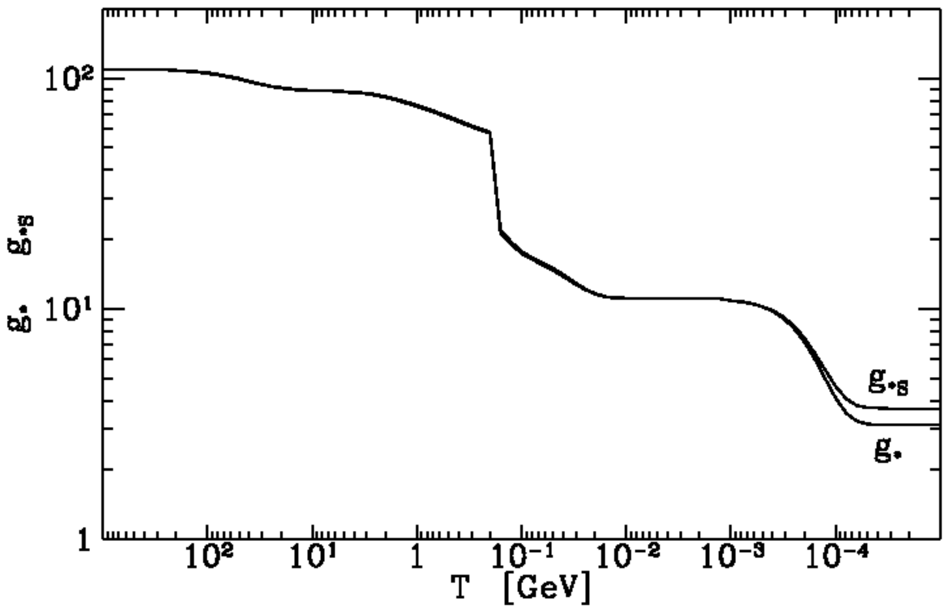
The contribution to the ***total energy density*** ρc^2 (as well as the total pressure p) of the non-relativistic species is negligible (due to the exponential cut), so ρc^2 can be well approximated only by the contribution of relativistic species

$$\rho c^2 \cong \rho_R c^2 = \frac{1}{2} a_B T^4 \underbrace{\left[\sum_{i=bosons,rel} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=fermions,rel} g_i \left(\frac{T_i}{T}\right)^4 \right]}_{g_*(T)} = \frac{1}{2} a_B \, g_*(T) T^4$$

where $g_*(T)$ represents the total, ***effective number of degrees of freedom*** of relativistic) particles.

For $kT<<1$ *MeV* the only relativistic species are photons and the three neutrinos (if m_ν is negligible); since (see the proof below) $T_\nu=(4/11)^{1/3}T_\gamma$, $g_*=2+7/8\cdot 2\cdot 3\cdot(4/11)^{4/3}=3.36$ ($2=\nu+\bar{\nu}$, $3=N_\nu$). For 1 *MeV* $\leq kT \leq 100$ *MeV* we add e^+ and e^- and $T_\nu=T_\gamma$, $g_*=43/4=10.75$. Above *300 GeV* all particles included in the Standard Model are relativistic, and $g_*=474/4=106.75$. At energies higher than $E_{EW} \sim M_W c^2 \sim 100$ *GeV* (Electroweak breacking) g_* depends on the adopted theory (for instance, in the minimal model of *GUT*, *SU(5)*, for $kT > E_{GUT} \sim 10^{16}$ *GeV* , $g_* \sim 160$).

In supersymmetric models, at each particle corresponds a supersymmetric partner, and g^* approximately doubles. If some *sparticles* have mass smaller than the Higgs boson, then there may be some changes in the following graph representing the behaviour of g^* as a function of temperature for the Standard Model of particle physics.



Time scale: In the Radiation Dominated (*RD*) era the Universe is well approximated by an *EdS* model, so $\rho=3/(32\pi G t^2)$, $E \sim 3kT$, $\rho=\rho_R$ and

$$t^2 \approx \frac{3c^2}{32\,\pi\,G\,\rho_R c^2} \approx \frac{45c^5\hbar^3}{16\,\pi^3\,G\,g_*}\cdot\frac{3^4}{(3kT)^4}$$

$$t(\mathrm{sec}) \approx \frac{2.2\times10^{-5}}{g_*^{1/2}E_{GeV}^2} \approx \frac{2.4}{g_*^{1/2}(kT)_{MeV}^2}$$

Thermodynamic equilibrium (TE): The Universe turns out to be in TE for

$$1\ MeV \le kT \le 10^{-3}\,M_{Pl}\,c^2 \sim 10^{16} GeV\ (\sim E_{GUT})$$

The upper limit is set by interactions mediated, at very high energy, by ultrarelativistic gauge bosons. The lower limit corresponds to interactions mediated by a massive gauge boson, like W^+, W^- and Z^0 below the scale of electroweak symmetry breaking (~ *100 GeV*). At a mean particle energy of ~ *1 MeV* these interactions are no more effective, are “***frozen out***”.

Neutrinos do not interact any more with matter and radiation: they ***decouple*** when the **mean energy** per particle is **about *1 MeV***.

Moreover, the mean free path of the particles is much greater than their average mutual distance ⇒ *perfect gas*.

Entropy

In thermodynamic equilibrium, the entropy S in a comoving volume element is preserved during the expansion (entropy can increase if processes like particle decay or phase transitions happen under condition which do not preserve thermodynamic equilibrium).

Entropy S and the first law of thermodynamics are related by (we use $d(pV)=pdV+Vdp$)

$$dQ = T\,dS = dU + dL \longrightarrow T\,dS = d(\rho c^2 V) + pdV = d[(\rho c^2 + p)V] - Vdp$$

If we consider $S=S(V,T)$

$$dS(V,T) = \frac{1}{T}d[\varrho(T)c^2 V] + \frac{p}{T}dV$$

$$dS(V,T) = \frac{V}{T}\frac{d[\varrho(T)c^2]}{dT}dT + \frac{\varrho(T)c^2 + p(T)}{T}dV$$

Since entropy is a function of state, its differential form is exact and the integrability condition $\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}$ gives

$$\frac{\partial}{\partial T}\left[\frac{\varrho(T)c^2 + p(T)}{T}\right] = \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{d[\varrho(T)c^2]}{dT}\right]$$

$$\frac{1}{T^2}\left[\left(\frac{d\,\rho c^2}{dT} + \frac{dp}{dT}\right)\cdot T - (\rho c^2 + p)\right] = \frac{1}{T}\frac{d\,\rho c^2}{dT}$$

$$\frac{dp}{dT} = \frac{\rho c^2 + p}{T} \;\;\rightarrow\;\; dp = (\rho c^2 + p)\frac{dT}{T}$$

We can use this result in the previous relation

$$T\,dS = d[(\rho c^2 + p)V] - V dp$$

and we get

$$dS = \frac{1}{T} d[(\rho c^2 + p)V] - V(\rho c^2 + p)\frac{dT}{T^2} = d\left[\frac{(\rho c^2 + p)V}{T} + const.\right].$$

So, up to an additive constant, the entropy S for a comoving volume $V=a^3$ (a is the scale factor) can be written as

$$S = \frac{a^3(\rho c^2 + p)}{T}$$

The **entropy density** s is defined as

$$s \equiv \frac{S}{V} = \frac{\rho c^2 + p}{T}$$

This (due to the exponential cut in number density) is dominated by the contribution of relativistic particles. For each relativistic species $s_i=(\rho_i c^2+1/3\cdot\rho_i c^2)/T=4\rho_i c^2/3T$, and the total contribution becomes, with good approximation

$$s = \frac{2}{3}a_B T^3 \underbrace{\left[\sum_{i=b,rel} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i,f,rel} g_i\left(\frac{T_i}{T}\right)^3\right]}_{g_{*S}(T)} = \frac{2\pi^2 k}{45} g_{*S}(T)\left(\frac{kT}{\hbar c}\right)^3$$

Note that if $T_i \equiv T$ for all relativistic particles, as it is for most of the time in the early Universe, then $\boldsymbol{g_*=g_{*S}}$ (see the figure above).
Also note that s is proportional to n_γ; in fact

$$s = \frac{\pi^4}{45\zeta(3)} k\, g_{*S}(T)\, n_\gamma = 1.8 k\, g_{*S}(T)\, n_\gamma$$

Today ($kT \leq 1\ MeV$) $g_{*S}=2+7/8 \cdot 2 \cdot 3 \cdot 4/11=3.909$ and

$$s \cong 7.04 \cdot k\, n_\gamma$$

Above ~ $1\ MeV$: $g_* \approx g_{*S}$ (**Note**: g_{*S} depends in general on $T \Rightarrow s$ *an*d n_γ cannot be always considered as proportional!)

Entropy S conservation implies $s \propto a^{-3}$, and also

$$g_{*S} \cdot T^3 \cdot a^3 = \text{constant}$$

while the Universe expands.

The physical size of a comoving volume is $\propto a^3$ and, since $s \propto a^{-3}$, it is also $\propto s^{-1}$. The number N of particles of a species inside a comoving volume (named *comoving number density*), $N \equiv n \cdot a^3$, is also equal (actually, proportional) to n/s, so we also write $\boldsymbol{N_i \equiv n_i/s}$. If particles are neither created nor destroyed, then $\boldsymbol{N_i \equiv n_i/s=const.}$ For *relativistic particles* in *TE* the comoving number density can be written as

$$N_i = F_{bf} \cdot \frac{g_i \zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \cdot \frac{45}{2\pi^2 k\, g_{*S}(T)}\left(\frac{kT}{\hbar c}\right)^{-3} = F_{bf} \cdot \frac{45\zeta(3)}{2\pi^4} \cdot \frac{g_i}{k\, g_{*S}(T)}$$

where F_{bf} is equal to 1 for bosons and to ¾ for fermions.

The baryon number N_B (the difference between baryons b and antibaryons $\bar{b}$) in a comoving volume is

$$N_B = \frac{n_B}{s} \equiv \frac{n_b - n_{\bar{b}}}{s}$$

As long as the interactions violating baryon number conservation (if they exist!) are very slow, n_B/s is conserved.

However, the ***baryon-photon ratio*** η, a crucial parameter in primordial (or Big Bang) nucleosynthesis,

$$\eta \equiv \frac{n_B}{n_\gamma} = 1.8 k\ g_{*S}(T) \cdot \frac{n_B}{s}$$

doesn't stay constant since g_{*S} depends on T. But after e^+ add e^- annihilation (at ~ $0.5\ MeV$) g_{*S} is constant ($=3.909$), so $\boldsymbol{\eta \approx 7.04\ k\ n_B/s}$ or n_B/s can be indifferently used.

We shall see that primordial nucleosynthesis requires that $\eta \approx 5\times10^{-10}$, so in our Universe there are today about 10^9 photons for each baryon. Also the entropy per baryon, $\boldsymbol{s/n_B=7.04\ k/\eta \approx 7\times10^{10}\ k/\eta_{10}}$, is extremely high ($\eta_{10}\equiv\eta\,/10^{-10}$)

The fact that $\boldsymbol{S = const.}$ implies

$$T \propto g_{*S}^{-1/3} \cdot a^{-1}$$

If g_{*S} is costant $\boldsymbol{T \propto a^{-1}}$. The $g_{*S}^{-1/3}$ factor enters the game when a species becomes non relativistic, annihilates and disappears (since annihilation is less and less balanced by pair creation): its entropy is transferred to photons and to the other interacting relativistic particles, so T decreases more gently.

If a relativistic particle decuples at time $t=t_D$, when $T=T_D$ and $a=a_D$, it doesn't benefits of the entropy exchange due to the annihilation (at $T<T_D$) of the other species. After decoupling $P\propto 1/a \Rightarrow P=(a_D/a)P_D$ and (if the particle is stable) $n=(a_D/a)^3 n_D$; since $P\propto 1/a$, n will be given by

$$n = \frac{g_i}{2\pi^2\hbar^3}\left(\frac{a_D}{a}\right)^3 \int_0^\infty \frac{P_D^2\, dP_D}{e^{\frac{c}{kT}\cdot\frac{a_D P_D}{a}} \pm 1}$$

which gives the right dependence on a if $T=(a_D/a)T_D$. The distribution function of momenta keeps its shape, but with $T\propto a^{-1}$ instead of $T\propto g_{*S}^{-1/3}a^{-1}$ which holds for particles still coupled. If the particle, for instance a "light" neutrino, becomes eventually non relativistic, the shape of the distribution function of its momentum is preserved, with $T\propto a^{-1}$.

This also explains the reason for *CMB* photons shows a black body spectrum even after the last scattering (at $z_{ls} \approx 1100$), when they decouple from baryons and are no more in thermodynamic equilibrium.

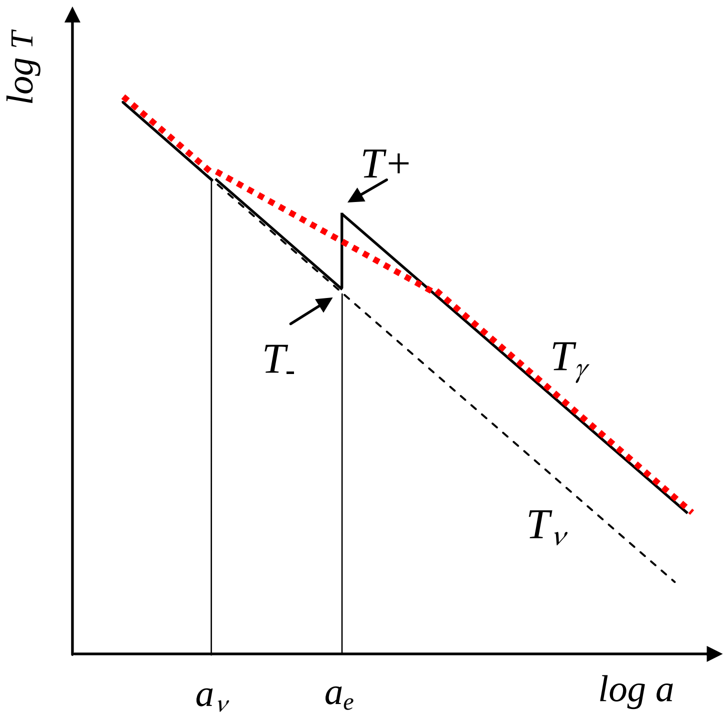
Neutrinos

We have already seen that at $kT\sim E\sim 1Mev$, when $a=a_\nu$, neutrinos (ν) decouple from other species, and so, while before $T_\nu=T_\gamma$, after decoupling $T_\nu = T_\gamma(a_\nu)a_\nu/a$. However, at a slightly lower energy, at $E\sim 0.5Mev$ $(a=a_e)$, electrons and positrons annihilate and their entropy goes to photons, but not to the decoupled neutrinos. Entropy is conserved ($g_{*S}\, T^3 a^3 = const.$) for still coupled particles (e^+, e^- e γ for $a < a_e$, only γ for $a > a_e$). We denote with a_-, T_- and a_+, T_+ the values just before and immediatleyafter electron-positron; we suppose that annihilation occurs instantaneously and we have[2] ($a_+ \approx a_e \approx a_-$):

[2] We could also add, both on the left hand side and on the right hand side, the contribution of neutrinos, but this contribution is the same immediately before and after annihilation, since neutrinos are decoupled. So we omit their contribution.

$$g_{*S}T^3a^3 = \overbrace{(\underbrace{2}_{\gamma}+\underbrace{\frac{7}{8}\cdot 2\cdot 2}_{e^+ + e^-})}^{before}\underbrace{\;}_{g_i}\,T_-^{\,3}a_-^{\,3} = \overbrace{\underbrace{2}_{\gamma}T_+^{\,3}a_+^{\,3}}^{after}$$

From this relation we get the the ratio (see also the following figure)



$$\frac{T_-}{T_+} = \left(\frac{4}{11}\right)^{1/3} = \frac{T_\nu}{T_\gamma}$$

After a_e both T_ν and T_γ scale as *1/a*, and their ratio stays constant until now. So, if $T_{\gamma 0}=2.73\ K$, $T_{\nu 0}=1.95\ K$.

Actually, the photon temperature does not rise abruptly at $a=a_e$, but decreases more slowly than *1/a* until the annihilation of e^+ and e^- ends (see the dotted line).

It is now easy to derive the present values of number densities of CMB photons and of cosmological neutrinos.

For today's ***CMB*** the density and the number density are easily derived:

$$\rho_{\gamma 0} = \frac{a_B T_{\gamma 0}^4}{c^2} = 4.67\cdot 10^{-34}\left(\frac{T_{\gamma 0}}{2.73}\right)^4 g\,cm^{-3}$$

$$\frac{s_0}{k} = \frac{2\pi^2}{45} g_{*S}(T_{\gamma 0})\left(\frac{kT_{\gamma 0}}{\hbar c}\right)^3 = 2934\left(\frac{T_{\gamma 0}}{2.73}\right)^3 cm^{-3}$$

$$n_{\gamma 0} = \frac{2}{\pi^2}\zeta(3)\left(\frac{kT_{\gamma 0}}{\hbar c}\right)^3 = 417\left(\frac{T_{\gamma 0}}{2.73}\right)^3 cm^{-3}$$

For each neutrino family, counting ν and $\bar{\nu}$,

$$\frac{n_{\nu+\bar{\nu}}}{n_\gamma} = \frac{\overset{g_\nu}{3}}{4} \cdot \frac{1 \cdot \overset{\nu+\bar{\nu}}{2}}{\underset{g_\gamma}{2}} \cdot \left(\frac{T_\nu}{T_\gamma} \right)^3 = \frac{3}{11}$$

$$n_{\nu+\bar{\nu},0} = 114 \left(\frac{T_{\gamma 0}}{2.73} \right)^3 \ cm^{-3}$$

COSMIC RELICS

The Universe seems to be neutral both from the point of view of electric charge and color charge. So Dark Matter candidates are thought to be indifferent to electromagnetic and strong forces.

It is possible to foresee the cosmological effect produced by weakly interacting massive particles (WIMPs) or, viceversa, to see the constraints posed by cosmological observations on the properties of such particles. Here we assume that these particles interact exactly as neutrinos do, but the term WIMP is also used for particles having much weaker, possible interactions beyond the Standard Model of Particle Physics.

There are two main cases: WIMPs can decouple when they are still relativistic (Hot Dark Matter, HDM, $kT_D \gg m_W c^2$) or when they are non relativistic (Cold Dark Matter, CDM, $kT_D \ll m_W c^2$).

(CR12

HDM If particles do not decay after decoupling, $n_0 a^3 \propto n/s \propto N \sim$ const.

(today, decouple.)

$$N(T_0) = N(T_D) = \begin{Bmatrix} 3/4 - f \\ 1 - b \end{Bmatrix} \frac{45\,\zeta(3)}{2\pi^4} \frac{g_i}{k\, g_{*S}(T_D)} = \frac{n_0}{s_0}$$

F_{bf}

remember

$$s = 1.8\, k\, g_{*S}(T)\, n_\gamma \;;\quad n_\gamma = \frac{2}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3$$

$$T_0 \sim 2.728\ \mathrm{K} \qquad g_{*S}(T_0) = 3.909$$

$$\to n_0 = \begin{Bmatrix} 3/4 \\ 1 \end{Bmatrix} \frac{\zeta(3)}{\pi^2} \frac{g_i\, g_{*S}(T_0)}{g_{*S}(T_D)} \cdot \left(\frac{kT_0}{\hbar c}\right)^3 =$$

$$\simeq \begin{Bmatrix} 3/4 \\ 1 \end{Bmatrix} \frac{g_i}{g_{*S}(T_D)} \cdot 813 \left(\frac{T_0}{2.728}\right)^3\ \mathrm{cm}^{-3}$$

If fermionic WIMPs:

$$n_0 \sim 610 \frac{g_i}{g_{*S}(T_D)}\ \mathrm{cm}^{-3}$$

[for neutrinos: $g_i = 2 = 1 \cdot 2\ (\nu + \bar\nu)$; $g_{*S}(T_D) = 10.75$
$\to n_0 \sim 114\ \mathrm{cm}^{-3}$]

If these WIMPs are non-relativistic today

$$\rho_0 \simeq n_0 m_W \to \Omega_W = \frac{n_0 m_W}{\rho_{0cr}}$$

$$\Omega_W h^2 \simeq 0.058 \frac{g_i}{g_{*S}(T_D)} \cdot \left(\frac{m_W}{1\,\mathrm{eV}/c^2}\right) \quad (*)$$

For 3 $\nu + \bar\nu$ families

$$\Omega_\nu h^2 \sim 0.011 \sum_{i=1}^{3} \left(\frac{m_{\nu_i}}{1\,\mathrm{eV}/c^2}\right)$$

From CMB + Large scale structure | CR | 3

$$\sum \left(\frac{m_{\nu i}}{1\,eV/c^2}\right) \lesssim 0.3 - 0.7$$

- The relation (*) can be applied also to other weakly interacting particles, provided they are relativistic at $T \sim T_D \sim 1$ MeV; for $m_W \gtrsim 1$ MeV/c^2 (*) is no more correct; $\Omega_W h^2$, for $m_W \sim 1$ MeV/c^2, is $\sim 10^4$!
- When HDM particles (ν) decouple, they are relativistic and $\langle E\rangle \sim 3kT_\nu$, $\langle E\rangle \sim \langle p\rangle \cdot c \rightarrow \langle p\rangle \cdot c \sim 3kT_\nu$. Since both $\langle p\rangle$ and $T_\nu \propto \frac{1}{a}$, this relation holds also when particles are no more relativistic and $\langle p\rangle \rightarrow m\langle v\rangle$, so
$m\langle v\rangle c \sim 3kT_\nu \rightarrow \langle v\rangle \sim \frac{3kT_\nu}{mc} \sim 150 \left(\frac{m}{1\,eV/c^2}\right)^{-1}$ km/s

This speed is high enough to prevent the formation of structures with escape velocities lower than $\langle v\rangle$, i.e. galaxies. Moreover, if we consider that structures were already formed at $z \sim 6$ (QSOs), $\langle v\rangle$ has to be multiplied by $(1+z) \sim 7$, giving values corresponding to the scale of big galaxy clusters ($> 10^{15} M_\odot$). In this scenario big structures form and then fragment to produce smaller structures [top-down scenario].

- Note that Ω_0 depends on T_D via $g_{*S}(T_D)$. If we consider a particle, still relativistic at decoupling, but with interactions (much) weaker than neutrinos, decoupling happens at $T_D \gg 1$ MeV, g_{*S} is larger, Ω_0 smaller, and higher values for m_W are allowed by cosmology.

CR14

If $T_D > 300$ GeV, $g_{*S} \gtrsim 100$ and

$$\Omega_W h^2 \sim 1.1 \times 10^{-3} \left(\frac{m_W}{1\ eV/c^2} \right) \sim \frac{m_W}{910\ eV/c^2}$$

Since m_W is larger than that of neutrinos, $\langle v \rangle$ decreases and smaller structures can form. In this case we speak of Warm Dark Matter (WDM)

Possible candidates are are photinos and gravitinos with masses $\lesssim 1$ keV, or sterile neutrinos with similar masses.

WDM alleviates some problems of CDM (see below), on small scales, such as an excess of small structures, with masses $\sim$ globular clusters or dwarf galaxies.

CDM In this case particles decouple when no more relativistic, the number density is cutted exponentially and, for a given Ω_W, masses can be much higher, and $\langle v \rangle \propto \frac{1}{m}$ much smaller: very small structures can form and we have a so called bottom-up scenario in which small structures merge, as time goes on, to form larger and larger structures. This scenario is in in quite good agreement with the observations.

- But in this case the final result depends on the details of the "freezing", at variance with the HDM case.
- Let's see the equation ruling the "freezing". If there no creation/destruction of particles, the proper number density $n \propto 1/a^3$: ($A \equiv$ const.)

$$n = \frac{A}{a^3} \rightarrow \dot{n} = -3 \underbrace{\frac{A}{a^3}}_{n} \cdot \frac{1}{a} \cdot \dot{a} = -3Hn$$

ICR15

So $\dot{n} + 3Hn = 0$

If we have annihilation and creation (with a source term S), the above eq. becomes

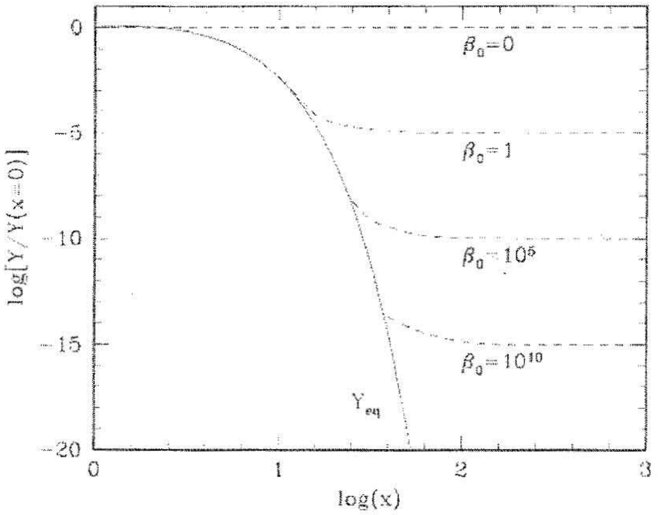
$$\dot{n} + 3Hn = -\langle \sigma v \rangle n^2 + S$$

where σ is the annihilation cross section and v the particle speed (n^2 because ann. is a two-body interaction)

If we were in stationary conditions ($H=0$), the value of n would be n_{eq}, the thermal equilibrium value at the mean temperature of the universe, so ($\dot{n}=0$, $H=0$)

$$-\langle \sigma v \rangle n_{eq}^2 + S = 0 \rightarrow S = \langle \sigma v \rangle n_{eq}^2$$

This relation can be derived directly from Boltzmann eq. (see, if you want, MoVdBW)



$\beta_0 \equiv \sigma_{ann}$

Fig. 3.5. The solution of Eq. (3.1n?) assuming a constant annihilation cross-section: $\beta = \beta_0$ (dashed curves). The solid curve shows the equilibrium abundance.

Note that the higher the value of β_0, the lower the final density of relics. The weakest wins!

$\dot{n} + 3Hn = -\langle\sigma v\rangle n^2 + S$; If $H = 0$ $\dot{n} = 0$ $-\langle\sigma v\rangle n_{eq}^2 + S = 0$

annihil. creazione

$S = \langle\sigma v\rangle n_{eq}^2$

CR | 6

the collision rate is $\Gamma = n\langle\sigma v\rangle$; likewise, the source term for thermal particle creation is $S = \langle\sigma v\rangle n_T^2$; thus, the continuity equation changes to read

$\dot{n} + 3Hn = -\langle\sigma v\rangle [n^2 - n_{eq}^2]$

$$\dot{n} + 3Hn = -\Gamma n + S = -\Gamma n\left(1 - \frac{n_T^2}{n^2}\right) \tag{1.115}$$

- we now introduce the comoving number density $N := a^3 n$; substituting from $\dot{N} = a^3(3Hn + \dot{n})$ in (1.115) yields

$$\dot{N} = -\Gamma N\left(1 - \frac{N_T^2}{N^2}\right) \tag{1.116}$$

substituting further

$$\frac{d}{dt} = \dot{a}\frac{d}{da} = aH\frac{d}{da} = H\frac{d}{d\ln a} \tag{1.117}$$

yields

$$\frac{d\ln N}{d\ln a} = -\frac{\Gamma}{H}\left(1 - \frac{N_T^2}{N^2}\right) \tag{1.118}$$

- thus, if the comoving number density is thermal, $N = N_T$, it does not change; if N deviates from N_T, it needs to change for re-adjustment to its thermal equlibrium value N_T; this is impossible if $\Gamma \ll H$ because then the rate of change becomes too small; then, the particles freeze out of thermal equilibrium

- for relativistic particles, $n \propto T^3 \propto a^{-3}$, thus $N = a^3 n = \text{const.}$; according to the freeze-out equation (1.118),

$$\frac{d\ln N}{d\ln a} = 0 \Rightarrow N = N_T \tag{1.119}$$

this implies that relativistic particle species retain their thermal-equilibrium density regardless of Γ/H, i.e. even after freeze-out

- for non-relativistic particles, the comoving number density in thermal equilibrium is

$$N_T \propto T^{-3/2} e^{-mc^2/kT} \tag{1.120}$$

$n_i \propto T^{3/2} e^{-\frac{mc^2}{kT}}$

$N_i \propto n_i a^3$ $a \sim \frac{1}{T}$

$N_i \propto \frac{T^{3/2}}{T^3} e^{-\frac{mc^2}{kT}}$

for $kT \lesssim mc^2$, N_T drops exponentially, i.e. very quickly $N_T \ll N$, then

$$\frac{d\ln N}{d\ln a} \approx -\frac{\Gamma}{H} \to 0 \tag{1.121}$$

as the collision rate falls below the expansion rate; the actual comoving number density of particles then remains constant, while its thermal-equilibrium value drops to zero

$\frac{\Gamma}{H} \sim 1$ good approximation for "freezing"

CR17

To estimate ~~Per stimare~~ Γ we need ~~occorre~~ $\langle \sigma_A \cdot v \rangle$.

- ~~Per~~ For $m_\psi < m_{Z^0}$ $\quad \langle \sigma v \rangle \sim \sigma_0 c$

~~con~~

$$\sigma_0 \sim \frac{5}{2\pi} \frac{G_F^2 m_\psi^2}{\hbar^4} = \frac{5}{2\pi} \frac{G_F^2 (m_\psi c^2)^2}{(\hbar c)^4} \propto m_\psi^2$$

The condition ~~La condizione~~ $\Gamma / H \simeq 1$ leads to ~~porta alla fine~~ (see details on ~~vedi dettagli in~~ "The Early Universe" by Kolb & Turner p. 129 e segg.) or MVdBW

$$\Omega_\psi h^2 \sim 3 \left[1 + \frac{1}{6} \ln\left(\frac{m_\psi c^2}{GeV}\right) \right] \cdot \left(\frac{m_\psi c^2}{GeV}\right)^{-2} \sim 3 \left(\frac{m_\psi c^2}{GeV}\right)^{-2}$$

$\Downarrow$

$$0.1 < \Omega_\psi h^2 < 1 \Rightarrow 2 \lesssim m_\psi \lesssim 5 \text{ GeV}, \quad \boxed{m_\psi \sim 1 \text{ GeV}}$$

~~Limite di~~ Lee-Weinberg Limit (See fig. 3.6 of MVdBW ~~vedi figura pag successiva~~)

- ~~Per~~ For $m_\psi > m_{Z^0}$ $\quad \sigma_0 \propto m_\psi^{-2}$

$$\sigma_0 \sim \frac{5}{2\pi} \frac{G_F^2}{(\hbar c)^4} (m_\psi c^2)^2 \cdot \left(\frac{m_Z c^2}{m_\psi c^2}\right)^4$$

$$\Gamma / H \sim 1 \Rightarrow \Omega_\psi h^2 \approx 0.1 \left[1 - \frac{\ln (m_\psi c^2)_{TeV}}{30} \right] (m_\psi c^2)^2_{TeV}$$

$$\boxed{\Omega_\psi h^2 \approx 0.1 \, (m_\psi c^2)^2_{TeV}} \Rightarrow 0.1 \lesssim \Omega_\psi h^2 \lesssim 1 \quad \text{implica}$$

$$1 \lesssim m_\psi \lesssim 3 \text{ TeV}, \quad \boxed{m_\psi \sim 1 \text{ TeV}}$$

We see that CDM candidates, interacting like neutrinos, must have $m_\psi \sim 1$ GeV or $m_\psi \sim 1$ TeV to be in agreement with cosmological observation, or to be a relevant contributor to dark matter.

Lee-Weinberg limit

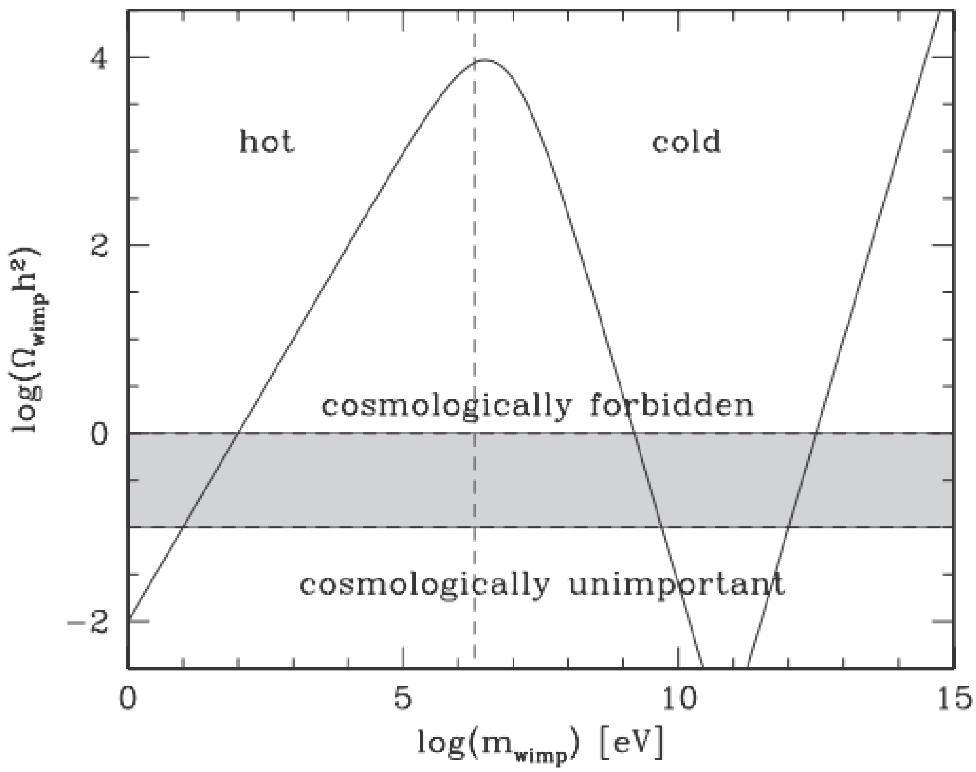


Fig. 3.6. Cosmological constraints on the mass of weakly interactive dark matter particles under the assumption that they interact as a Dirac-type neutrino. The solid curve shows the predicted cosmological density parameter of the WIMPs as a function of WIMP mass, while the shaded area roughly brackets the observed range of the cosmological density parameter. The mass ranges in which the particles make up 'hot' and 'cold' dark matter are indicated.

If DM particles candidates are taken from (CR/8
SUSY, estimates of their density largely depend
on the many free parameters of theories.
If R-parity is conserved, the lightest (LSP)
neutralino (χ) is the most natural DM candidate.
Neutralino states are a mixture of the bino ($\tilde{B}$),
wino ($\tilde{W}_3$) and two higgsinos ($\tilde{H}_0^1$, $\tilde{H}_0^2$). Bino and
Wino are SUSY partners of the gauge bosons B
and W_3 which, during EW symmetry breaking, convert
to photons and Z^0's.

All the relics mentioned above are thermal relics:
there was a time when they were in thermal equilibrium
with the energy bath, until they decoupled and freezed.
But there are also non-thermal relics, particles that
were never in thermal equilibrium, and were produ-
ced with negligible velocities. Their evolution
is then similar to CDM.
The axion is one of these relics: it has very low mass,
but its velocity is low, and

can form very small structures.

Recombination and Last Scattering

When the temperature of the Universe drops below $kT\sim 13.6\ eV$ (the ionization potential of hydrogen in the ground level) protons and electron begin to combine and form neutral hydrogen. This is the epoch of the ***recombination*** (actually, recombination is the name of the radiative process involved; for the Universe "first combination" would be more appropriate). But, due to the very large number of photons for each baryon (about 10^9, as we have seen), hydrogen becomes (almost) neutral at a lower temperature ($kT\sim 0.3\ eV$, $T\sim 3000\ K$)[3]. We neglect recombination of He, which takes place earlier.

There are different mechanisms involved in the making of neutral hydrogen. If recombination takes place in an isolated cloud of ionized hydrogen (HII cloud), two processes are dominant: direct recombination to the ground state, and the capture of an electron to an excited state which then cascades to the ground level. In the first case, a Lyman continuum photon (with energy larger than $13.6\ eV$) is produced, while in the second case one of the recombination photons must have an energy higher than or equal to that of Ly-α. If the cloud is optically thin (optical depth $\tau \ll 1$), all recombination photons can escape and do not contribute to further ionization.

In the case of cosmological recombination, however, recombination photons will be absorbed again because they cannot escape from the Universe. In fact, the direct capture of electrons to the ground state does not contribute to the net recombination, because the resulting photon is energetic enough to ionize another hydrogen atom from its ground state. The normal cascade process is also ineffective, because the Lyman series photons produced can excite hydrogen atoms from their ground states, so that multiple absorptions lead to re-ionization. Therefore, recombination in the early Universe must have proceeded by different means.

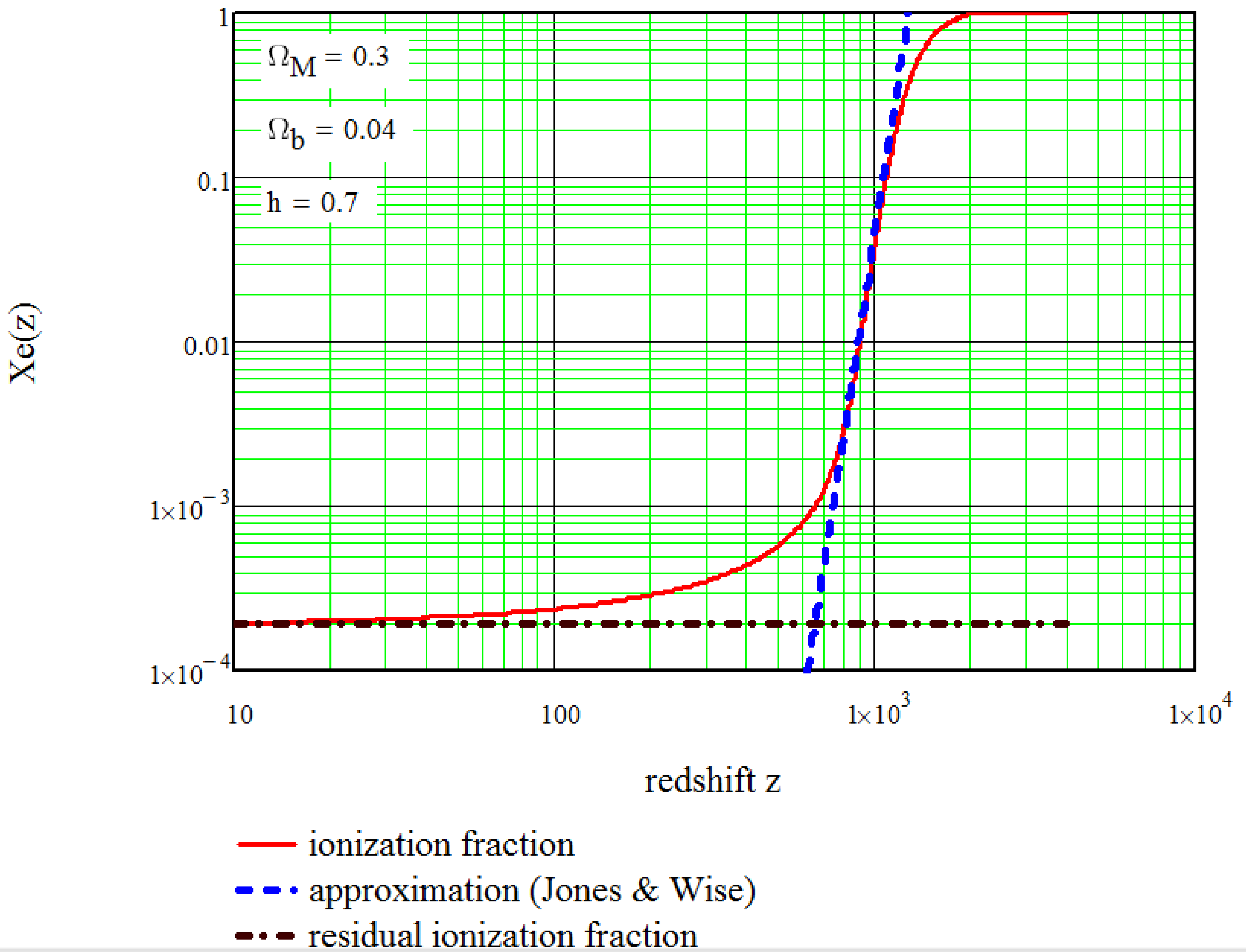
That leaves two main processes for the production of neutral, atomic hydrogen. One is ***two-photon decay*** from the metastable $2s$ level to the ground state, at the rate $\Gamma_{2\gamma} \approx 8.23\ s^{-1}$ (in this process two photons must be emitted in order to conserve both energy and angular momentum, and the energies of the two photons may not be able to contribute to ionization). The second is the loss of the Lyman-α resonance photons by the cosmological redshift. Two-photon decay turns out to be the dominant process.
Moreover, since expansion dilutes proton and electrons, at a certain time (redshift) recombination stops, is frozen, and a tiny fraction of ionized hydrogen remains.

We use the following definitions and relations: ionization fraction $X_e \equiv n_p/(n_p+n_H)$, $\eta \equiv n_B/n_\gamma = const. = 2.7\times10^{-8}\Omega_b h^2$, $n_B = n_p+n_H = \rho_{0b}/m_p$, m_p proton mass, $\rho_b=\rho_{0b}(1+z)^3$, $\rho_{0b}=\Omega_b\,\rho_{0cr}$, $T=T_{\gamma 0}(1+z)$. So electron (and proton) density is given by

$$n_e(z) = X_e(z) n_B = X_e(z)\rho_b / m_p = X_e(z)\times 1.13\times10^{-5}\Omega_b h^2 (1+z)^3$$

[3] For order of magnitude estimates, the Kelvin temperature T_K can be linked to energy by: $T_K \sim 10^{13} E_{GeV} \sim 10^4 E_{eV}$

The following figure shows the evolution of the ionization fraction versus redshift for $\Omega_M=0.3$, $\Omega_b=0.04$ and $h=0.7$.



Conventionally recombination corresponds to $X_e=0.1$. We see in the figure that $X_e \sim 0.1$ at a redshift around 1100. The figure also shows that recombination is never complete. The recombination process freezes, and a residual ionization remains (at z ~ 10):

$$X_{residual} \approx 10^{-5} \frac{\sqrt{\Omega_M h^2}}{\Omega_b h^2}$$

on the order of 10^{-4} .

The dependence on cosmological parameters is due to the balance between the recombination rate, proportional to n_p (equal to n_e), and the expansion rate H. So

$$\Gamma_{rec} \propto n_e \propto X_e(z)\Omega_b h^2 (1+z)^3 \qquad H(z) = H_0\sqrt{\Omega_M}(1+z)^{3/2} \text{ (MD EdS)}$$

$$\Gamma_{rec} \sim H \implies X_e(z)\Omega_b h^2 \propto H_0\sqrt{\Omega_M} \implies X_e \propto \frac{H_0\sqrt{\Omega_M}}{\Omega_b h^2} \propto \frac{\sqrt{\Omega_M h^2}}{\Omega_b h^2}$$

An approximation for $X_e(z)$, good for *800 <z <1200* is given by (Jones & Wise, 1985):

$$X_e(z) \cong 2.4\times10^{-3}\frac{\left(\Omega_M h^2\right)^{1/2}}{\Omega_b h^2}\left(\frac{z}{1000}\right)^{12.75}$$

Recombination is also associated to the last scattering of CMB photons, since after recombination the Universe becomes finally transparent.

A useful parameter is the optical depth: since $d\tau = -n_e\sigma_T c\,dt$ (τ grows starting from us, cosmic time increases toward us), where σ_T is the *Thomson scattering* cross section (***σ_T=6.65×10^{-25}cm^2***). When we integrate we have $[dt/dz = -1/(1+z)H(z)]$

$$\int_\tau^0 d\tau' = -\int_t^{t_0} n_e\sigma_T c\,dt' = -\int_z^0 n_e(z')\sigma_T c\frac{-1}{(1+z')H(z')}dz'$$

$$\tau(z) = \int_0^z n_e(z')\sigma_T\, c\frac{1}{(1+z')H_0E(z')}dz'$$

When estimating the optical depth τ, the dependence on cosmological parameters disappears since $n_e(z)=X_e(z)\,n_B(z) \sim X_e(z)\,\Omega_B h^2$, and $H_0\,E(z) \sim \Omega_M^{1/2}\,h$, so

$$\tau(z) \cong 0.37\left(\frac{z}{1000}\right)^{14.25}$$

The probability of receiving a photon from the optical depth τ is equal to $e^{-\tau}$. The probability of receiving a photon from the interval between τ and $\tau+d\tau$ corresponds to the probability of receiving it from the interval between z and $z+dz$:

$$e^{-\tau}d\tau = g(z)dz \Rightarrow g(z) = e^{-\tau}\frac{d\tau}{dz}$$

With the above approximation for $\tau(z)$

$$g(z) = 5.26\times10^{-3}\left(\frac{z}{1000}\right)^{13.25}\exp\left[-0.37\left(\frac{z}{1000}\right)^{14.25}\right]$$

which has a maximum for *z=1067*, and conventionally we assume that the ***last***

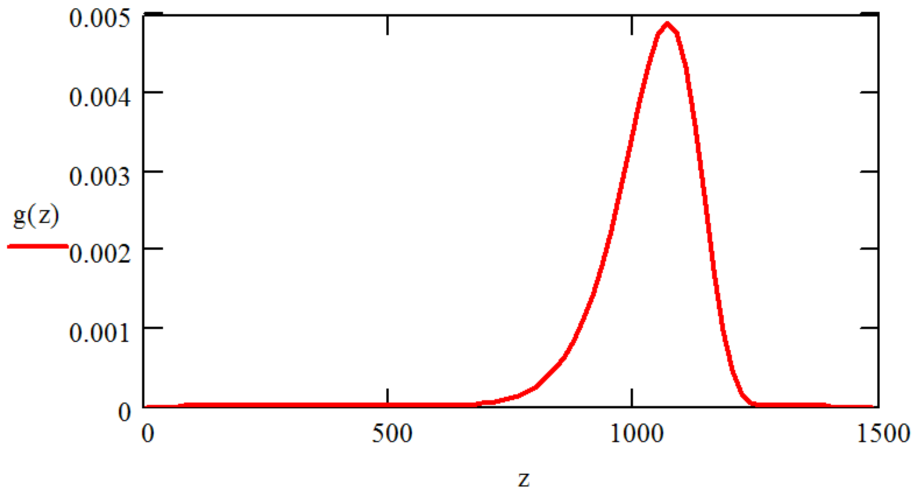
scattering corresponds to this redshift ($z_{ls} \approx 1067$). The following figure shows the probability distribution for the last scattering redshift. The 68% probability is included in a $\Delta z \approx 170$ around the maximum, so the last scattering event is not instantaneous and does not correspond to a single redshift. This means that the last scattered photons have a spread in their temperatures, but this is compensated by the higher redshift suffered by photon which decoupled earlier.

The age of the Universe at the last scattering can be derived, approximately, by using a *MD – EdS* model with $\Omega_M = 0.3$ and $h=0.7$, which gives

$$t(z_{ls}) \approx \frac{2}{3H_0\sqrt{\Omega_M}(1+z_{ls})^{3/2}} \approx 4.8 \times 10^5 \; years$$

while a better approximation gives about 4×10^5 years.

Probability distribution for the last scattering redshift



BIG BANG NUCLEOSYNTHESIS (BBN)

At times earlier than about 1 second the temperature was greater than 10^{10} K, corresponding to an average kinetic energy per particle of more than an MeV. At such energies, nuclear physics processes like

$$n + \nu_e \leftrightarrow p + e^-, \qquad n + e^+ \leftrightarrow p + \bar{\nu}_e, \qquad n \leftrightarrow p + e^- + \bar{\nu}_e \qquad (15.50)$$

and other processes involving leptons and photons such as (Fig. 15.4)

$$\nu + \bar{\nu} \leftrightarrow e^+ + e^- \leftrightarrow \gamma + e^+ + e^- \qquad (15.51)$$

can all be in thermal equilibrium. For the nucleons, $kT \ll mc^2$ and (neglecting the irrelevant chemical potential) (15.36) gives the energy density

$$\rho c^2 = mc^2 \frac{g}{\hbar^3}\left(\frac{mkT}{2\pi}\right)^{3/2} e^{-mc^2/kT}, \qquad (15.52)$$

the last being the usual Boltzmann suppression factor. Hence, the ratio of the number of neutrons to protons will be

$$r \equiv \frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-\frac{(m_n - m_p)c^2}{kT}} \cong e^{-\frac{(m_n - m_p)c^2}{kT}}$$

where m_n and m_p are neutron and proton masses, and $(m_n\text{-}m_p)c^2=1.293$ *MeV*. The rate of the interactions exchanging n into p and vice versa is (G_F = Fermi weak coupling constant) :

$$\Gamma_{n\leftrightarrow p} \cong 2(kT)^5_{MeV}\ s^{-1} \propto G_F^2 T^5$$

Compare this with $H=1/2t$ (*EdS* in RD era), where

$$t(\mathrm{sec}) \approx 2.4\, g_*^{-1/2}\, (kT)^{-2}_{MeV}$$

($g_* \approx 10$). $\Gamma_{n\leftrightarrow p} \approx H$ for $kT_{D\nu} \approx 0.7\ MeV$, $t_{D\nu} \approx 1.5\ sec$. The neutron to proton ratio freezes at

$$r_0=n_{n,0}/n_{p,0} \cong exp(-1.293/0.7) \cong 0.16.$$

Only neutron β *decay* is now possible, with $\tau_n=885.7\pm0.8\ sec$ (about 15 minutes).

. The neutrons will then start to decay, $n \rightarrow p + e^- + \bar{\nu}_e$, but the lifetime for this is long ($\approx$15 min) compared to the age of the universe at this point. Alternatively, they can combine with protons through very fast processes such as

$$p + n \leftrightarrow {}^2H + \gamma, \quad {}^2H + n \rightarrow {}^3H + \gamma, \quad {}^3H + p \rightarrow {}^4He + \gamma,$$
$$ {}^2H + p \rightarrow {}^3He + \gamma, \quad {}^3He + n \rightarrow {}^4He + \gamma. \quad (15.54)$$

The key process is the formation of deuterium ${}^2H$, which has a binding energy B_D = 2.23MeV. Because of the relatively large number of photons with respect to baryons, the high energy tail of the distribution of photons immediately dissociates the deuterium which is formed, and this until the number of dissociating photons n_γ^{diss} becomes comparable with that of baryons, n_B . We will have:

$$\frac{n_\gamma^{diss}}{n_B} = \frac{n_\gamma^{diss}}{n_\gamma} \cdot \frac{n_\gamma}{n_B} = \frac{1}{\eta} \cdot \frac{n_\gamma^{diss}}{n_\gamma}$$

with

$$n_\gamma = \frac{2}{\pi^2}\zeta(3)\left(\frac{kT}{\hbar c}\right)^3$$

The density of the dissociating photons is obtained by putting $P = E/c$ in the relation that gives the density of photons, placing B_D as the lower limit in the integration:

$$n_\gamma^{diss} = \frac{2}{2\pi^2(\hbar c)^3}\int_{B_D}^{\infty}\frac{E^2 dE}{e^{E/kT}-1} \approx \frac{1}{\pi}\left(\frac{kT}{\hbar c}\right)^3\left(\frac{B_D}{kT}\right)^2 e^{-B_D/kT},$$

a good approximation since $E/kT > B_D/kT \gg 1$. For $1 < \eta_{10} < 10$, $n_\gamma^{diss}/n_B \approx 1$ if $kT \approx 0.1$ *MeV*, $\boldsymbol{T \approx 10^9 K}$. ($\eta_{10} = \eta / 10^{-10}$)

At this time the deuterium is no longer destroyed by photons and quick reactions occur leading to the formation ^{4}He: this is the era of BBN. The universe has an age of about ($g_* = 3.36$ at $kT = 0.1$ *MeV*)

$$t_{BBN}(\text{sec}) \approx 2.4\, g_*^{-1/2}\,(0.1)_{MeV}^{-2} \approx 150\,s$$

That is about **<u>three minutes</u>**.

Between the freezing, $t_{D\nu} \approx 1.5$ *sec*, and t_{BBN} neutrons decay to protons and, from $r_0 \approx 0.16$, we arrive to

$$r_{BBN} \cong \frac{n_{n,0}e^{-t_{BBN}/\tau_n}}{n_{p,0} + n_{n,0}\left(1 - e^{-t_{BBN}/\tau_n}\right)} \cong 0.13$$

After the bottleneck of deuterium, all neutrons that did not decay end up embedded in the nuclei of ^{4}He. Since it takes two neutrons for each ^{4}He nucleus and this has atomic weight 4, the abundance in mass Y_{BBN}, of ^4He is

$$Y_{BBN} \equiv \frac{mass\,of\ \ ^4He}{mass\,of\ \ ^4He + mass\,of\ \ free\ \ protons} = \frac{4\cdot n_n/2}{4\cdot n_n/2 + 1\cdot\left(n_p - n_n\right)}$$

$$Y_{BBN} = \frac{2\,r_{BBN}}{1 + r_{BBN}} \approx 0.23$$

The detailed calculation, much more complicate, provides similar values, in agreement with the experimental data that suggest Y_{obs} around *0.24-0.25*.

As shown in the following figure, the predicted abundance of ^{4}He does not vary much with the baryon-to-photon ratio $\boldsymbol{\eta}$, because τ_n is long (compared to the age of the universe) and neutrons decay slowly. However, Y_{BBN} depends strongly on $T_{D\nu}$, which depends on *H*, which in turn depends on g_* at a temperature of about *1 MeV*:

$$g_* = 2 + \frac{7}{8}(4 + 2\cdot N_\nu)$$

where N_ν is the number of neutrino species. The higher the value of N_ν the higher is $T_{D\nu}$ and so the greater are r_0 and Y_{BBN} ($\Delta N_\nu = 1 \Rightarrow \Delta Y_{BBN} \cong 0.013$, see the lines in the figure). The observational limits on Y_{BBN} give $N_\nu = 3 \pm 1$. In the *80s*, until *LEP* at *CERN* measured the decay (width) of Z^0 and obtained $N_\nu = 2.994 \pm 0.012$, the best estimate of N_ν was given by BBN. We notice that BBN and *LEP* are sensitive to different kinds of particles: BBN is sensitive to particles that were relativistic at *kT~1 MeV*; the width of Z^0 is sensitive to neutrinos with masses $m_\nu < M_{Z^0}/2$. So they measure different things.[4]

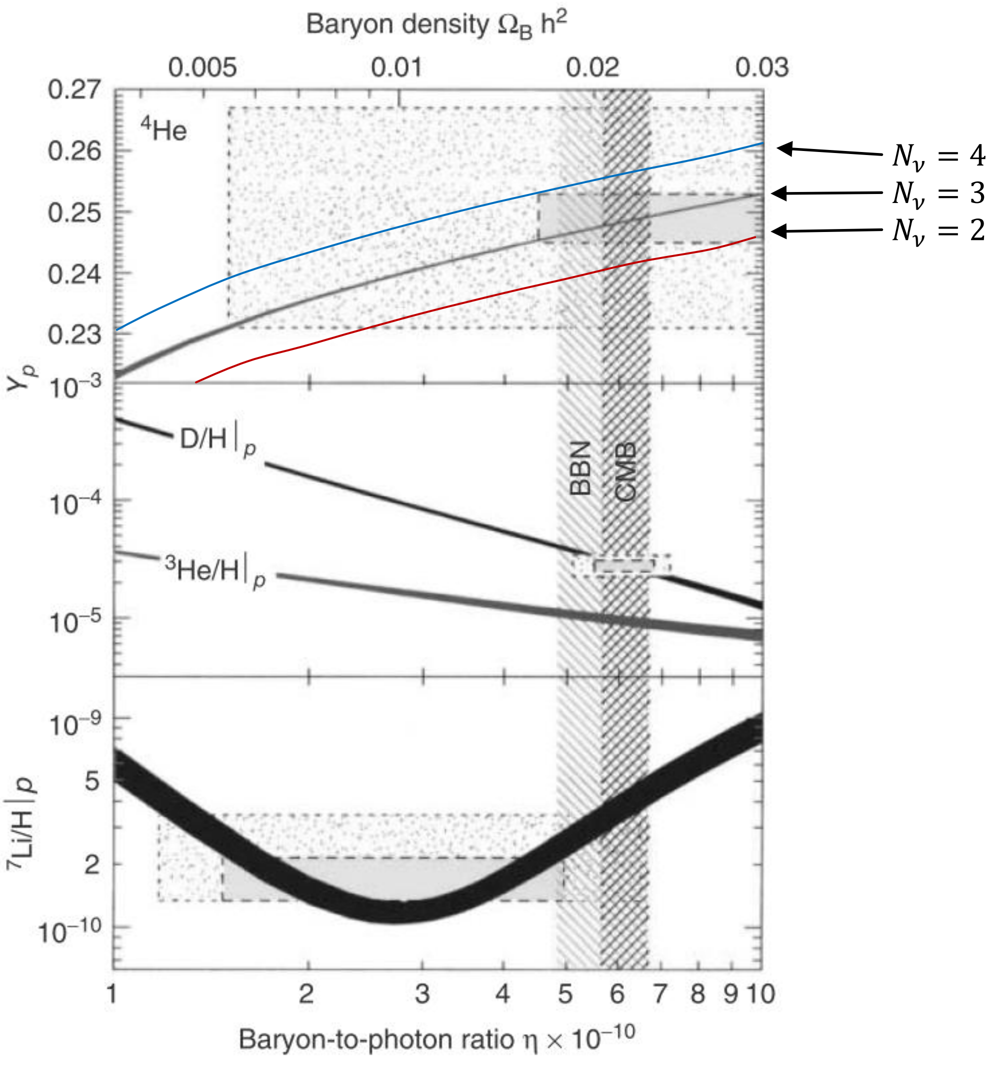


Figure 8.7 Baryon components of the universe. The unit of abscissa is $\eta_{10} \equiv n_B/n_\gamma \times 10^{10}$. Values determined from observation of light elements agree well with those determined from the nucleosynthesis [733, 739] and CMB (cosmic microwave background radiation) spectrum by WMAP [705, 740]. (Reproduced with permission of [7].)

[4] If you are interested in the possibilities offered by BBN to explore physics beyond the Standard Model, look at the Particle Data Group site (*http://pdg.lbl.gov/*), and in particular the review on BBN (*http://pdg.lbl.gov/2015/reviews/rpp2015-rev-bbang-nucleosynthesis.pdf*).

Although the bulk of the neutrons end up in ^{4}He, it is predicted that some ^{2}H, ^{3}H, and ^{3}He will remain, and, despite the absence of stable nuclides with atomic mass number $A = 5$ or 8, and the large Coulomb barrier between higher-Z nuclei, some heavier elements will be formed by processes such as

$$^3\text{He} + {^4\text{He}} \rightarrow {^7\text{Be}} + \gamma, \qquad {^4\text{He}} + {^3\text{H}} \rightarrow {^7\text{Li}} + \gamma, \qquad (15.56)$$

but the rates depend more critically on the baryon density, as Fig. 15.7 illustrates. Of course later, when matter condensed into stars, its temperature again became sufficiently high for nuclear reactions to start up

So, for the production of Carbon, Nitrogen Oxigen and so on, we have to wait for the formation and evolution of stars.

Concordance, Dark Matter, and the CMB

We now use the observed light element abundances to test the theory. We first consider standard BBN, which is based on Standard Model physics alone, so $N_\nu = 3$ and the only free parameter is the baryon-to-photon ratio η. Thus, any abundance measurement determines η, while additional measurements overconstrain the theory and thereby provide a consistency check. Also observations of the CMB constrain the value of η.

First we note that the overlap in the η ranges spanned by the larger boxes (which include systematic errors) in the Figure above indicates overall concordance. More quantitatively, when we account for theoretical uncertainties, as well as the statistical and systematic errors in observations, there is acceptable agreement among the abundances when

$$5 \leq \eta_{10} \leq 6.5 \ (95\%\ CL).$$

However, the agreement is much less satisfactory if we use only the quoted statistical errors in the observations. In particular, as seen in the Figure, D and ^{4}He are consistent with each other, but favor a value of η which is higher than that indicated by the ^{7}Li abundance determined in stars. Actually, there is a possible problem with Lithium, which maybe requires new physics; the discrepancy could also be explained by astrophysical processes during stellar evolution.

Even so, the overall concordance is remarkable: using well-established microphysics we have extrapolated back to an age of ∼ 1 s to correctly predict light element abundances spanning 9 orders of magnitude. This is a major success for the standard cosmology, and inspires confidence in extrapolation back to still earlier times. This concordance provides a measure of the baryon content

$$0.019 \leq \Omega_b h^2 \leq 0.024 \ (95\%\ CL),$$

a result that plays a key role in our understanding of the matter budget of the Universe.

Primordial Baryosynthesis

The above picture still leaves us, however, with the problem of where the net baryon number and lepton number of the universe have come from, i.e., why there are more quarks than antiquarks, and more electrons than positrons. We know that the solar system is made of matter not antimatter. The very small proportion of antimatter in the cosmic radiation ($\approx 10^{-4}$), the failure to observe the X-rays that would result if matter–antimatter annihilation occurred at all commonly in the collisions of stars, gas clouds, or galaxies, and the lack of any very convincing mechanism for separating matter and antimatter on a cosmic scale, all suggest that the universe is made just of matter. If in the early universe the numbers of quarks and antiquarks had been equal, as Fig. 15.4 suggests, their final annihilation once $kT < 1$ GeV is estimated to yield only

$$\eta \equiv \frac{n_B}{n_\gamma} = \frac{n_{\bar{B}}}{n_\gamma} \approx 10^{-19}, \qquad \text{rather than} \qquad \eta \approx 10^{-9} \gg \frac{n_{\bar{B}}}{n_\gamma}, \tag{15.57}$$

which is actually observed.

7.6 Criteria for a baryon asymmetry

If we assume unitarity (all probabilities of interactions add up to 1) and CPT is a good symmetry, then a nonzero baryon asymmetry can be generated if the following conditions hold†, (Sakharov, 1967; Kuzmin, 1970).

i) baryon number is not conserved
ii) C and CP are not conserved
iii) there is departure from thermal equilibrium

Condition (i) is necessary if we are to pass from a state with baryon number (B), zero to $B \neq 0$; however, it is not a sufficient condition. The C operator changes n_q—$n_{\bar{q}}$ so if C is conserved we must have $n_q = n_{\bar{q}}$ in the system and

†C : charge conjugation, C (particle) = antiparticle; P parity reversal, P (right hand) = left hand; T - time-reversal.

hence $B=0$. Since the P operator leaves both n_q and $n_{\bar{q}}$ unchanged the CP operator also requires $n_q = n_{\bar{q}}$ and hence $B=0$. Hence the condition (ii) is necessary. Finally, if thermal equilibrium obtains then T is a good symmetry and so CPT symmetry would imply CP symmetry and $B=0$ by (ii): therefore, we require condition (iii). [For more detail see especially Kolb and Wolfram (1980)].

The asymmetry could be linked to the breaking of GUTs or of the electro-weak interaction. The question is open and, very likely, requires new physics.

INFLATION

We have already mentioned two problems affecting the Hot Big Bang model: the ***flatness problem*** and the ***horizon problem***. To them one can add the ***magnetic monopoles problem*** (monopoles are zero-dimensional topological defects, that are produced at the time of the phase transition corresponding to the breaking of GUTs; their number density, coupled with their very high mass, would produce a value of Ω clearly unacceptable).

The paradigm of ***inflation***, which solves these problems, has been proposed by *Alan Guth* in 1981. It assumes that there has been an accelerated expansion phase between the times t_i and t_f (with $t_{Pl} < t_i < t_f \ll t_{eq}$), produced by an equation of state that mimics that of a cosmological constant:

$$t_i < t < t_f \qquad a(t) \approx a(t_i)e^{H(t-t_i)}$$

(H ~ constant). The scale factor grows as in a de Sitter model (which has $\Lambda \neq 0$ and density of matter negligible), instead of growing as $a(t) \sim t^{1/2}$, like an EdS model in the RD era.

The exponential growth, if sufficiently prolonged, produces a growth of the particle horizon d_H sufficient to solve the horizon problem; Ω converges towards unity (as in models dominated by the cosmological constant), resolving the flatness problem (remember also that the curvature of the spatial section scale as $a(t)^{-2}$, and the exponential growth of $a(t)$ force this curvature towards zero). The problem of monopoles is resolved through a strong dilution of their number density.

If inflation occurs around the time of the breaking of grand unification (GUT), the above problems are solved provided

$$\ln\left(\frac{a_f}{a_i}\right) \equiv \mathcal{N} \geq 60$$

where $\mathcal{N}$ is the number of ***e-foldings***.

Lagrangian formulation of Field Equations

As we have seen, to have a phase of inflation is necessary that the universe possesses, for a certain time interval, an equation of state of the type $P = -\rho c^2$.

This can be achieved in a natural way by means of a scalar field present in the early stages of the early universe.To understand the mechanism it is necessary to introduce some concepts used in Quantum Field Theory.

In Classical Mechanics the equations of motion of a dynamical system can be derived from a ***Lagrangian function L***

$$L(q_i, \dot{q}_i) = T(\dot{q}_i) - V(q_i)$$

where q_i are the generalized coordinates, *T* is the kinetic energy and *V* is the potential energy. The action *S*, involved in the motion of the system from one configuration at time t_1 to another at the time t_2, is given by

$$S = \int_{t_1}^{t_2} L dt$$

and, according to the ***principle of least action***, the evolution of the system between the two configurations is that which corresponds to the minimum value of *S*. This condition leads to the *Euler-Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

These relations describe the motion of particles, that is, of localized objects. A field instead occupies a certain region of space, and Field Theory wants to calculate one (or more) functions of position and time: $\phi = \phi(x, y, z, t)$ (eg .: temperature, electric potential, the three components of the magnetic field in a room). While, in the mechanics of particles, the Lagrangian ***L*** is a function of the coordinates q_i and of their derivatives, Field Theory works with a ***Lagrangian density*** $\boldsymbol{\mathcal{L}}$ which is a function of the field ϕ and of its derivatives with respect to *x, y, z*, and *t*. To keep the relativistic covariance of physics more apparent, we use space-time coordinates $x_0 = ct$ and $x_1, x_2, x_3 = x, y, z$, so that the Lagrangian is the volume integral of $\boldsymbol{\mathcal{L}}$

$$L = \int \boldsymbol{\mathcal{L}} \, d^3x$$

and the action is

$$S = \frac{1}{c}\int \boldsymbol{\mathcal{L}} \, d^4x$$

(the factor *1/c*, inessential, serves to keep the dimensions of the action).

In relativistic field theory q_i is replaced by the field ϕ, and the index *i* is replaced by space-time coordinates x^α. Since each time derivative can be associated to a similar term involving a gradient, we use all the covariant derivatives $\partial\phi/\partial x^\alpha = \partial_\alpha \phi$ and Euler-Lagrange equations become

$$\partial_\alpha \left[\frac{\partial(\boldsymbol{\mathcal{L}})}{\partial(\partial_\alpha \phi)}\right] - \frac{\partial(\boldsymbol{\mathcal{L}})}{\partial \phi} = 0$$

Actually this writing is correct in a Euclidean space and in orthogonal coordinates; to take account of a more general choice of coordinates (e.g. co-moving spatial coordinates) the volume element d^4x is replaced with $\sqrt{-g}\, d^4x$ where g is the determinant of the metric $g_{\alpha\beta}$. So the Euler-Lagrange equation becomes

$$\partial_\alpha \left[\frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial(\partial_\alpha \phi\,)} \right] - \frac{\partial\left(\sqrt{-g}\,\mathcal{L}\right)}{\partial\phi} = 0$$

In a flat, static Minkowski space, the metric is $g_{\alpha\beta} = \eta_{\alpha\beta} = \eta^{\alpha\beta} = diag(1,-1,-1,-1)$. Then

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}; \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\partial_0 ; \overline{\nabla} \right)$$

$$\partial^\alpha = \eta^{\alpha\nu}\partial_\nu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}; -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \left(\partial^0 (\equiv \partial_0); -\overline{\nabla} \right)$$

$$\partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box^2$$

If we use comoving coordinates ($r = a\,x$) in a flat, expanding space: $g_{\alpha\beta} = diag(1, -a^2, -a^2, -a^2)$ and $\sqrt{-g} = a^3$. We have then ($\overline{\nabla}_x$ is the gradient referred to the co-moving coordinate x)

$$\partial_\alpha = \left(\partial_0 ; \overline{\nabla}_x \right) \qquad \partial^\alpha = \left(\partial^0 ; -\frac{1}{a^2}\overline{\nabla}_x \right) \qquad \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{a^2}\nabla_x^2$$

Let's consider, for instance, the following Lagrangian (density):

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha\phi\right)\left(\partial^\alpha\phi\right) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2 = \frac{1}{2}\left(\partial_\alpha\phi\right)\left(\partial^\alpha\phi\right) - \frac{1}{2}\mu^2\phi^2$$

where ϕ is a real, single scalar field. In this case, *i.e.* Minkowski space, ($\sqrt{-g} = 1$),

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha\phi\right)} = \partial^\alpha\phi \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -\left(\frac{mc}{\hbar}\right)^2\phi = -\mu^2\phi$$

and hence Euler-Lagrange formula requires[5]

$$\partial_\alpha\partial^\alpha\phi + \mu^2\phi = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \mu^2 \right]\phi = \left[\Box^2 + \mu^2 \right]\phi = 0$$

[5] The first result above can be derived by expanding completely the Lagrangian: $\mathcal{L} = \frac{1}{2}[\partial_0\phi\,\partial_0\phi - \partial_1\phi\,\partial_1\phi - \partial_2\phi\,\partial_2\phi - \partial_3\phi\,\partial_3\phi\,] - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$, so that $\frac{\partial}{\partial(\partial_0\phi)} = \partial_0\phi = \partial^0\phi$, $\frac{\partial}{\partial(\partial_1\phi)} = -\partial_1\phi = \partial^1\phi$ and so on.

which is the ***Klein-Gordon equation***, describing (in Quantum Filed Theory) a particle of spin 0 and mass *m*.

In analogy with *L = T - V*, in the Lagrangian written above the first term, $\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$, is named kinetic energy term, while the second term, in this case quadratic in ϕ (the term corresponding to the mass), is the potential energy term. For a scalar field we will write the Lagrangian in the general form

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\phi)(\partial^\alpha\phi) - V(\phi)$$

where $V(\phi)$ is a suitable potential ($V(\phi)=1/2\ \mu\ \phi^2$ in *Klein-Gordon* case). If $\mathcal{L}$, written as above, depends on *x* only through ϕ and its derivatives $\partial_\alpha\phi$, the following quantity (***energy-momentum tensor***) is preserved (i.e., has four-divergence equal to zero):

$$T^{\alpha\beta} \equiv \partial^\alpha\phi\,\partial^\beta\phi - g^{\alpha\beta}\mathcal{L}$$

In the case of a **perfect fluid** we have seen that the energy-momentum tensor has the form

$$T^{\alpha\beta} = (p+\rho c^2)u^\alpha u^\beta - p\,g^{\alpha\beta}$$

where *P* is the pressure, ρc^2 the Energy density and u^α is the four-velocity ($u^\alpha \equiv dx^\alpha/ds$); in the co-moving reference frame $u^\alpha=(1,0,0,0)$. In a flat space, by using co-moving coordiantes, the comparison of the two relations gives:

$$\begin{cases} \rho c^2 \equiv T^{00} = \frac{1}{2}\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\frac{1}{a^2}\left(\overline{\nabla}_x\phi\right)^2 + V(\phi) \\ p \equiv a^2\left(\frac{T^{11}+T^{22}+T^{33}}{3}\right) = \frac{1}{2}\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{6}\frac{1}{a^2}\left(\overline{\nabla}_x\phi\right)^2 - V(\phi) \end{cases}$$

(see the following scanned page for the proof).

In the case in which the field is spatially homogeneous (from which $\overline{\nabla}_x\phi = 0$; even if $\overline{\nabla}_x\phi$ is different from zero the term containing $\overline{\nabla}_x\phi$ becomes rapidly negligible due to the a^{-2} factor [6]) and the term $1/2c^2\ (\partial\phi/\partial t)^2$ is negligible compared to the potential $V(\phi)$, we have

$$\rho c^2 = V(\phi)$$
$$p = -V(\phi) = -\rho c^2$$

that is, an equation of state that mimics that which corresponds to the cosmological constant!

[6] Actually there are small fluctuations on the scale of the Hubble radius, which are the "seeds" of the large-scale structure of the universe

$$T^{\alpha\beta} = \partial^\alpha\phi\,\partial^\beta\phi - g^{\alpha\beta}\mathcal{L} \quad \Big| \quad \mathcal{L} = \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)$$

and

$$T^{\alpha\beta}_{fl} = (p + \rho c^2)u^\alpha u^\beta - p\,g^{\alpha\beta} \qquad \boxed{r \equiv a \cdot x} \text{ comoving coord.} \qquad \partial^0 = \partial_0$$

$$g_{\alpha\beta} = (1, -a^2, -a^2, -a^2)\ ; \quad g^{\alpha\beta} = \left(1, -\frac{1}{a^2}, -\frac{1}{a^2}, -\frac{1}{a^2}\right); \quad \partial^i = -\frac{1}{a^2}\partial_i$$

$$T^{00} = \partial^0\phi\,\partial^0\phi - \underbrace{g^{00}}_{\equiv 1}\left[\frac{1}{2}\left(\partial_0\phi\partial^0\phi + \partial_1\phi\partial^1\phi + \partial_2\phi\partial^2\phi + \partial_3\phi\partial^3\phi\right) - V(\phi)\right] =$$

$$= \left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left\{\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{a^2}\underbrace{\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]}_{\overline{D}_x\phi\cdot\overline{D}_x\phi}\right\} + V(\phi)$$

$$\boxed{T^{00} = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2a^2}(\overline{D}_x\phi)^2 + V(\phi) = \rho c^2 = T^{00}_{fl}}$$

$$T^{11}_{fl} = -p \cdot -\frac{1}{a^2} = \frac{p}{a^2} = T^{22}_{fl} = T^{33}_{fl}$$

$$T^{11} = \partial^1\phi\,\partial^1\phi - \left(-\frac{1}{a^2}\right)\left[\qquad\qquad\right] =$$

$$= \frac{1}{a^4}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{a^2}\left[\frac{1}{2}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\frac{1}{a^2}(\overline{D}_x\phi)^2\right] - \frac{1}{a^2}V(\phi)$$

$$+\ T^{22} = \frac{1}{a^4}\left(\frac{\partial\phi}{\partial y}\right)^2 + \quad " \quad "$$

$$+\ T^{33} = \frac{1}{a^4}\left(\frac{\partial\phi}{\partial z}\right)^2 + \quad " \quad "$$

$$T^{11} + T^{22} + T^{33} = \frac{1}{a^4}(\overline{D}_x\phi)^2 + \frac{3}{2a^2}\left[\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{a^2}(\overline{D}_x\phi)^2\right] - \frac{3}{a^2}V(\phi) = \frac{3p}{a^2} \qquad \Big/ \frac{a^2}{3}$$

$$p = \frac{1}{a^2}\cdot\frac{1}{3}(\overline{D}\phi)^2 + \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2a^2}(\overline{D}\phi)^2 - V(\phi) \qquad \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$$

$$\boxed{p = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{6a^2}(\overline{D}_x\phi)^2 - V(\phi)}$$

If $\forall\alpha : \partial_\alpha\phi$ is negligible with respect to *V(ϕ)*, which is then *constant*,

$$T^{\alpha\beta} \approx -p\, g^{\alpha\beta} \approx \rho c^2 g^{\alpha\beta} \approx V(\phi) g^{\alpha\beta} \equiv \frac{\Lambda_{eff} c^4}{8\pi G}\, g^{\alpha\beta}$$

and the term of potential energy corresponds to an effective cosmological constant.

Phase transitions and Symmetry Breaking

In the history of the early universe one or more phase transitions have occurred. At high energies, according to the unified theory of the electroweak interaction, the weak and electromagnetic interactions were manifestations of a single force. Then, due to the progressive cooling produced by cosmic expansion, at a certain time (around a critical temperature T_{EW}~$10^{15}K$, E_{EW}~$10^2 GeV$) the universe has undergone a phase transition, after which the two interactions separated.

The Grand Unified Theories (GUTs), which attempt to unify electromagnetism and weak and strong interactions, in turn, require a phase transition in the universe at critical temperature T_{GUT}~10^{28}-$10^{29}K$, E_{GUT}~10^{15}-10^{16} GeV, above which there was symmetry between the three interactions.

Let's consider an analogy with the magnetization of a ferromagnetic material. Above the Curie temperature T_C the magnetic moments linked to the spins of atoms are randomly oriented and rapidly fluctuating, there is rotational symmetry around each point of the material and the expectation value of (the mean value) of the spin is null ($<\boldsymbol{S}> = 0$). However, falling the temperature below T_C, alignemet of spins becomes energetically more favorable, and there is a phase transition to a magnetized state, with $<\boldsymbol{S}> \neq 0$ in a certain direction $\hat{\boldsymbol{\imath}}$. The original symmetry is lost, broken, because the different domains that begin to form, independently of each other, have spins with different directions. In the end, when the whole mass has turn into domains, defects form at the borders of the different regions.

In a similar way, while above T_{GUT} there was symmetry between the three interactions, below T_{GUT} it is broken. Going back to the case of the ferromagnetic material, the way in which the rotational symmetry is broken in the different portions of the mass can be measured by the growth of the spin $\boldsymbol{S}$ and the orientation of the different domains. Similarly, the way in which the symmetry between the three interactions breaks down can be characterized by the acquiring of non-null values of parameters named ***Higgs fields***; this phenomenon is called ***spontaneous symmetry breaking*** (*SSB*). The symmetry is present when the Higgs fields have zero expectation value; it is spontaneously broken when at least one of the boson fields acquires an expectation value other than zero. As in the case of ferromagnetic domains, defects remain at the boundaries of the different regions in which the symmetry is broken in different ways, assuming different sets of values for the Higgs fields. These defects are called ***topological defects***, and may be two-dimensional (domain walls), uni-dimensional (cosmic strings) and zero-dimensional (magnetic monopoles). During the phase transition that leads to the breaking of the symmetry a period of exponential expansion may also occur: the inflation. Let's see how.

For simplicity we consider a single Higgs field, the scalar field ϕ. We consider again a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) - V(\phi)$$

The equation of motion, a generalization of *Klein-Gordon* (*KG*) equation, becomes

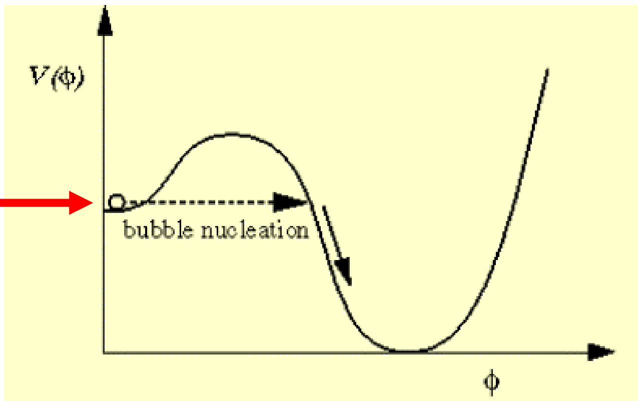
$$\Box^2 \phi + \frac{\partial V}{\partial \phi} = 0$$

Free particle states are the solution of this equation with only a quadratic term in ϕ in the potential $V(\phi)$, like in *KG* case; the coefficient of this term specifies the mass m of the particle: $V(\phi)=1/2\ \mu^2\ \phi^2$, $\mu = m\ c/\hbar$. The "***vacuum***" state, which by definition is the state in which there are no particles, occurs when $\partial V/\partial\phi = 0$; in the *KG* case this occurs at $\phi = 0$.

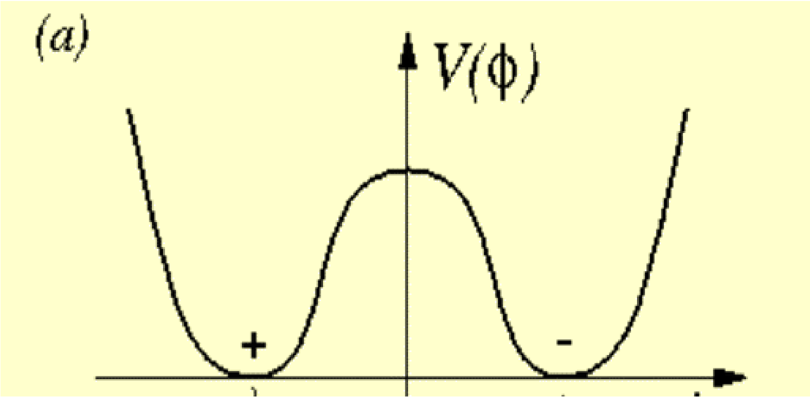
Higher-order terms in $V(\phi)$ correspond to the interactions between these particles. The equation written above admits the solution ϕ=*constant* at any value of ϕ for which $\partial V/\partial\phi=0$. The vacuum (no particles) state will therefore be one of those in which the expevtation value of ϕ assumes one of these constant values. There are several possibilities:

- It may be that the $\partial V/\partial\phi = 0$ has only one solution. In order for the energy to be bounded below, this should correspond to a minimum of $V(\phi)$ and also corresponds to the unique vacuum of the theory
- On the other hand there may be multiple solutions of $\partial V/\partial\phi = 0$. The maxima of the potential are unstable, but all the minima are possible vacua of the theory. If there is more than a minimum, the lowest would be the ultimate vacuum, the "***true vacuum***" of the universe.
- However, the universe may be, at a certain moment, in a local minimum with a higher value of the potential; it would be in a "***false vacuum***", with the possibility, by tunnel effect, to move to the *true vacuum*.

Inflation, for this potential, corresponds to the trapping of the field in the well at ϕ=*0*.



- In some cases there may be several such minima that have the same value of the potential, and the vacuum is degenerate.



This Figure corresponds to a potential of the form

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 + V(0)$$

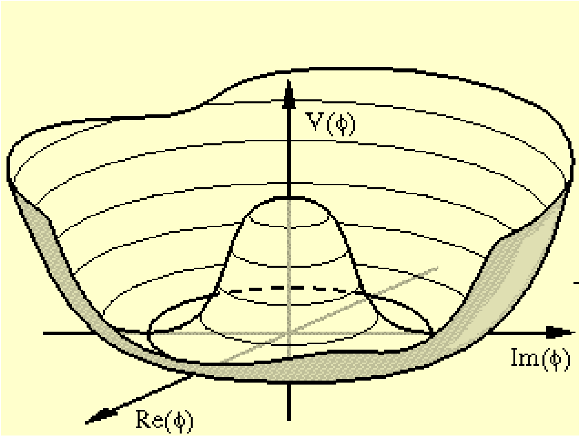
with μ and λ real constants ($\lambda > 0$ if the potential is bounded from below). The first term on the r.h.s. looks like a mass term and the second like an interaction, but the sign of the mass term is wrong, the mass should be imaginary! However, $\phi=0$ is a maximum for the potential, and we have two, degenerate, minima corresponding to possible "vacua" or groud states for

$$\phi = \pm\mu/\sqrt{\lambda} \equiv \pm v$$

Peturbation theory involves an expansion of $\mathcal{L}$ in ϕ around a minimum of the potential. We arbitrarily choose one of the two minima, for instance $+v$, and define a new field $\eta \equiv \phi - v$. We write the potential as a function of the new field η and now the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha\eta)(\partial^\alpha\eta) - [\mu^2\eta^2 + \text{terms cubic and higher order in } \eta + \text{const.}]$$

which possess the right sign of the mass term and complicated interactions. If we chose the other minimum the mass term remains the same (only the η^3 term changes his sign).



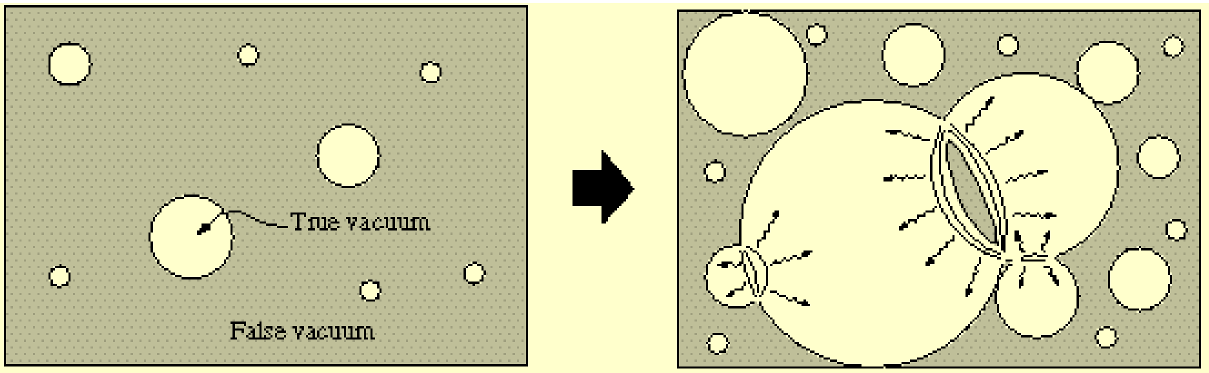
This is an example with only two possible values for the true vacuum, but more general potentials can lead to an infinite number of possible values in which the true vacuum may end. Here we see a two-dimensional (complex) case for the potential [we substitute $\phi\phi^*$ to ϕ^2 and $(\phi\phi^*)^2$ to ϕ^4, where ϕ^* is the complex conjugate of ϕ]. True possible vacua correspond to points belonging to the circle of the minimum values of $V(\phi)$.

The random choice of one of the minima generates a ***spontaneous symmetry breaking*** (*SSB*), similar to the formation of a domain with a particular orientation of the spins of its atoms in a portion of a ferromagnetic material that cools below T_C.

The potential written above has this form, with a negative coefficient for ϕ^2, at a temperature $T = 0$. But in the early universe, when the temperature is very high, to take into account this effect, corrections to $V(\phi)$ produce an effective potential with additional terms proportional to $\phi^2 T^2$. Therefore, at high enough temperature, the minimum of the potential is at $\phi = 0$, and the symmetry is unbroken.

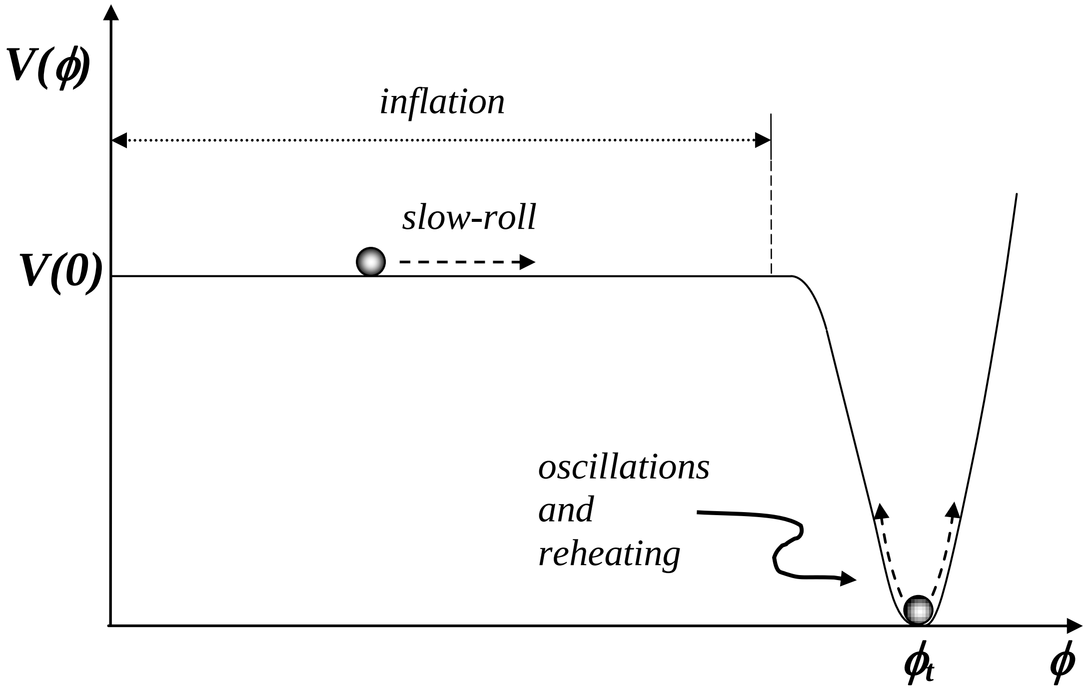
During the cooling of an expanding universe, according to details that depend on the particular shape of $V(\phi)$, the spontaneous symmetry breaking will take place:

- Through a ***phase transition of first order***, in which the field, initially in $\phi = 0$, crosses, by tunnel effect, a potential barrier within which it remains trapped for a certain time; inflation, with $\partial\phi/\partial t = 0$, occurs during this trapping phase. This is the model initially proposed by Guth, named ***old inflation***, which presents, however, some problems. In fact, a phase transition of first order occurs through the formation of bubbles of the new phase in the middle of the old phase; these bubbles expand, collide and coalesce until the new phase completely replaces the old one.



  But in the model of Guth, to have a phase of inflation sufficiently long, the probability of forming bubbles is low and, since the false vacuum expands exponentially, the bubbles can not coalesce and the transition to the true vacuum does not occur.

- Through a ***phase transition of the second order***, in which the field evolves smoothly from one phase to another. This is the model of ***new inflation***, proposed by *Linde, Albrecht and Steinhardt* in 1982, in which the field evolves very slowly (***slow-roll***) from the condition of false vacuum at $\phi = 0$ to the true vacuum. Again, if the evolution from $\phi = 0$ takes place slowly and $V(\phi) \simeq V(0)$ for a long enough time before falling into the true vacuum, we have a phase of inflation (we will see, later, what are the conditions for this to happen).



Orders of magnitude for Inflation

The expansion in the Early Universe, if we neglect the curvature (which, however, tends rapidly towards zero due to the enormous growth of the scale factor), will be given by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

and the energy density is given by

$$\rho c^2 = \rho_R c^2 + \rho_\Lambda c^2$$

with

$$\rho_R c^2 = \frac{\pi^2}{30}\cdot\frac{g_*(T)(kT)^4}{(\hbar c)^3}$$

$$\rho_\Lambda c^2 = V(\phi = 0)$$

Where $V(\phi)$ corresponds to the energy density of the field ϕ that, at high temperature, has its minimum in $\phi = 0$.

Until ρc^2 is dominated by $\rho_R c^2$, the universe behaves as in the model of EdS dominated by radiation, with $a(t) \sim t^{1/2}$. But while $\rho_R c^2$ scale as $1/a^4$, $\rho_\Lambda c^2$ remains constant. At a certain time t_i, $\rho_R c^2 \sim \rho_\Lambda c^2$, and, from that moment, the expansion becomes dominated by an "effective" cosmological constant Λ_{eff}, as in the exponentially expanding de Sitter model:

$$a(t) = a_i \exp\left[\sqrt{\frac{8\pi G}{3c^2}V(\phi=0)}\left(t-t_i\right)\right]$$

$$H = \cos t = \sqrt{\frac{8\pi G}{3c^2}V(\phi=0)} \equiv \sqrt{\frac{\Lambda_{eff}c^2}{3}}$$

where a_i *is the scale factor at* t_i. At the equality time, at $T=T_\Lambda$,

$$\rho_\Lambda c^2 = \rho_R c^2 = \frac{\pi^2}{30}\cdot\frac{g_*(T_\Lambda)\left(kT_\Lambda\right)^4}{\left(\hbar c\right)^3}$$

$$\approx 1.1\times10^{40}\left(kT_\Lambda\right)^4_{GeV}\ \text{erg/cm}^3$$

$$\approx 1.1\times10^{100}\left(kT_{\Lambda15}\right)^4\ \text{erg/cm}^3$$

where $kT_{\Lambda15}$ is the energy scale in units of 10^{15}*GeV*. To this energy density corresponds an "effective" cosmological constant

$$\Lambda_{GUT} = \frac{8\pi G}{c^4}\rho_\Lambda c^2 = 2.3\times10^{-8}(kT_\Lambda)^4_{GeV}\ cm^{-2}$$

$$\approx 2.3\times10^{52}(kT_{\Lambda15})^4\ cm^{-2}$$

If we compare this value (Λ_{GUT}) for $kT_{\Lambda15}=1$, with that of the cosmological constant today ($\Lambda_0 \cong 10^{-56}$ cm^{-2}) we get a huge ratio (fine-tuning?):

$$\frac{\Lambda_{GUT}}{\Lambda_0} \approx 10^{108}$$

The Hubble constant, during the phase in which the system is trapped in the false vacuum and the expansion occurs exponentially, is

$$H = \sqrt{\frac{\Lambda}{3}}\cdot c = 2.6\times10^6(kT_\Lambda)^2_{GeV} = 2.6\times10^{36}(kT_{\Lambda15})^2\ s^{-1}$$

If we take $kT_{\Lambda15}=1$, and we want that $Ht_f \geq 60$ to solve the horizon and flatness problems, then we have that

$$t_f \geq \frac{60}{H} \approx 2\times10^{-35}\ s$$

as the epoch of the end of inflation, while the start, using a model of EdS dominated by radiation, is given by

$$t_i \approx \frac{1}{2H} \approx 2\times10^{-37}\ s$$

These are order of magnitude estimates and depend on the value of kT_Λ adopted.

Dynamics of the Inflaton

Let us derive the evolution equation of ***inflaton***, i.e. the scalar field ϕ, starting from the Lagrangian density:

$$S_\phi = \frac{1}{c}\int\sqrt{-g}\ \mathcal{L}\, d^4x$$

For an expanding universe, spatially flat, in orthogonal coordinates, $\sqrt{-g} = a^3$ and the Euler-Lagrange equations are applied to the quantity $a^3\mathcal{L}$:

$$a^3\mathcal{L} = \frac{a^3}{2}\partial_\mu\phi\partial^\mu\phi - a^3V(\phi)$$

If the inflaton is solely dependent on time, and not on the spatial coordinates, only ∂_0 will be different from zero and

$$a^3\mathcal{L} = \frac{a^3}{2}\cdot\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - a^3V(\phi)$$

and the Euler-Lagrange equations give:

$$\frac{\partial\left(\mathcal{L}a^3\right)}{\partial\left(\partial_\mu\phi\right)} = \frac{\partial\left(\mathcal{L}a^3\right)}{\partial\left(\frac{\partial\phi}{c\,\partial t}\right)} = c\cdot\frac{a^3}{2c^2}\cdot 2\frac{\partial\phi}{\partial t} = \frac{a^3\dot{\phi}}{c}$$

$$\partial_\mu\left(\frac{\partial\left(\mathcal{L}a^3\right)}{\partial\left(\partial_\mu\phi\right)}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{a^3\dot{\phi}}{c}\right] = \frac{1}{c^2}\left[a^3\ddot{\phi} + \dot{\phi}\cdot 3a^2\dot{a}\right]$$

$$\frac{\partial\left(\mathcal{L}a^3\right)}{\partial\phi} = -a^3\frac{dV}{d\phi}$$

Putting together and simplifying we get ($\dot{a}/a = H$)

$$\ddot{\phi}+3H\dot{\phi}+c^2\frac{dV}{d\phi}=0$$

which represents the evolution of the inflaton.
This equation, if we refer to the typical potential of *new inflation*, has two different regimes, one called "***slow roll***" and one during which rapid oscillations around the minimum develop. Let's look at them in more detail.
a) ***Slow-roll regime***: is this phase there is a slow "rolling" of the non-accelerated field ϕ which corresponds to the ***phase of inflation***. In this regime the term $\ddot{\phi}$ is negligible ($|\ddot{\phi}| \ll |3H\dot{\phi}|$) and the equation of motion reduces to

$$3H\dot{\phi}=-c^2\frac{dV}{d\phi}$$

that is, the "friction" due to the expansion is dynamically balanced by the acceleration due to the slope of the potential.

By using the time derivative of the above relation, since H is essentially constant during inflation, and naming V'' the $d^2V/d\phi^2$, the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ gives

$$\left|\frac{c^2V''\dot{\phi}}{3H}\right|<<\left|3H\dot{\phi}\right|\rightarrow\left|V''\right|<<\frac{9H^2}{c^2}\approx\frac{9}{c^2}\cdot\frac{8\pi G}{3}\cdot\frac{V}{c^2}\approx\frac{24\pi GV}{c^4}$$

$$\eta(\phi)\equiv\frac{c^4}{24\pi G}\left|\frac{V''}{V}\right|<<1$$

Another crucial condition to have $p=-\rho c^2$ is that $\dot{\phi}^2/2c^2 \ll V(\phi)$, which leads to

$$\left(\frac{c^2V'}{3H}\right)^2<<2c^2V\rightarrow c^2V'^2<<2V\cdot 9H^2\approx 2V\cdot 9\cdot\frac{8\pi GV}{3c^2}$$

$$\varepsilon(\phi)\equiv\frac{c^4}{48\pi G}\left(\frac{V'}{V}\right)^2<<1$$

The two constraints on the potential, $\eta \ll 1, \varepsilon \ll 1$ are the *slow-roll conditions*.

b) ***Fast oscillations***: At the end of the inflation phase, the potential "falls" in the true vacuum and the inflaton oscillates rapidly around the minimum. If nothing more happened, we would have oscillations undergoing redshift as time goes on, in a universe that has already cooled dramatically during the inflationary adiabatic expansion. In order that the thermal history of the universe evolves as suggested by the evidence (e.g. BBN) it is required that the energy of the false vacuum is converted into matter and radiation with a certain efficiency. This process is named ***reheating***. We have already noted that inflation rapidly diluted magnetic monopoles because the energy density of the scalar field remains constant, while the density of monopoles decreases as $1/a^3$ (this does not mean that they disappeared completely, one day they will return within the horizon) . However, in order not to be recreated by reheating, it is necessary that this does not bring again the temperature of the universe to values able to remake them.

The number of e-foldings: It is immediate to calculate the number $\mathcal{N}$ of *e-foldings*. We start from

$$H = \frac{\dot{a}}{a} = \frac{da}{a\,dt} \rightarrow \frac{da}{a} = H\,dt \rightarrow \int_{a_i}^{a_f} d\ln(a) = \int_{t_i}^{t_f} H\,dt$$

$$\mathcal{N} \equiv \ln\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} \frac{H^2}{H}\,dt = -\int_{t_i}^{t_f} \frac{8\pi G}{3} \cdot \frac{V(\phi)}{c^2} \cdot \frac{3\dot{\phi}}{c^2 V'(\phi)}\,dt$$

$$\mathcal{N} = -\frac{8\pi G}{c^4} \int_{\phi_i}^{\phi_f} \cdot \frac{V(\phi)}{V'(\phi)}\,d\phi .$$

Other models of inflation

In the model of inflation that we have described, a spontaneous breaking of the symmetry occurs, and the field starts from zero or from a small value (in this case one refers to ***small scale models***). However it is possible to achieve inflation even without SSB and with large initial values of the field, which then rolls toward zero (***large scale models***). This is the case of the so-called ***chaotic inflation model*** proposed by Linde (1983), in which the potential $V(\phi)$ is simply

$$V(\phi) = \lambda\phi^4 \quad or \quad V(\phi) = 1/2\,\mu^2\phi^2$$

and the potential has a minimum at $\phi = 0$. The phase of inflation takes place if, within the horizon, the field, due to quantum fluctuations, assumes a value different from zero in a region of the universe and then returns toward the minimum. This is more likely to happen at the end of Planck era, rather than at the time of the breaking of great unification.

To solve the problems of the standard model it is not necessary a stage with exponential expansion of the scale factor; it is sufficient that

$$a(t) \propto t^p \qquad p > 1$$

(***power-law inflation***) The required potential has the shape

$$V(\phi) \propto e^{\alpha\,\phi}$$

In the above discussions we have assumed that space is flat, homogeneous and isotropic. What happens if it is not the case? It can be seen (see for example chap. 8, paragraph 6, in "The Early Universe", Kolb and Turner) that, unless the initial space curvature is so high to force the universe to recollapse before inflation, this phase produces, for a wide class of models, huge regions uniform and flat, which exceed in size the current Hubble radius, and then solve the problems of the standard model. Inhomogeneity and/or anisotropy are, however, only delayed and will eventually reappear.

Cosmological constant and Dark Energy

How can we interpret today's cosmological constant? In Einstein's equations (and those of Friedmann), if we remove all matter and radiation, the cosmological constant is the only source of the field: Λ corresponds to the ***density of the vacuum***.
But, according to Quantum Field Theory, the vacuum is not the nothingness of metaphysics, but the ground state of minimum energy, with no particles, of the field itself. We have seen that the cosmological constant behaves as a perfect fluid with $\rho_\Lambda c^2=\varepsilon_\Lambda=\Lambda c^4/8\pi G$ and $p_\Lambda=-\varepsilon_\Lambda=-\rho_\Lambda c^2$, and the energy-momentum tensor is diagonal

$$T^{\alpha\beta} \equiv \begin{pmatrix} \varepsilon_\Lambda & 0 & 0 & 0 \\ 0 & p_\Lambda & 0 & 0 \\ 0 & 0 & p_\Lambda & 0 \\ 0 & 0 & 0 & p_\Lambda \end{pmatrix}$$

Moreover, it must be expected that the values of ε_Λ and p_Λ, that define the state of vacuum, are the same in any, not accelerated, reference frame, so they have to be relativistic invariants.

If, for example, we make a Lorentz transformation with velocity $v=\beta c$ [$\gamma^2=1/(1-\beta^2)$] along the axis x^1, $T^{\alpha\beta}$ changes according to the rule

$$T'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta T^{\gamma\delta}$$

where

$$\Lambda^\alpha_\gamma \equiv \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $T^{\gamma\delta}$ is diagonal, as above. We get (see the following scanned page for the proof):

$$T'^{00} = \varepsilon'_\Lambda = \frac{\varepsilon_\Lambda + \beta^2 p_\Lambda}{1-\beta^2}$$

$$T'^{01} = \gamma^2 \beta(\varepsilon_\Lambda + p_\Lambda) = T'^{10}$$

$$T'^{11} = \frac{\beta^2 \varepsilon_\Lambda + p_\Lambda}{1-\beta^2}$$

$$T'^{22} = T'^{33} = p_\Lambda$$

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \gamma^2 = \frac{1}{1-\beta^2} \qquad \rho_\Lambda c^2 = \varepsilon_\Lambda$$

$$T^{\alpha\beta} \begin{pmatrix} \rho_\Lambda c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

Se $\alpha = \beta$ $T^{\alpha\beta} \neq 0$

$$T'^{00} = \Lambda^0_\alpha \Lambda^0_\beta T^{\alpha\beta} = \Lambda^0_\alpha \Lambda^0_\alpha T^{\alpha\alpha} = \gamma^2 \rho_\Lambda c^2 + \gamma^2 \beta^2 p = \gamma^2 [\varepsilon_\Lambda + \beta^2 p]$$

$$\boxed{\varepsilon'_\Lambda = \frac{\varepsilon_\Lambda + \beta^2 p}{1-\beta^2}}$$

$$T'^{01} = \Lambda^0_\alpha \Lambda^1_\beta T^{\alpha\beta} = \Lambda^0_\alpha \Lambda^1_\alpha T^{\alpha\alpha} = \gamma^2\beta\,\varepsilon_\Lambda + \gamma^2\beta\, p = \boxed{\gamma^2\beta(\varepsilon_\Lambda + p)} = T'^{10}$$

$$T'^{11} = \Lambda^1_\alpha \Lambda^1_\beta T^{\alpha\beta} = \Lambda^1_\alpha \Lambda^1_\alpha T^{\alpha\alpha} = \gamma^2\beta^2\varepsilon_\Lambda + \gamma^2 p = \gamma^2[\beta^2\varepsilon_\Lambda + p] = \boxed{\frac{\beta^2\varepsilon_\Lambda + p}{1-\beta^2}}$$

$$T'^{02} = \Lambda^0_\alpha \Lambda^2_\alpha T^{\alpha\alpha} = 0 \ \ldots$$

$$T'^{22} = \Lambda^2_\alpha \Lambda^2_\alpha T^{\alpha\alpha} = p = T'^{33}$$

Se $p = -\varepsilon_\Lambda$ $\qquad \varepsilon'_\Lambda = \dfrac{\varepsilon_\Lambda [1-\beta^2]}{1-\beta^2} = \varepsilon_\Lambda$

$$T'^{01} = T'^{10} = 0 \qquad T'^{11} = \frac{p[1-\beta^2]}{1-\beta^2} = p$$

and all the other terms of $T'^{\alpha\beta}$ are null. In order that $\varepsilon'_\Lambda = \varepsilon_\Lambda$, $p'_\Lambda = p_\Lambda$ and $T'^{\alpha\beta}$ is diagonal, $p_\Lambda = -\varepsilon_\Lambda = -\rho_\Lambda c^2$ is required. Thus we see that the "false" vacuum of inflation, and the "true" vacuum share a similar equation of state.

We can also consider the vacuum as a "substance" with given ε_Λ and p_Λ, in the sense that the relation $dU = -dL = -pdV$ (since $dQ = 0$) is satisfied. Indeed $dU = d(\varepsilon_\Lambda V) = \varepsilon_\Lambda dV = -p_\Lambda dV$ *if* $p_\Lambda = -\varepsilon_\Lambda$.

But can we estimate the expected value for ε_Λ? (Note that in the following pages $c=1$).

5.3 The Λ problem

We now return to the contribution of quantum fluctuations to the vacuum energy density. First consider a quantum mechanical harmonic oscillator. Its energy eigenvalues are given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, \ldots . \tag{5.29}$$

The vacuum ($n = 0$) therefore has a finite amount of energy (zero point energy). A relativistic field can be considered as a sum of harmonic oscillators of all possible frequencies ω. In the simple case of a scalar field with mass m, the vacuum energy is the sum of all contributions:

$$E_0 = \sum_j \frac{1}{2}\hbar\omega_j . \tag{5.30}$$

This summation can be carried out by putting the system in a box of size L^3 and then considering the limit as L tends to infinity (see e.g. [Car92]). Assuming periodic boundary conditions, equation (5.30) becomes (using again $\hbar = 1$)

$$E_0 = \frac{1}{2}L^3 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\omega_k \tag{5.31}$$

From periodic boundary conditions (n=integer number) $L = n\frac{\lambda}{2} = n\frac{\pi}{k}$, so $n = \frac{kL}{\pi}$ and similar relations hold for the two other directions. If we move from discrete to continuous values of n (a good approximation if n is not very small), $\mathrm{d}n = \frac{L}{\pi}\mathrm{d}k$, $d^3n = \left(\frac{L}{\pi}\right)^3 d^3k$; since only positive values of n are possible, the volume in n or k space is $1/8$ of the total space and we get (5.31). From $E^2 = m^2c^4 + p^2c^2, E = \hbar\omega, p = \hbar k$ we get $\omega^2 = m^2 + k^2$ if $\hbar = c = 1$. From spherical symmetry $\mathrm{d}^3k = 4\pi k^2\mathrm{d}k$ and (5.32) and (5.33) are obtained.

where $k = 2\pi/\lambda$ corresponds to the wave vector. Now, using the relation

$$\omega_k^2 = k^2 + m^2 \tag{5.32}$$

and a maximum cut-off frequency $k_{max} \gg m$, the integration can be carried out, resulting in

$$\rho_V = \lim_{L\to\infty} \frac{E_0}{L^3} = \int_0^{k_{max}} \frac{4\pi k^2}{(2\pi)^3} dk \frac{1}{2}\sqrt{k^2+m^2} = \frac{k_{max}^4}{16\pi^2}. \tag{5.33}$$

Assuming the validity of the general theory of relativity up to the Planck scale ($l_{Pl} \simeq (8\pi G)^{-\frac{1}{2}}$), $l_{Pl}^{-1} = k_{max}$ results in a value which lies 121(!) orders of magnitude above the experimental value. This would lead to a vacuum energy of [Car92]

$$\rho_V = 10^{74}\ \mathrm{GeV}^4 \approx 10^{92}\ \mathrm{g\ cm}^{-3}. \tag{5.34}$$

It is rare for an estimate to be quite so incorrect. In addition, a value for k_{max} at approximately the electroweak scale of about 200 GeV, leads to a discrepancy of 54 orders of magnitude. Even with a k_{max} of order Λ_{QCD}, the prediction is still 42 orders of magnitude away from observation.

This is the so-called ***cosmological constant problem*** from the point of view of Field Theory. There is a second problem linked to the coincidence between matter-energy density and cosmological constant: why do they are comparable today?
Many attempts have been done to find a solution to these problems.
The discovery of SUSY led to the hope that, since bosons and fermions (of identical mass) contribute equally but with opposite sign to the vacuum expectation value, the cosmological constant should be zero. But SUSY is today broken, so Λ could be zero only in the early universe. Attempts has been done to produce a (almost) vanishing cosmological constant also with broken SUSY.
Anthropic explanations have been proposed. In several cosmological theories the observed big bang is just one member of an ensemble. The ensemble may consist of different expanding regions at different times and locations in the same spacetime, or of different terms in the wave function of the universe. If the vacuum energy density ρ_V varies among the different members of this ensemble, then the value observed by any species of astronomers will be conditioned by the necessity that this value of ρ_V should be suitable for the evolution of intelligent life.
The anthropic bound on a positive vacuum energy density is set by the requirement that ρ_V should not be so large as to prevent the formation of galaxies (the accelerated expansion stops the growing of the amplitude of density fluctuations). A negative value for the cosmological constant, as we have seen, acts as an additional self-gravity and forces the recollapse of the universe; if this recollapse happens too early, no intelligent life can develop.

Dynamical models of Dark Energy

Many ideas have been proposed to solve the problem of Dark Energy (if you are interested in this subject you can refer to the book “DARK ENERGY, Theory and Observations” by Luca Amendola and Shinji Tsujikawa, 2010, Cambridge University Press).
There are basically two approaches for the construction of dark energy models. The first approach is based on “modified matter models” in which the energy-momentum tensor $T_{\mu\nu}$ on the r.h.s. of the Einstein equations contains an exotic matter source with a negative pressure. The second approach is based on “modified gravity models” in which the Einstein tensor $G_{\mu\nu}$ on the l.h.s. of the Einstein equations is modified.
Here we mention the so-called ***quintessence***[7] model as one of the representative modified matter models.
Quintessence is a canonical scalar field Q with a potential $V(Q)$ responsible for the late-time cosmic acceleration. Unlike the cosmological constant, the equation of state of quintessence dynamically changes with time: $p_Q = w_Q \rho_Q c^2$ with

$$\rho_Q c^2 = \frac{\dot{Q}^2}{2c^2} + V(Q)$$
$$p_Q = \frac{\dot{Q}^2}{2c^2} - V(Q)$$
$$w_Q = \frac{\frac{\dot{Q}^2}{2c^2} - V(Q)}{\frac{\dot{Q}^2}{2c^2} + V(Q)}$$

where w_Q can be in the range from -1 to +1. Here we can use relations similar to those use when working on inflation. We assume a flat, $k=0$, universe. The evolution of the field and the dynamics of the universe are given by the already known relations

$$\ddot{Q} + 3\frac{\dot{a}}{a}\dot{Q} + c^2\frac{dV(Q)}{dQ} = 0$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_Q + \rho)$$
$$\rho_Q c^2 = \frac{\dot{Q}^2}{2c^2} + V(Q)$$

Peebles and Ratra proposed a potential like

[7] According to ancient Greek science, the quintessence (from the Latin “fifth element”) denotes a fifth cosmic element after earth, fire, water, and air.

$$V(\phi) \simeq \frac{\varkappa}{Q^{\alpha}}$$

and assumed that at high redshift the density of the field is “subdominant” with respect to that of matter/radiation, in order to preserve the BBN. We assume that the scale factor grows as

$$a(t) \propto t^q$$

The equation of the field is

$$\frac{dV(Q)}{dQ} = \frac{d}{dQ}\left[\frac{\varkappa}{Q^{\alpha}}\right] = -\varkappa\alpha Q^{-\alpha-1}$$

$$\ddot{Q} + 3\frac{\dot{a}}{a}\dot{Q} - \frac{c^2\varkappa\alpha}{Q^{\alpha+1}} = 0$$

$$a \sim t^q \Rightarrow \frac{\dot{a}}{a} = \frac{q\,t^{q-1}}{t^q} = \frac{q}{t}$$

$$\ddot{Q} + 3\frac{q}{t}\dot{Q} = \frac{c^2\varkappa\alpha}{Q^{\alpha+1}}$$

the solution is

$$Q = At^p \rightarrow \begin{cases} \dot{Q} = pAt^{p-1} \\ \ddot{Q} = Ap(p-1)t^{p-2} \end{cases}$$

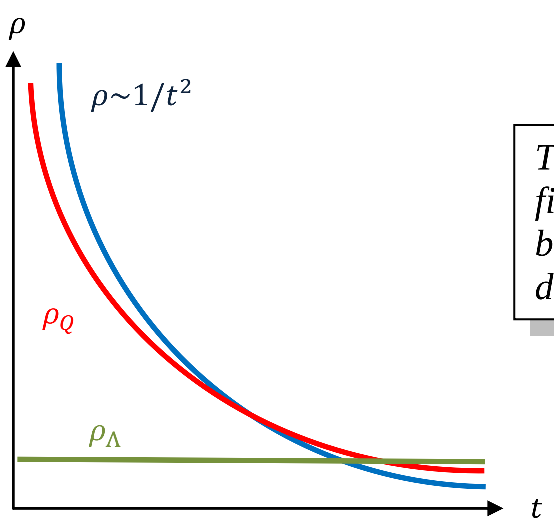
$$Ap(p-1)t^{p-2} + 3\frac{q}{t}pAt^{p-1} = \frac{c^2\varkappa\alpha}{A^{\alpha+1}(t^p)^{\alpha+1}}$$

$$A\left[p(p-1) + 3qp\right]t^{p-2} = \frac{c^2\varkappa\alpha}{A^{\alpha+1}}\,t^{-p(\alpha+1)}$$

$$\Rightarrow p-2 = -p(\alpha+1) \rightarrow p + p(\alpha+1) = 2 \rightarrow p(\alpha+2) = 2$$

$$p = \frac{2}{\alpha+2} \qquad Q \sim t^{\frac{2}{\alpha+2}} \longleftarrow$$

← doesn’t depend on q !!!



The energy density associated to the scalar field is negligible at early times, but it becomes finally dominant. See next page for details.

$$\rho_Q c^2 = \frac{1}{2c^2}\dot{Q}^2 + V(Q) = \frac{1}{2c^2}A^2 p^2 t^{2(p-1)} + \frac{\varkappa}{A^\alpha t^{\alpha p}} =$$

$$= \left[\frac{A^2p^2}{2c^2} + \frac{\varkappa}{A^\alpha}\right] t^{2p-2} \sim t^{-\frac{2\alpha}{\alpha+2}}$$

$$p = \frac{2}{\alpha+2}$$
$$\alpha + 2 = \frac{2}{p}$$
$$\alpha = \frac{2}{p} - 2 = \frac{2-2p}{p}$$
$$\alpha = \frac{2(1-p)}{p}$$
$$\alpha p = 2(1-p)$$

Since, in EdS, $\rho \propto 1/t^2$, both for matter and radiation

$$\frac{\rho_Q}{\rho} \propto \frac{t^{2p-2}}{1/t^2} \sim t^{2p} \sim t^{\frac{4}{\alpha+2}}$$

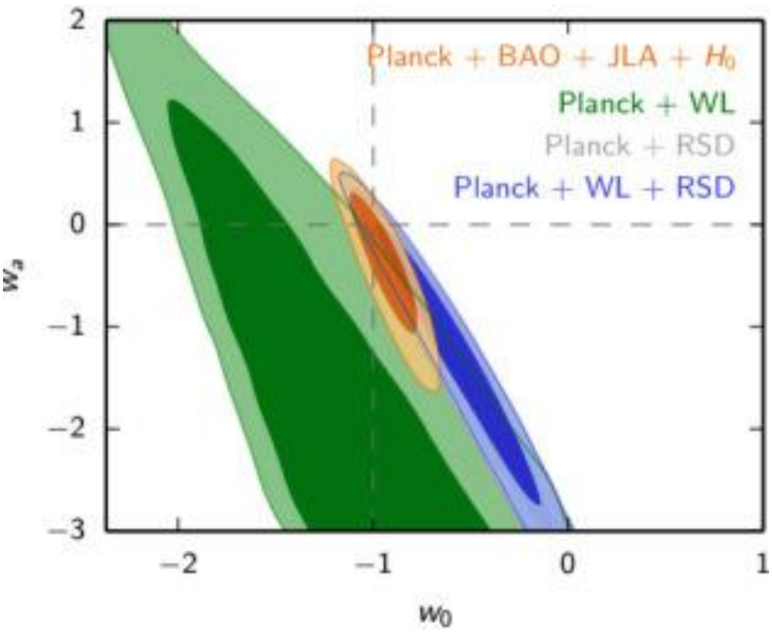
α=0, *V(Q)=const.*, corresponds to the cosmological constant, $\rho_Q \propto t^0$.
For ***α>0*** the scalar field becomes finally the dominant component, even if it was negligible at high redshift.

While *Λ* or vacuum are spatially homogeneous, quintessence feels the gravitational field of the other components, like dark matter and baryons, and fluctuations δQ develop. But, for a slowly varying eq. of state (compared to the expansion rate), those fluctuations rapidly dissipate on scales smaller than the Hubble radius R_H and so the Q field is smooth on scale relevant for structure formation.

In any case, the first thing to understand is if the vacuum energy density is constant or varies over time. To do that all the available sets of cosmological observations are used to fit, for instance, a linear dependence on scale of the equation of state

$$w(a)=w_0 + w_a\,(1 - a/a_0)$$

The results are not conclusive, and a cosmological constant is still consistent with the data (plot taken from Planck satellite 2015 results).



SHORT COSMIC HISTORY

Era	t (sec)	E(3kT)	T (K)	Events
Planck	10^{-44}	10^{19}GeV	5×10^{31}	Quantum Gravity
GUT	10^{-38} 10^{-36}	10^{16}GeV 10^{15}GeV	5×10^{28} 5×10^{27}	GUT's SSB Inflation? Baryogenesis?
Electroweak	10^{-10}	10^{2}GeV	5×10^{16}	Electroweak SSB
Adronic	10^{-4}	200 MeV	10^{12}	Quark-adrons transition
Leptonic	0.7	1 MeV	10^{10}	Decoupling of ν_e
	5	0.5 MeV	5×10^{9}	Annichilation e^+e^-
BBN	2-3 min	0.1 MeV	10^{9}	BBN: ^4He, ^3He, D, ^7Li
Radiation-Matter Equality	6×10^{4} yr	2 – 3 eV	10^{4}	Matter-dominated era begins
Recombination	4×10^{5} yr	0.7 eV	3000	The universo becomes neutral and transparent
Void	10 Gyr	10^{-3} eV	3.6	Vacuum-dominated era begins
Today	13.7 Gyr	7×10^{-4} eV	2.73	

yr	sidereal year (1900)	3.1558149984×10^{7} sec
ly	light year	9.4605×10^{17} cm
a.u.	astronomical unit	1.495985×10^{13} cm
pc	parsec	3.0856×10^{18} cm
H_0	Hubble constant	$3.241\times10^{-18}\,h\,sec^{-1}$
$1/H_0$	Hubble time	$3.086\times10^{17}\,h^{-1}$ sec
$M_\odot$	solar mass	1.989×10^{33} g
$R_\odot$	solar radius	6.9598×10^{10} cm
$L_\odot$	solar luminosity	3.90×10^{33} erg sec
$M_\oplus$	Earth mass	5.977×10^{27} g
$R_\oplus$	equatorial Earth radius	6.37817×10^{3} km