SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Harmonic oscillators & 2&N DOF systems: Coupled oscillators

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What is a **wave**?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → Harmonic Oscillation

Oscillatory Motion

The motion of an object can be predicted if the external forces acting upon it are known.

A special type of motion occurs when the force on the object is proportional to the displacement of the object from equilibrium.

If this force always acts towards the equilibrium position a back and forth motion will results about the equilibrium position.

This is known as periodic or oscillatory motion.

Familiar examples of periodic motion

1. Pendulum

2. Vibrations of a stringed instrument

3. Mass on a spring

Other examples include

1. Molecules in a solid

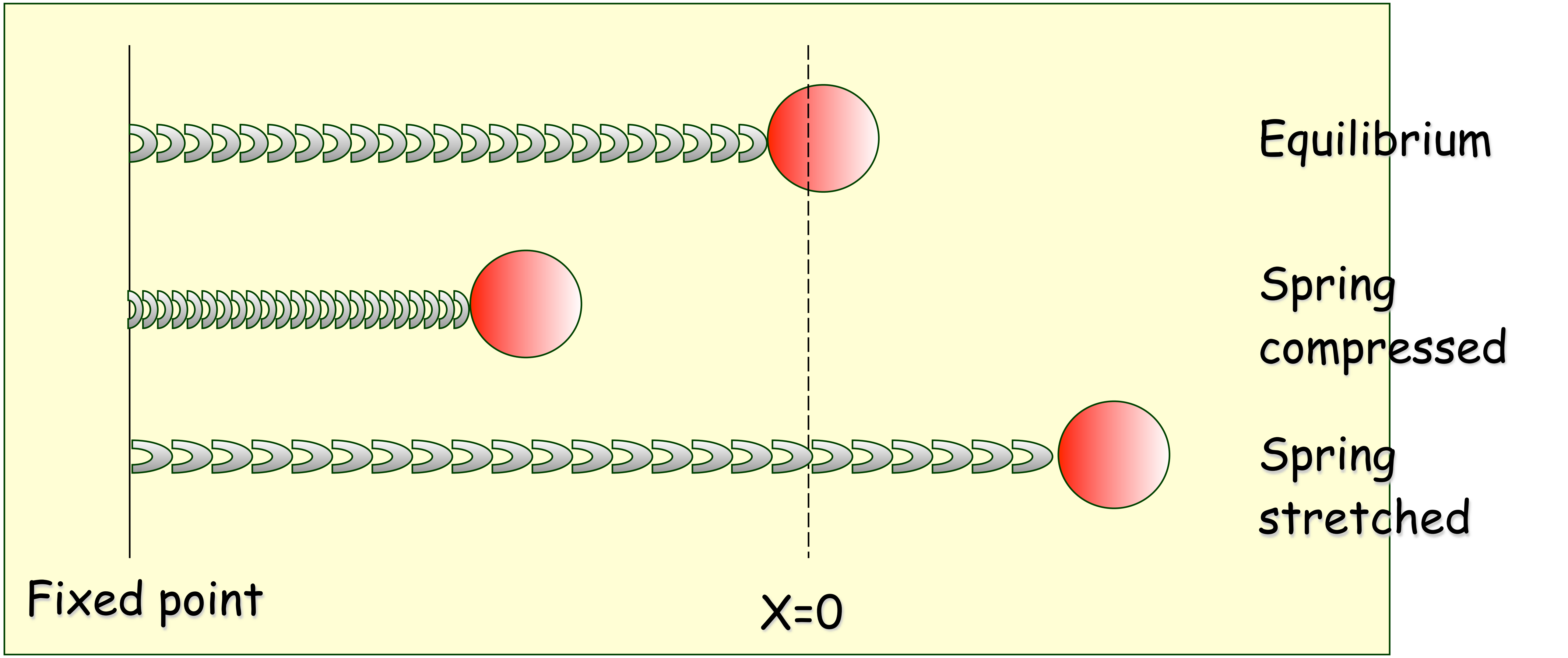
2. Air molecules in a sound wave

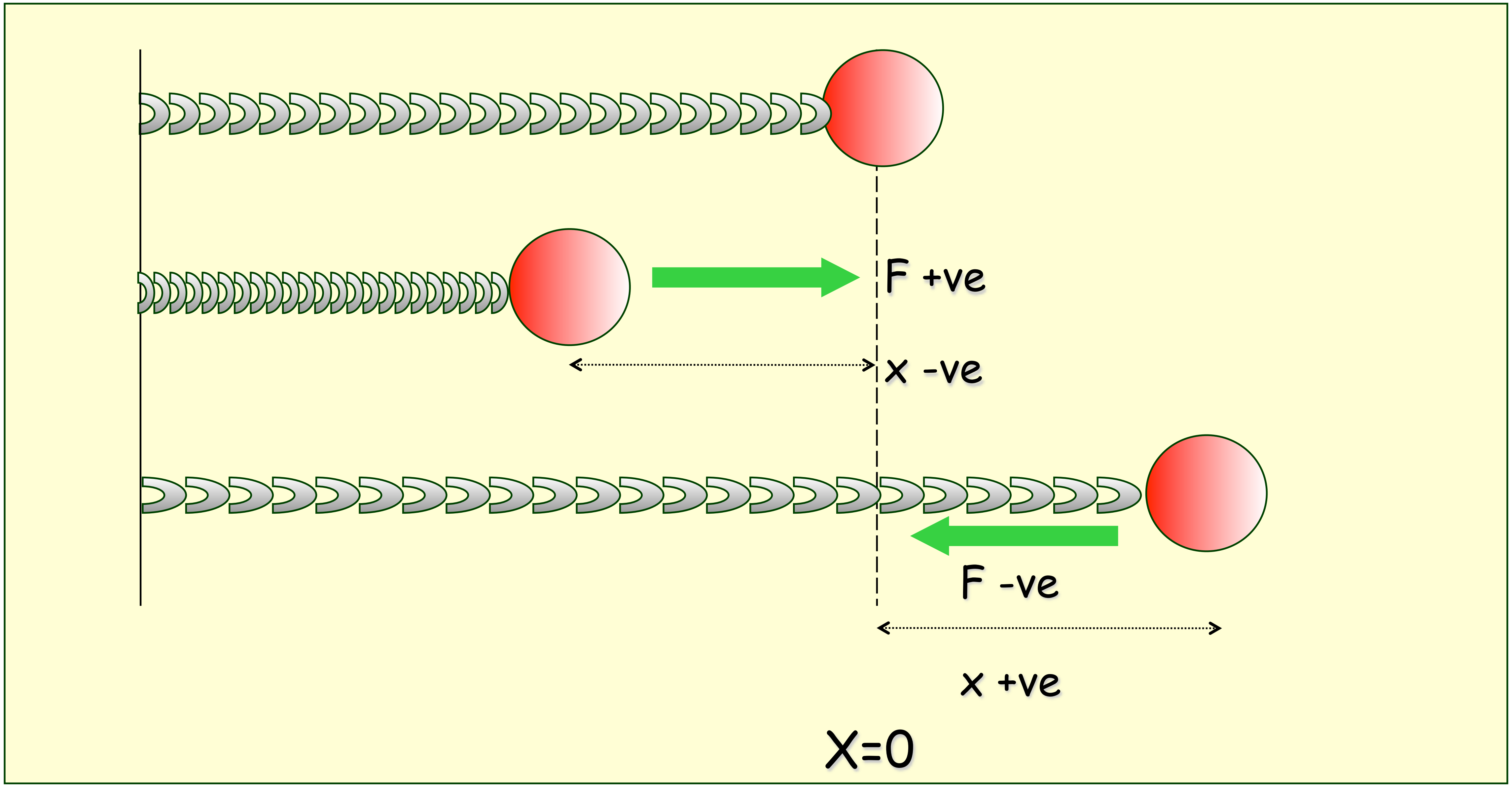
3. Alternating electric current

Simple Harmonic Motion

If an object oscillates between two positions for an indefinite length of time with no loss of mechanical energy the motion is said to be **simple harmonic motion**.

Example: Mass on a spring.





Spring exerts a force on the mass to restore it to its original position.

$$F \propto -x \quad \text{or} \quad F = -kx \qquad \text{(Hooke's Law)}$$

where k is a +ve constant, the **spring constant**

A mass under a restoring force

From Newton's 2nd Law $F = ma$

where $a = \dfrac{d^2x}{dt^2}$

therefore $-kx = m\dfrac{d^2x}{dt^2}$

or $$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

This is the condition for simple harmonic motion

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

An object moves with simple harmonic motion (SHM) when the acceleration of the object is proportional to its displacement and in the opposite direction.

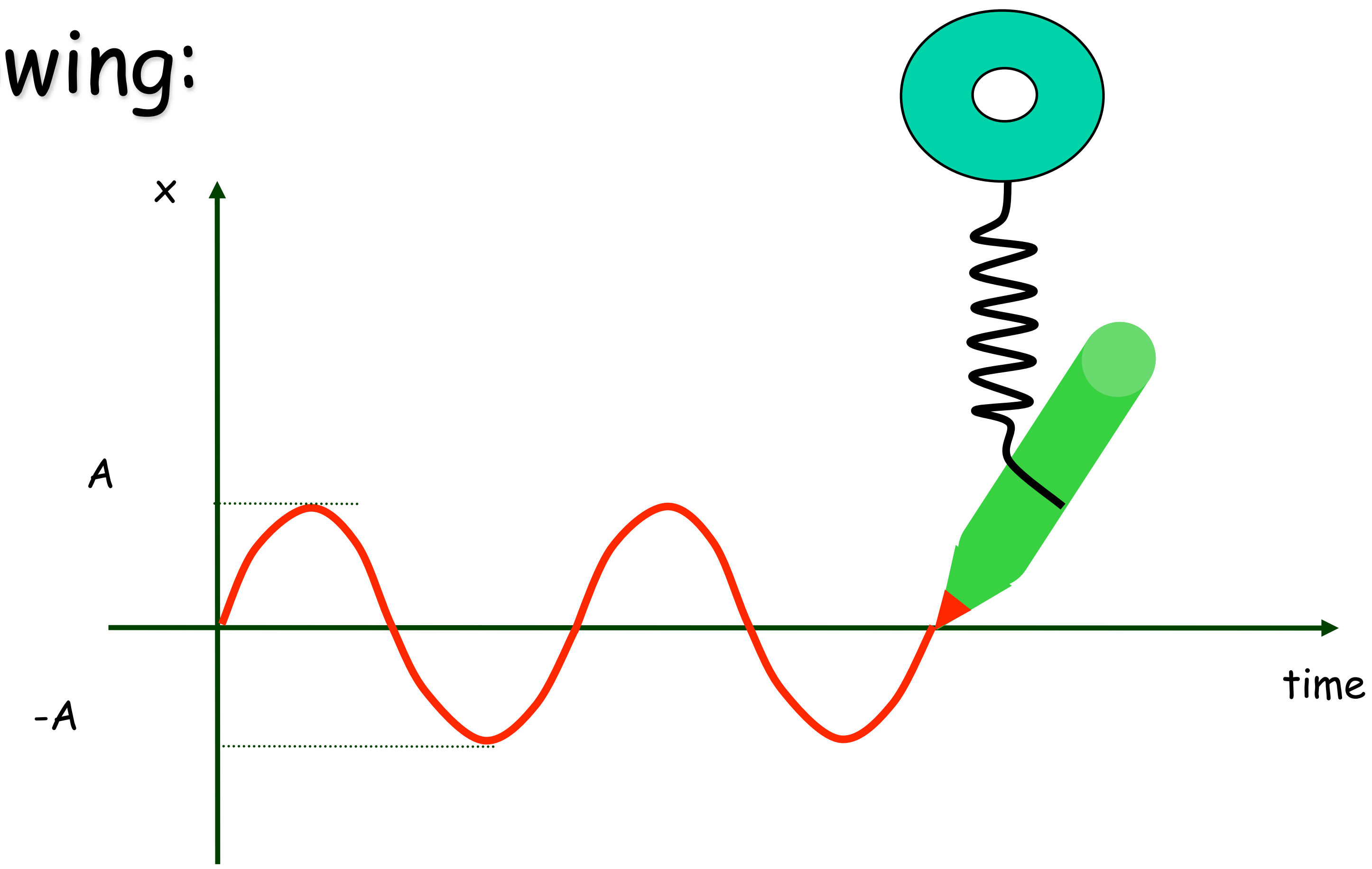
Some definitions:

The time taken to make one complete oscillation is the **period T**.

The **frequency of oscillation, f** = 1/T in s^{-1} or Hertz

The distance from equilibrium to maximum displacement is the **amplitude of oscillation, A**.

Consider the following:



The general equation for the curve traced out by the pen is $x = A \cos(\omega t + \delta)$

where $(\omega t + \delta)$ is the **phase** of the motion

and δ is the **phase constant**

We can show that the expression $x = A \cos(\omega t + \delta)$

is a solution of $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$ by differentiating wrt time

$$x = A\cos(\omega t + \delta)$$

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t + \delta)$$

$$a = \frac{dv}{dt} = -A\omega^2 \cos(\omega t + \delta)$$

$$\text{or} \qquad a = -\omega^2 x$$

Compare this to $a = -(k/m)x$

$x = A \cos(\omega t + \delta)$ is a solution if $\omega = \sqrt{\dfrac{k}{m}}$

We can determine the amplitude of the oscillation (A) and the phase constant (δ) from the initial position x_o and the initial velocity v_o

$$\text{if} \quad x = A\cos(\omega t + \delta) \text{ then } \quad x_o = A\cos(\delta)$$

$$\text{if } v = -A\omega\sin(\omega t + \delta) \quad \text{then} \quad v_o = -A\omega\sin(\delta)$$

The system repeats the oscillation every T seconds

therefore $$x(t) = x(t+T)$$

and
$$\begin{aligned} A\cos(\omega t + \delta) &= A\cos(\omega(t + T) + \delta) \\ &= A\cos(\omega t + \delta + \omega T) \end{aligned}$$

The function will repeat when $\omega T = 2\pi$

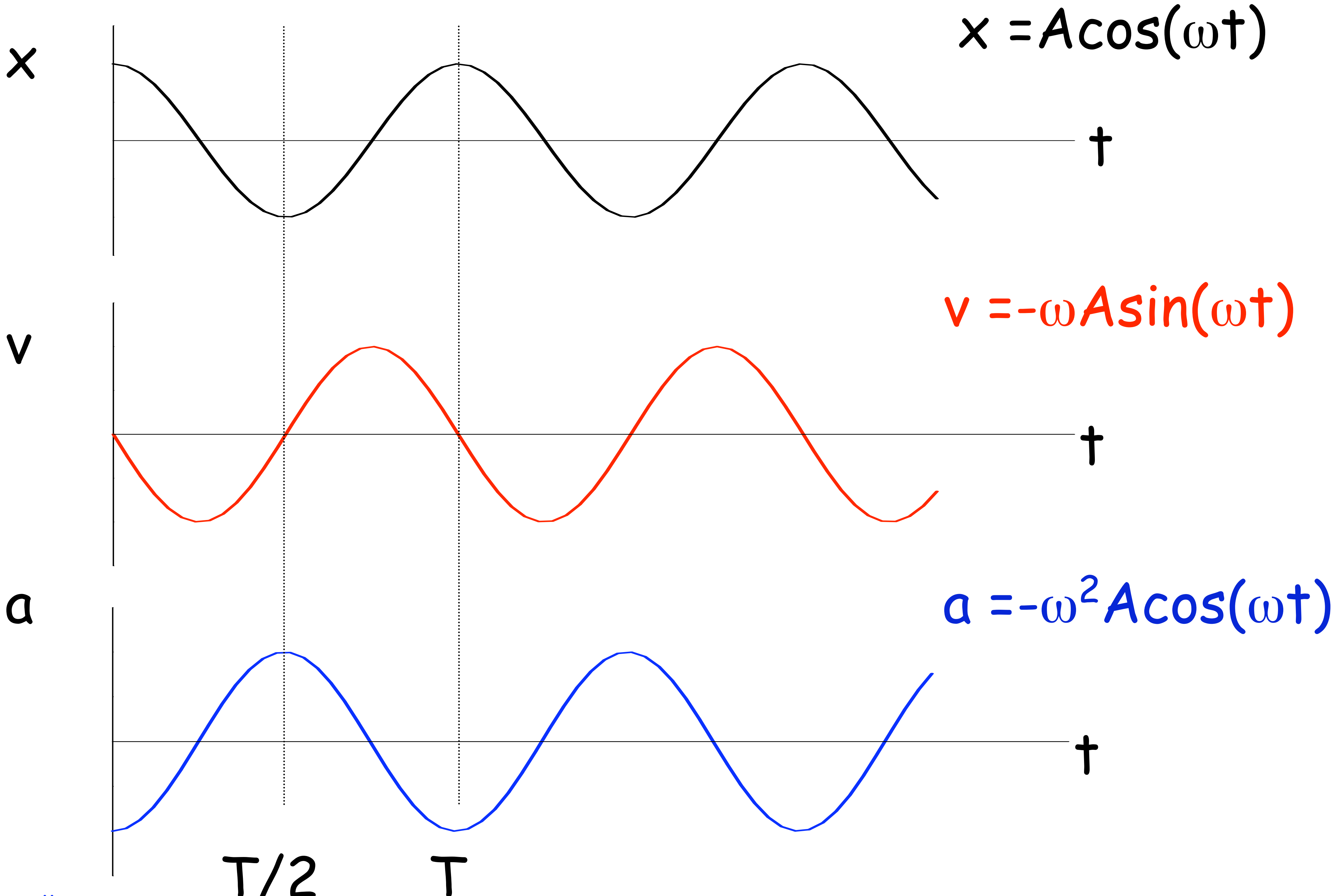
We can relate ω, f and the spring constant k using the following expressions.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

ω is known as the angular frequency and has units of rad.s^{-1}

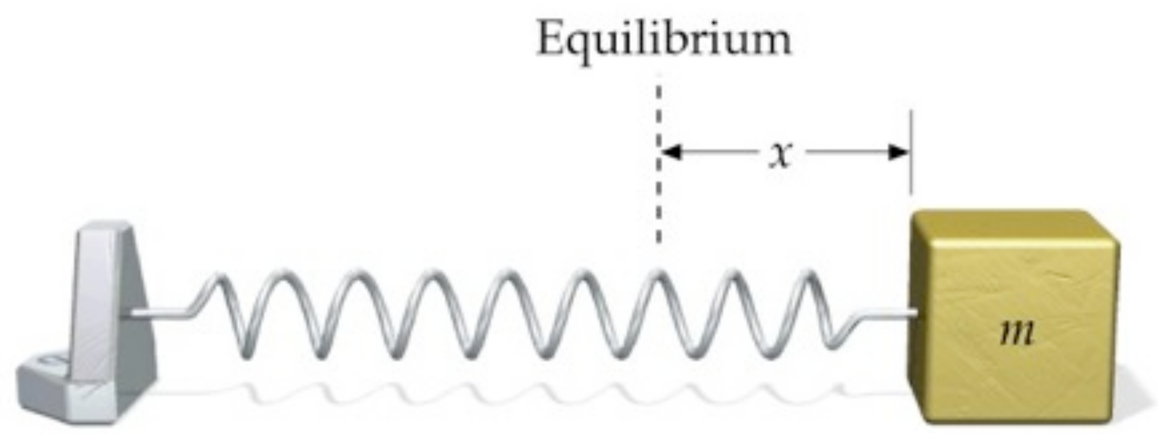
Time dependence of x, v and a of an object undergoing SHM



Energy in Simple Harmonic Motion

In SHM the total energy (E) of a system is **constant** but the kinetic energy (K) and the potential energy (U) vary wrt.

Consider a mass a distance x from equilibrium and acted upon by a restoring force



Kinetic Energy

$$K = \tfrac{1}{2}mv^2$$

$$v = -A\omega \sin(\omega t + \delta)$$

$$K = \tfrac{1}{2}mA^2\omega^2 \sin^2(\omega t + \delta)$$

Substitute $\omega^2 = k/m$

$$K = \tfrac{1}{2}kA^2 \sin^2(\omega t + \delta)$$

Potential Energy

$$U = \tfrac{1}{2}kx^2$$

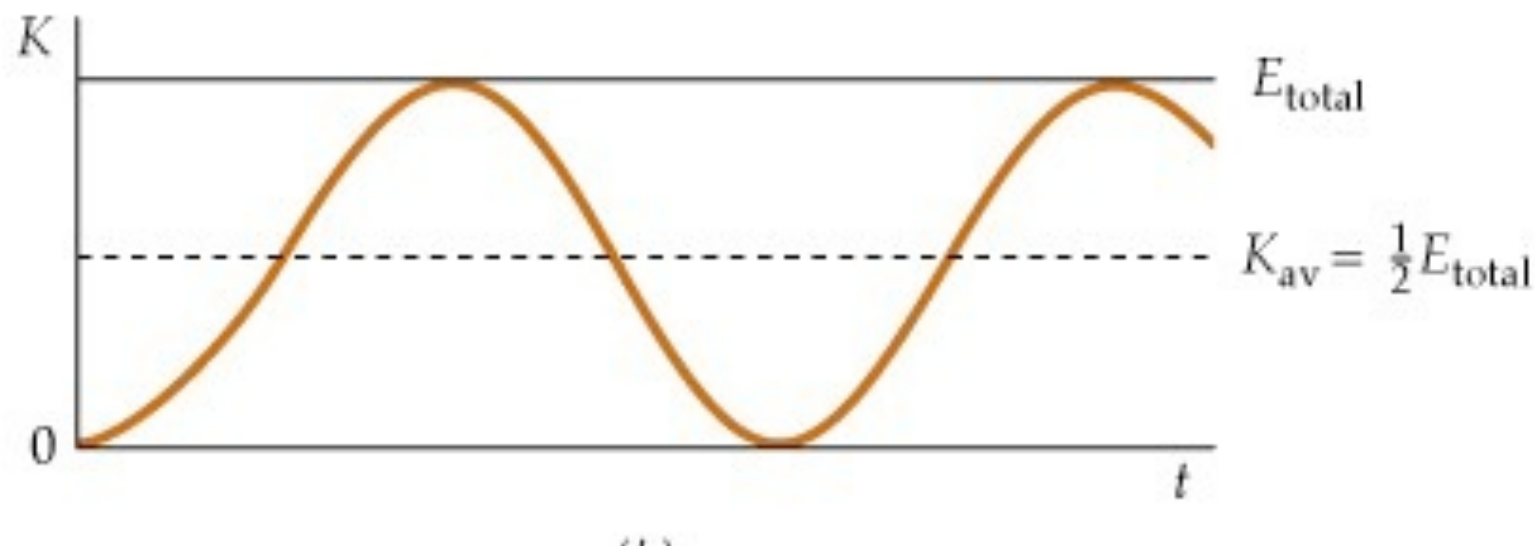
$$x = A\cos(\omega t + \delta)$$

$$U = \tfrac{1}{2}kA^2 \cos^2(\omega t + \delta)$$

Graphical representation

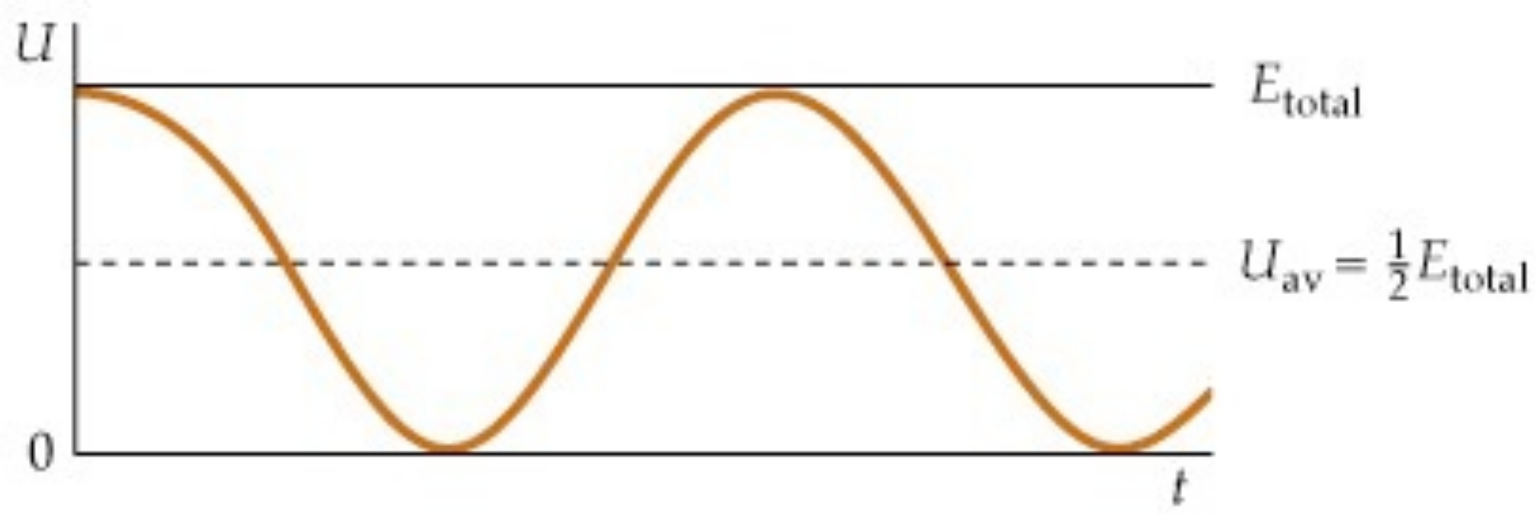
Kinetic

Energy



Potential

Energy



Total Energy

Total energy E = K + U

$$= \tfrac{1}{2}kA^2 \sin^2(\omega t + \delta) + \tfrac{1}{2}kA^2 \cos^2(\omega t + \delta)$$

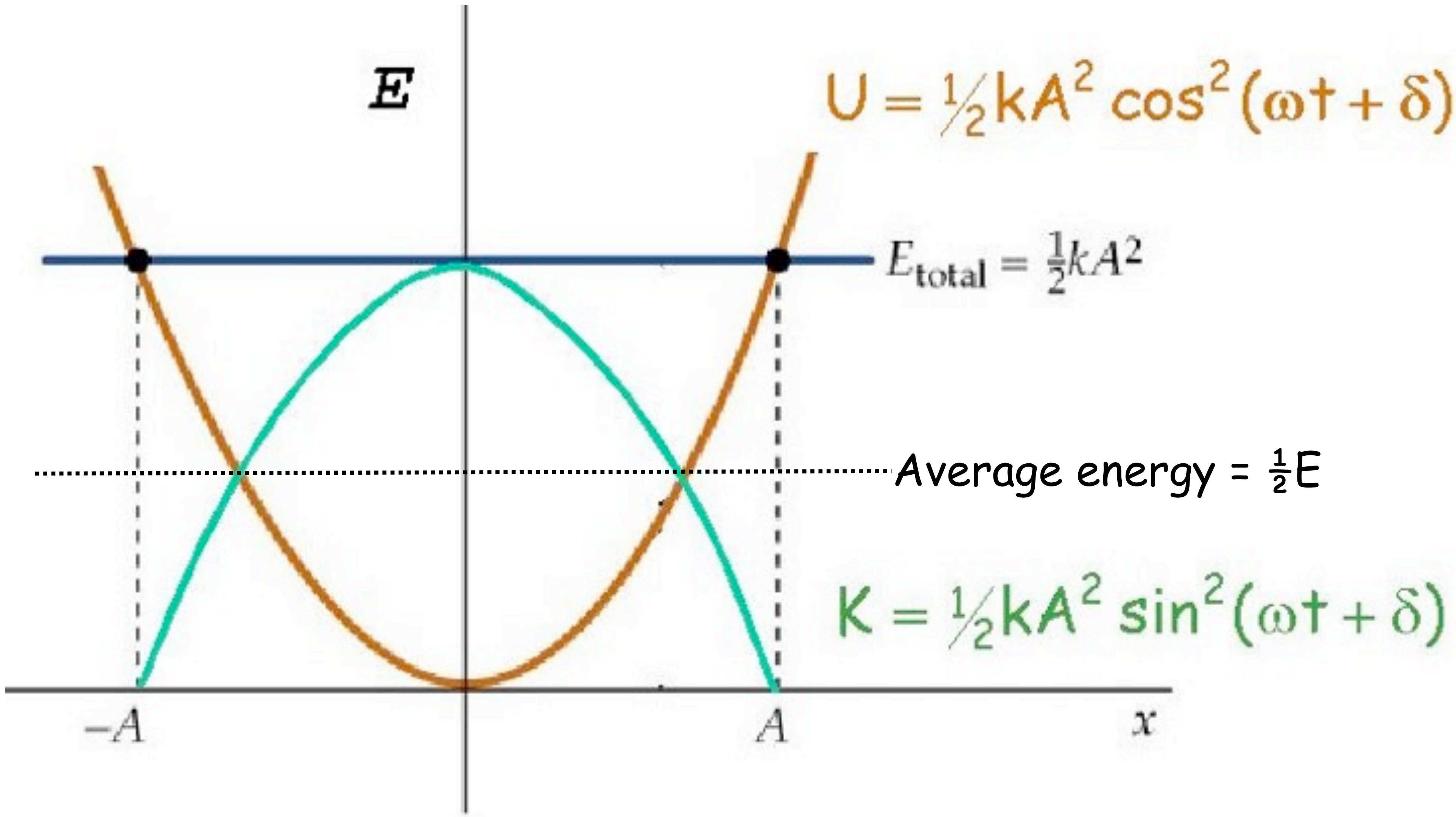
$$= \tfrac{1}{2}kA^2 \left(\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right)$$

but $\left(\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right) = 1$

$$\boxed{\therefore \quad E = \tfrac{1}{2}kA^2}$$

In SHM the total energy of the system is proportional to the square of the amplitude of the motion

Graphical representation



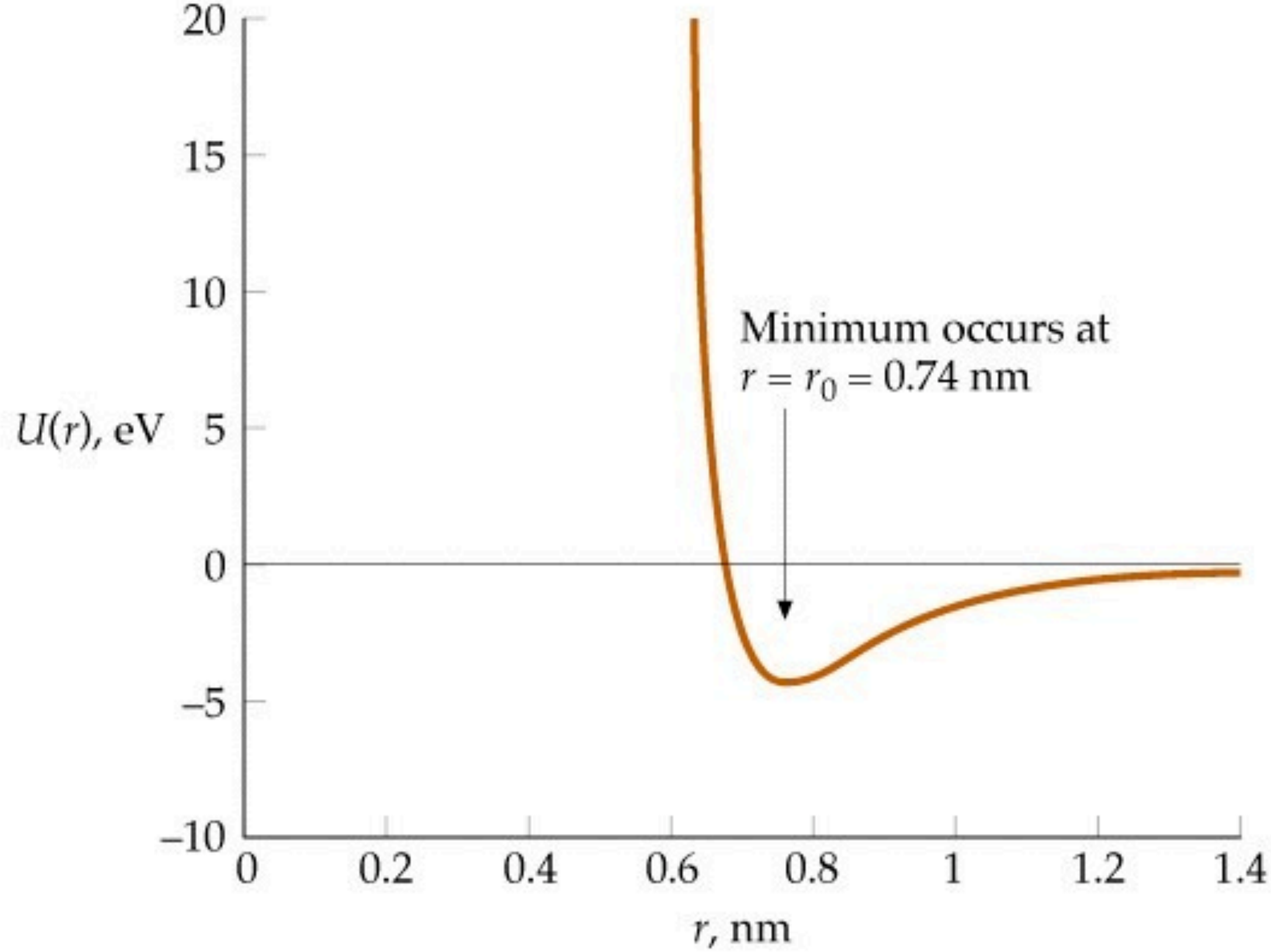
At maximum displacement K=0 $\therefore$ E = U

At equilibrium U=0 and $v=v_{max}$ $\therefore$ E = K

At all times E = K + U is constant

Lennard-Jones potential

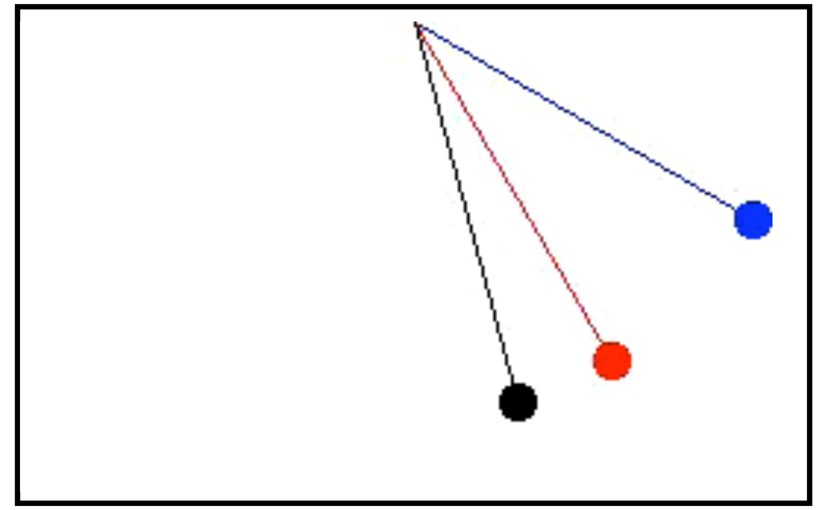
Potential energy



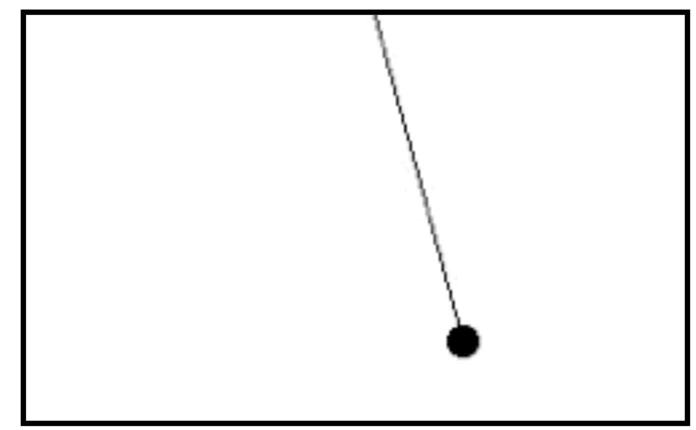
Interatomic distance

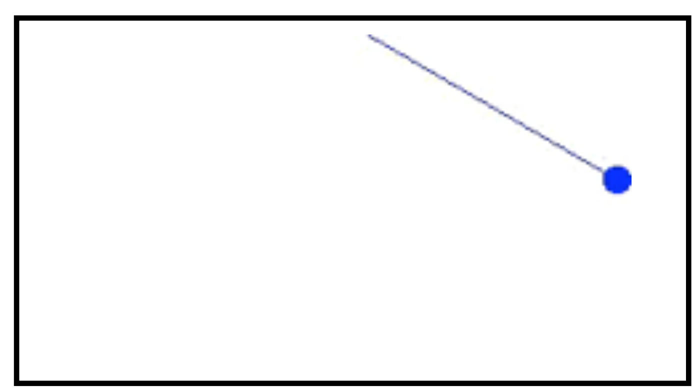
The Simple Pendulum

A simple pendulum consists of a string of length L and a bob of mass m.



When the mass is displaced and released from an initial angle ϕ with the vertical it will swing back and forth with a period T.





We are going to derive an expression for T.

Forces on mass:

mg (downwards)

tension (upwards)

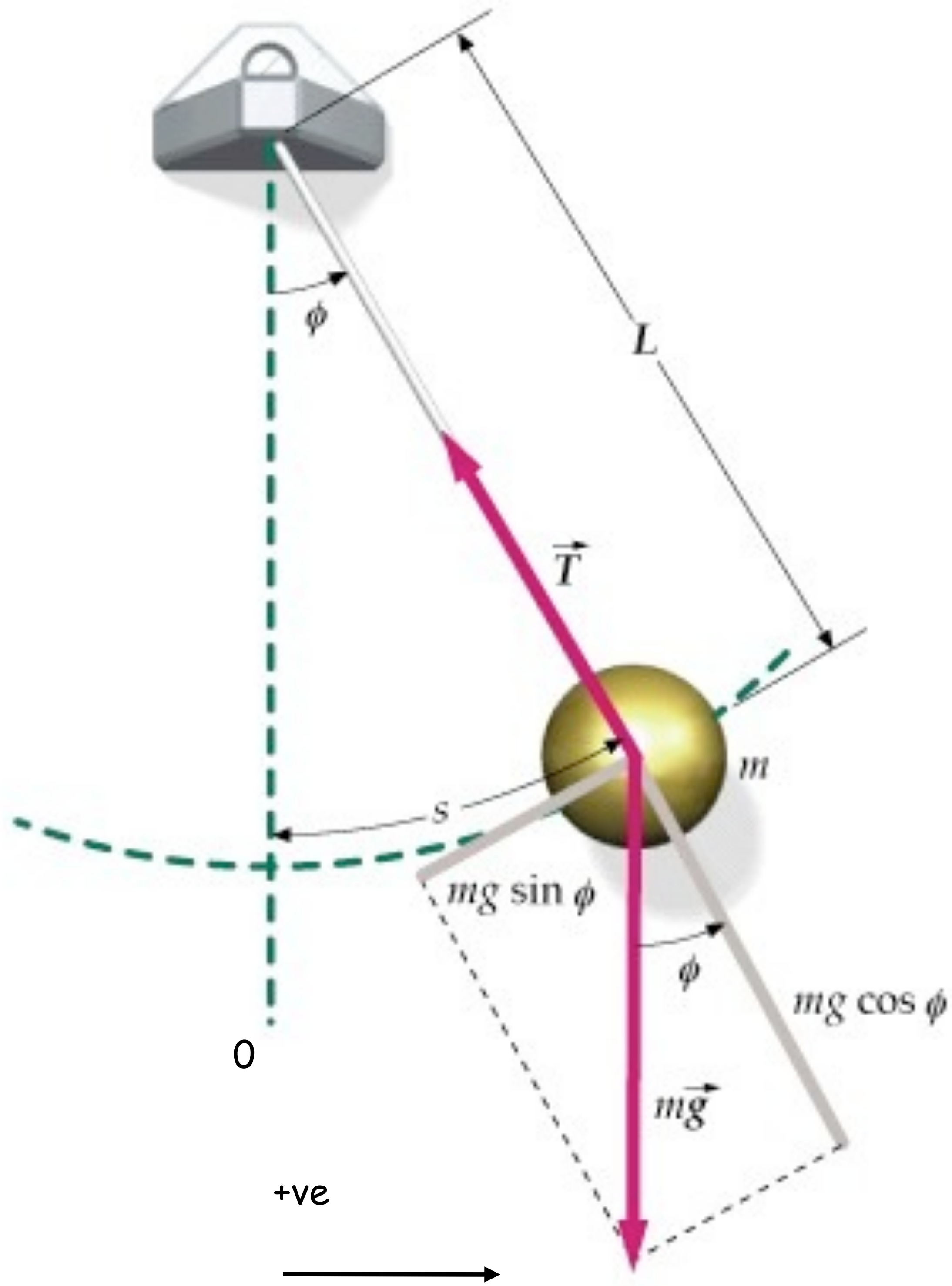
When mass is at an angle ϕ to the vertical these forces have to be resolved.

Tangentially:

weight = mg sin ϕ (towards 0)

tension = T cos 90 = 0

$$\sum F_{tang} = -mg\sin\phi$$



Using $\frac{\phi(\text{rads})}{2\pi} = \frac{s}{2\pi L}$

we find $s = L\phi$

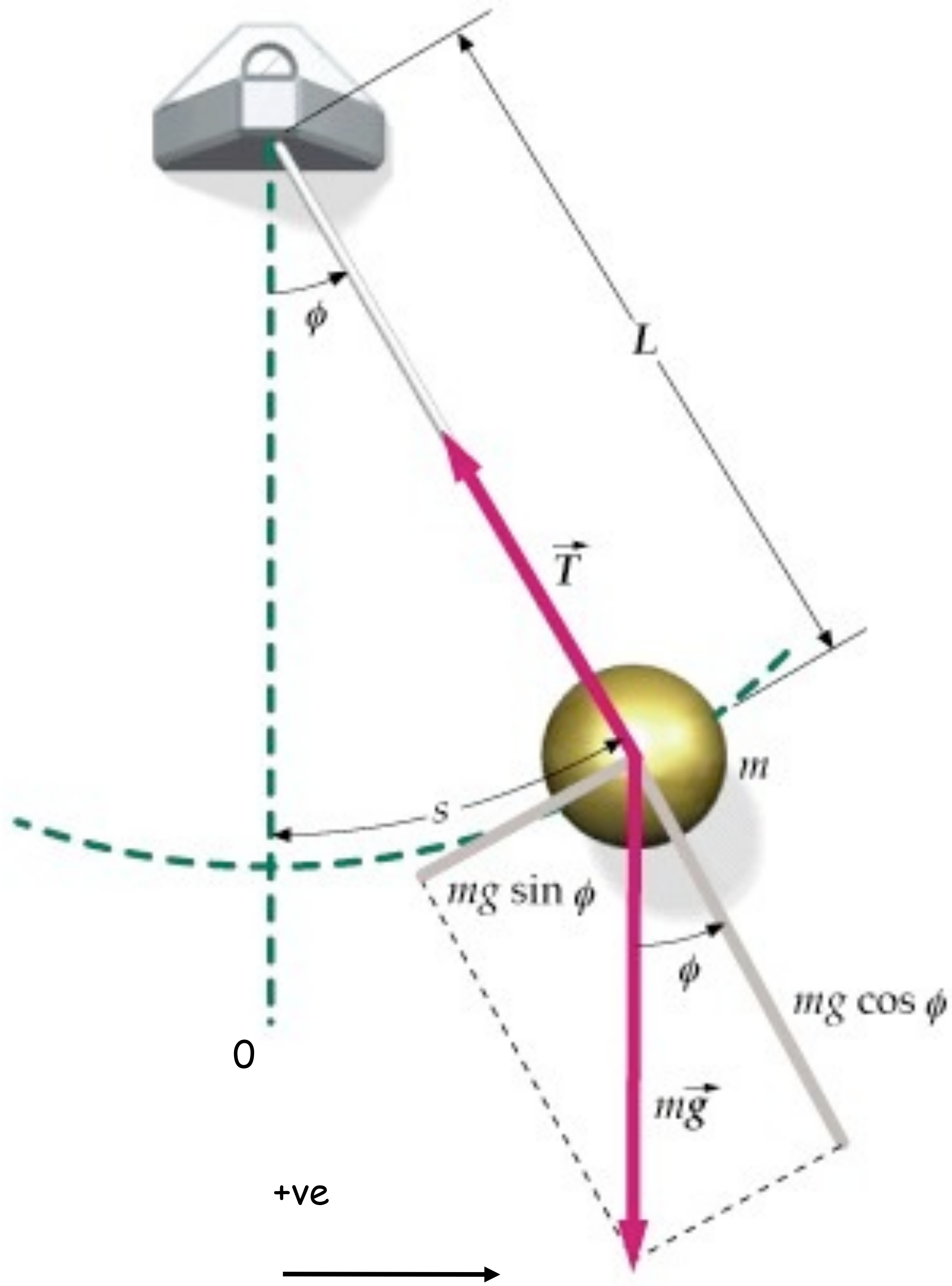
From Newton's 2nd Law (N2)

$$\sum F_{tang} = -mg\sin\phi$$

$$= ma$$

$$= m\frac{d^2s}{dt^2}$$

$$= mL\frac{d^2\phi}{dt^2}$$



$$-mg\sin\phi = mL\frac{d^2\phi}{dt^2}$$

or $$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin\phi$$

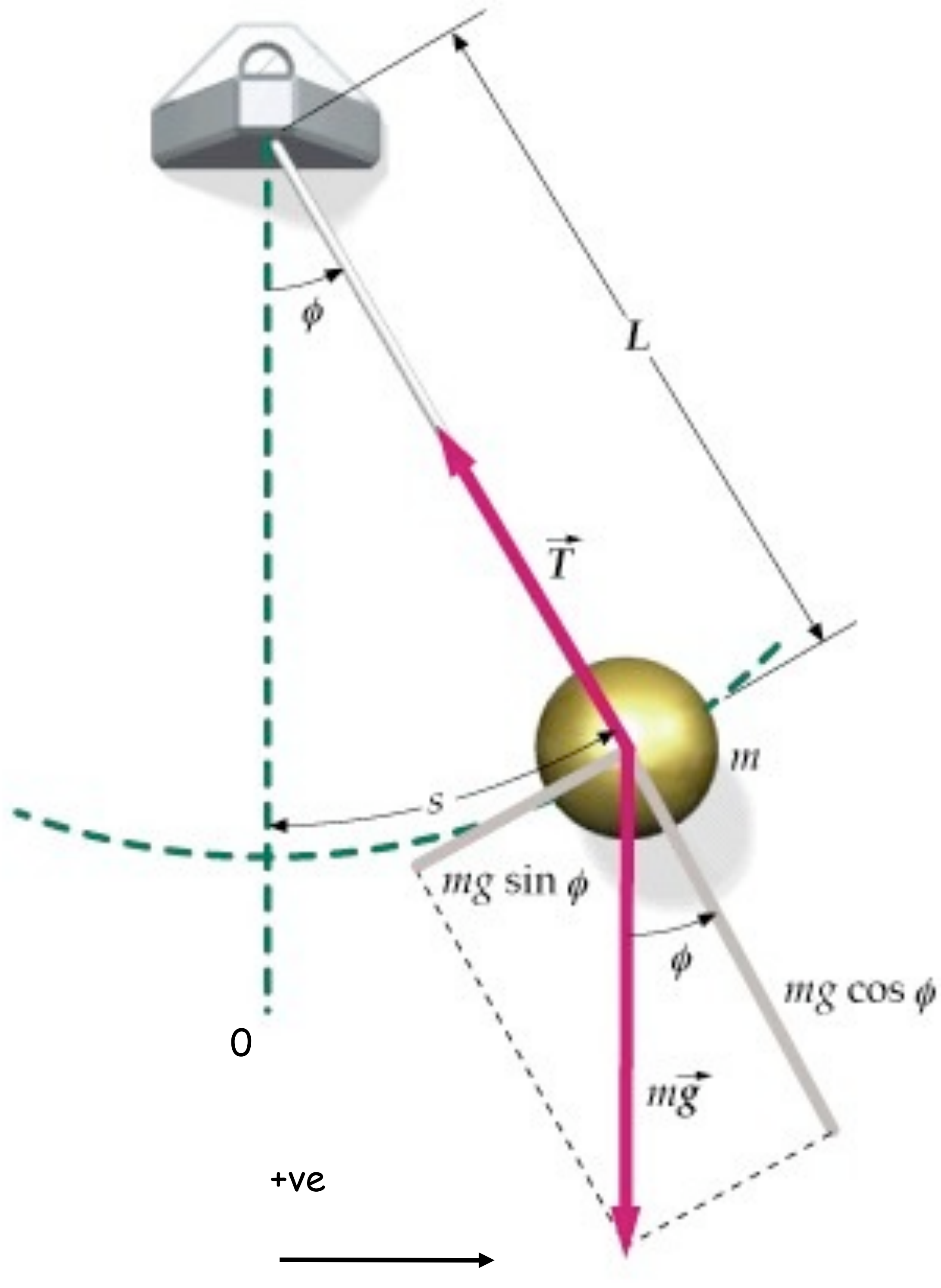
For small ϕ $\sin\phi \sim \phi$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\phi$$

ie SHM with $\omega^2 = \frac{g}{L}$

This has the solution

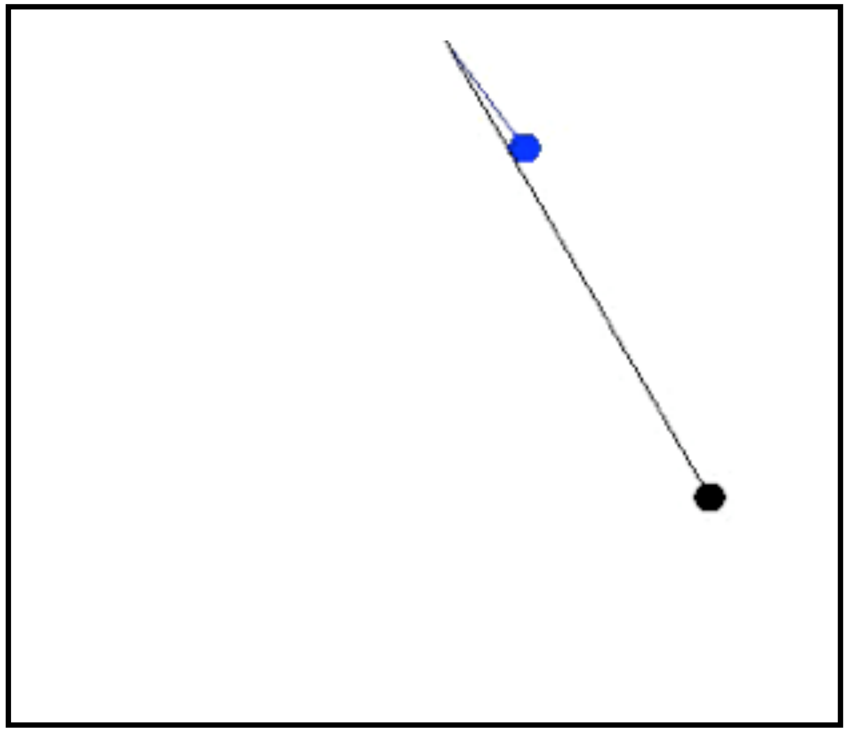
$$\phi = \phi_o \cos(\omega t + \delta)$$



Period of the motion

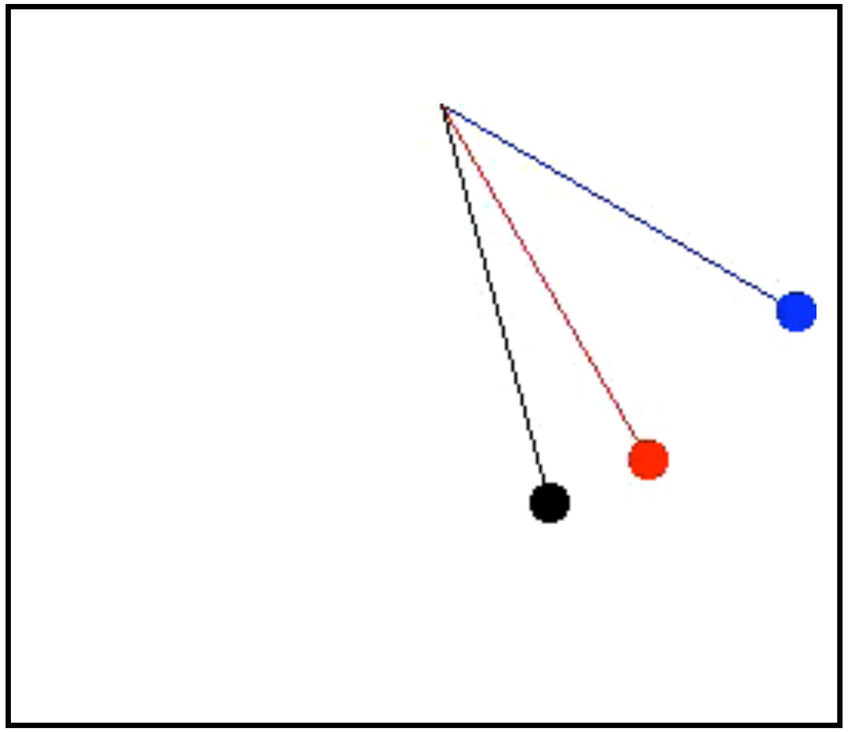
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

ie the longer the pendulum the greater the period



Note: T does not depend upon amplitude of oscillation

even if a clock pendulum changes amplitude it will still keep time

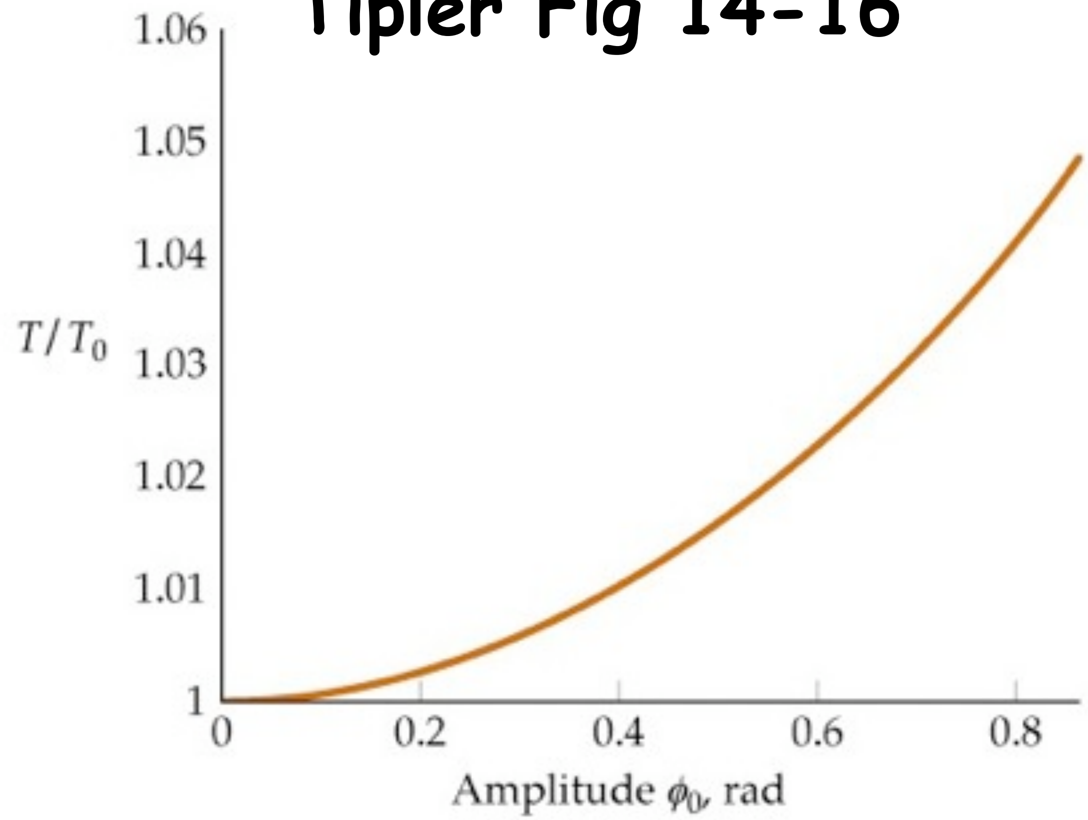


Period of the motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

This is only true for $\phi < 10^o$

Tipler Fig 14-16



Generally

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots\right)$$

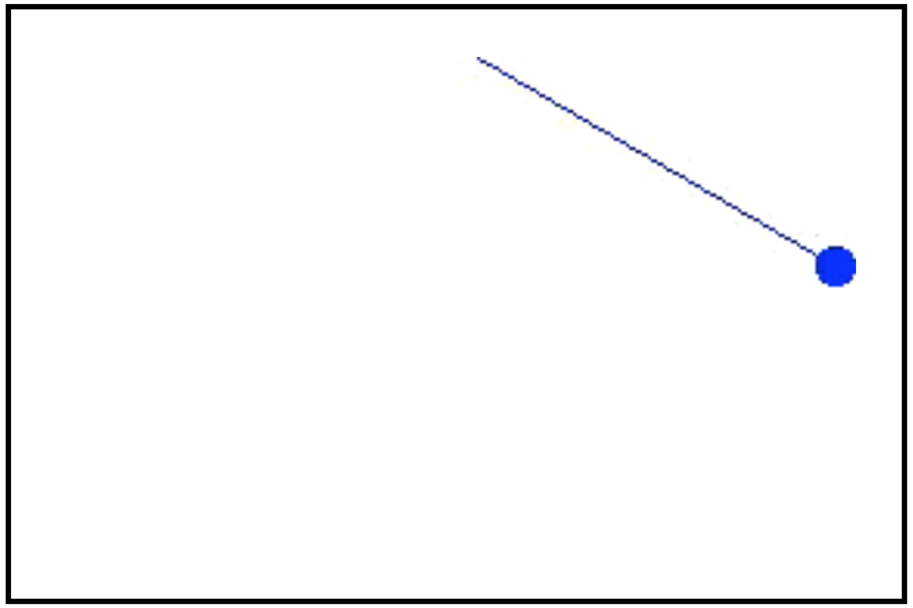
$$T = T_o\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots\right)$$

If the initial angular displacement is significantly large the small angle approximation is no longer valid

The error between the simple harmonic solution and the actual solution becomes apparent almost immediately, and grows as time progresses.

Dark blue pendulum is the simple approximation,

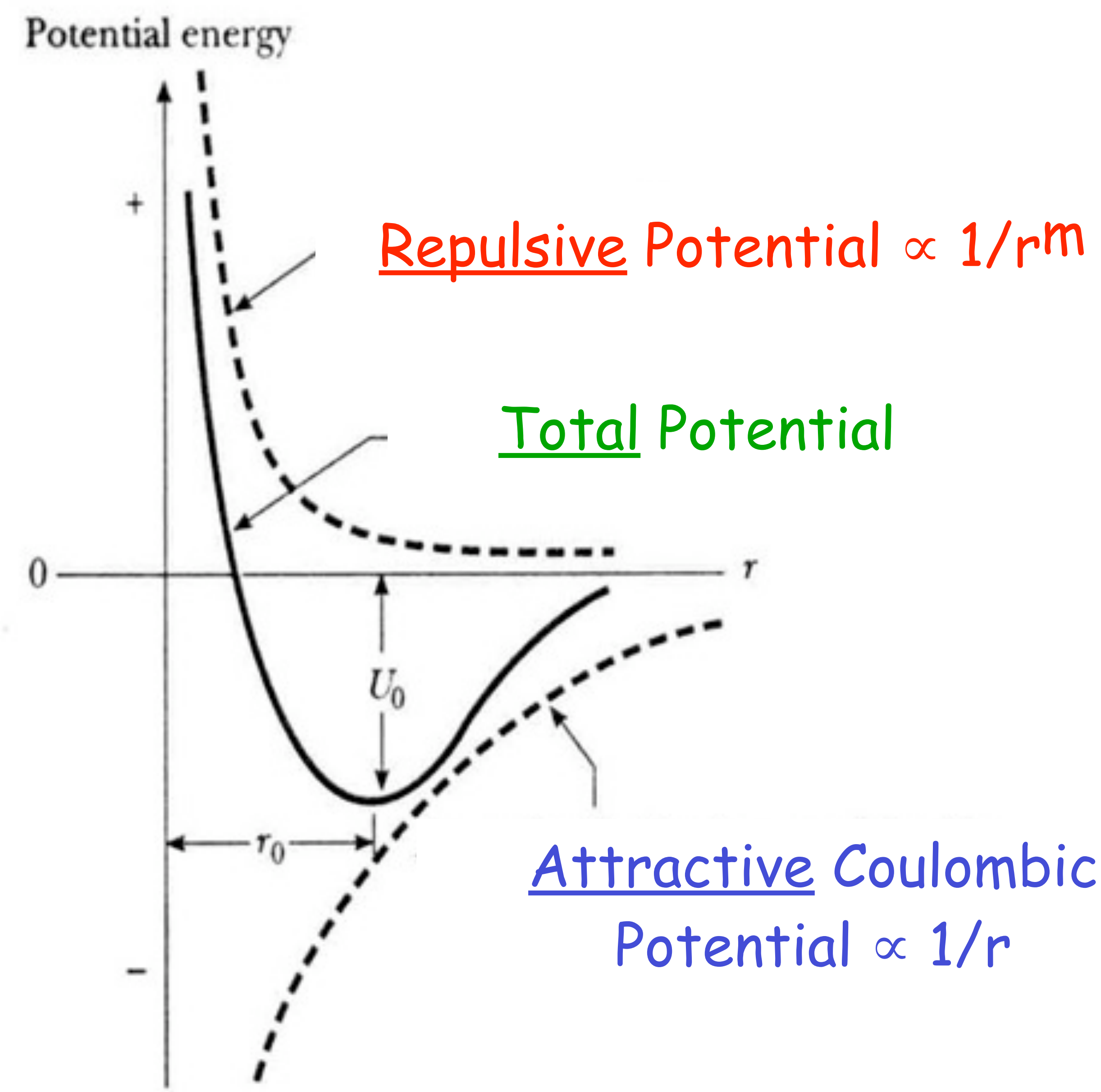
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

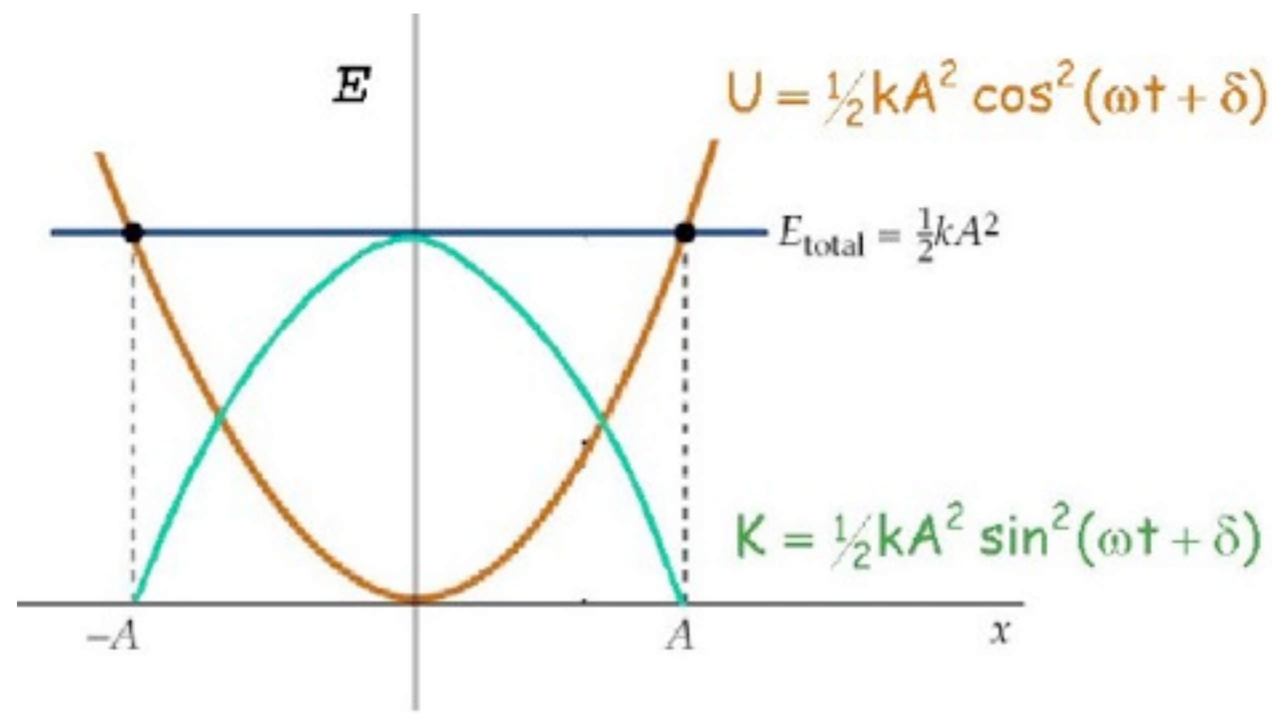


light blue pendulum shows the numerical solution of the nonlinear differential equation of motion.

$$T = T_o \left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\phi}{2}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\phi}{2}\right) + \ldots..\right)$$

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

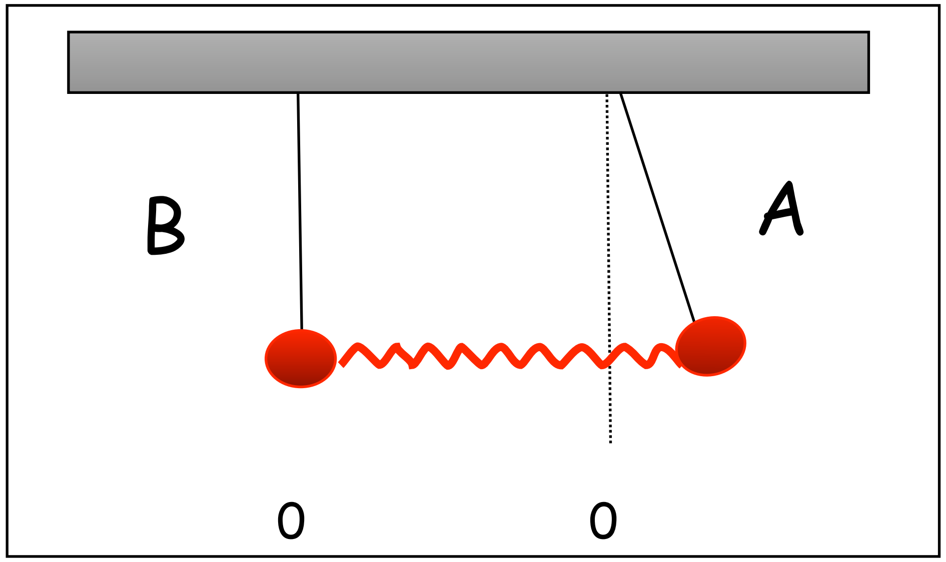




What is a wave? - 2

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

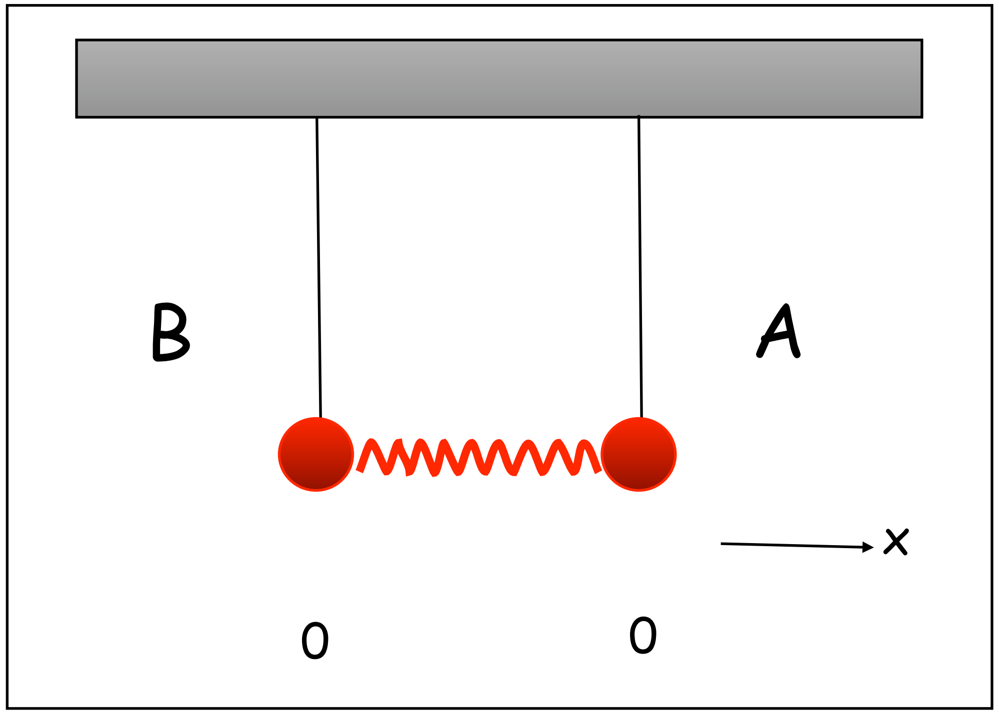
Coupling of harmonic oscillators → the disturbances can **propagate**

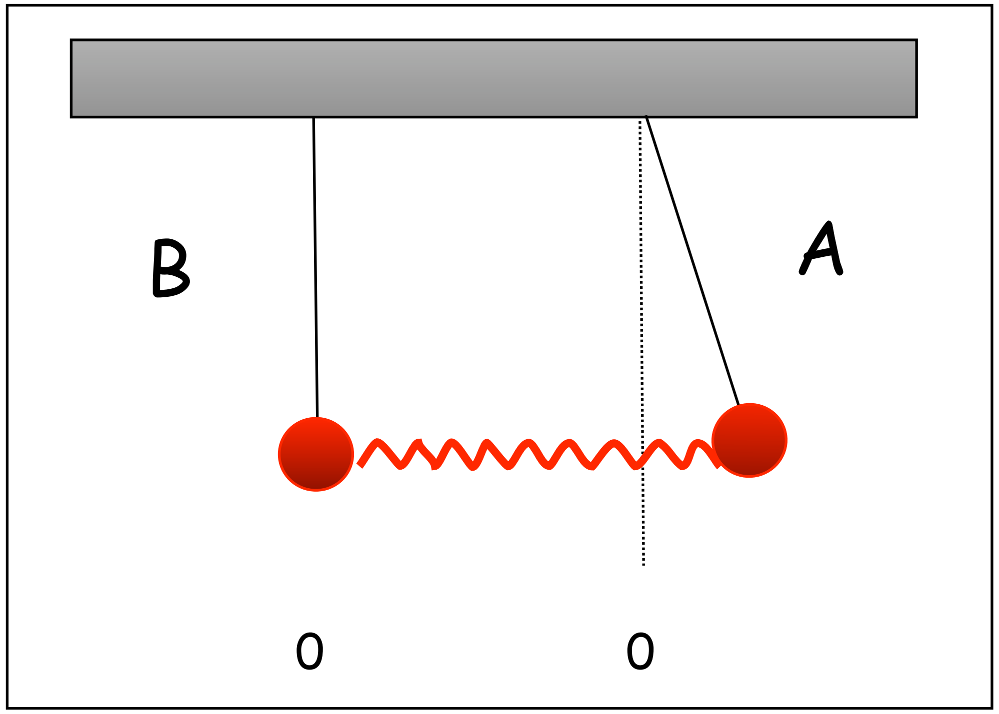


Simplest example: Two identical pendula A and B connected by a light unstretched spring.

(a) Move A to one side, while holding B fixed, then release.

A will oscillate with gradually decreasing amplitude.

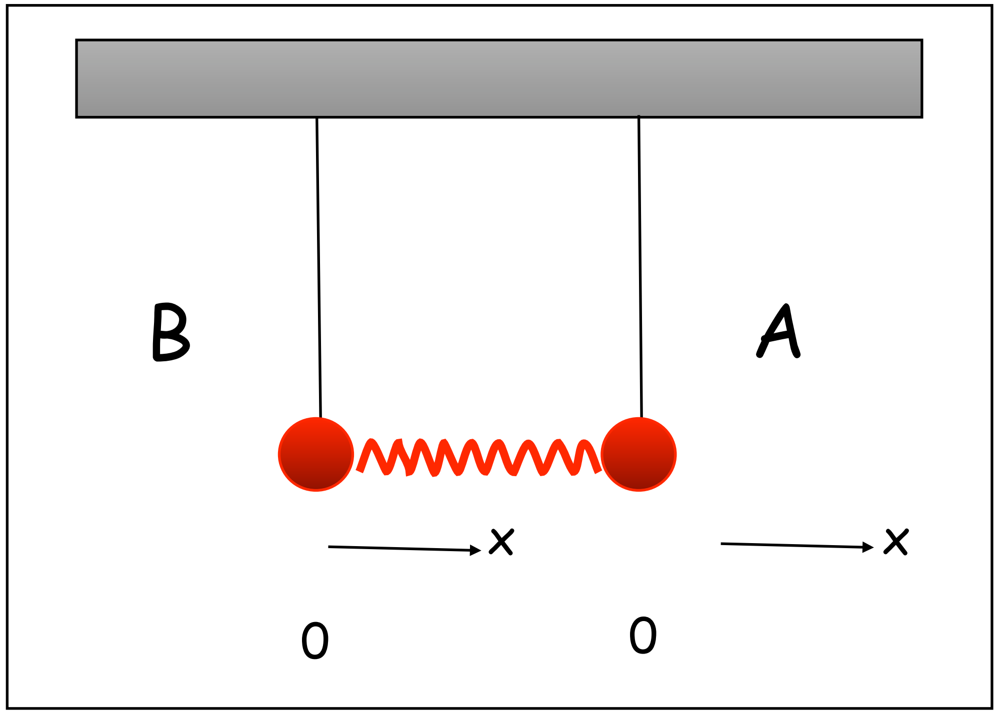


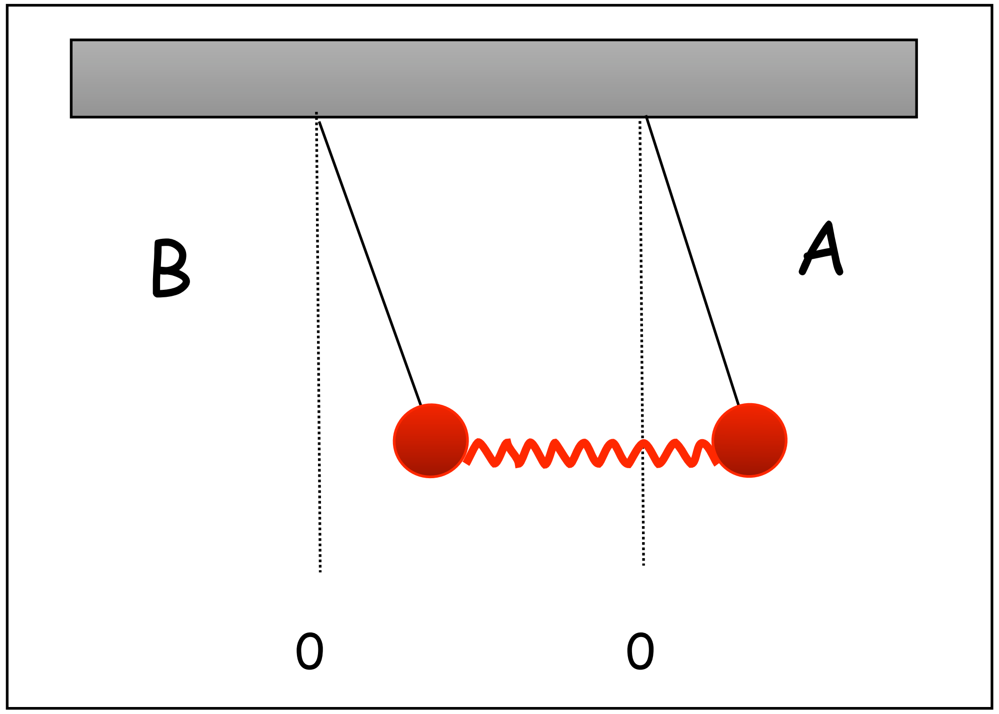


(b) Move A and B to same side by equal amounts then release.

The distance between A and B is constant and equal to the relaxed length of the spring, and the spring exerts no force on either mass.

Each pendulum is essentially free and oscillating with its natural frequency $\omega_o = \sqrt{\frac{g}{L}}$





If x_A and x_B are the displacements of A and B resp.

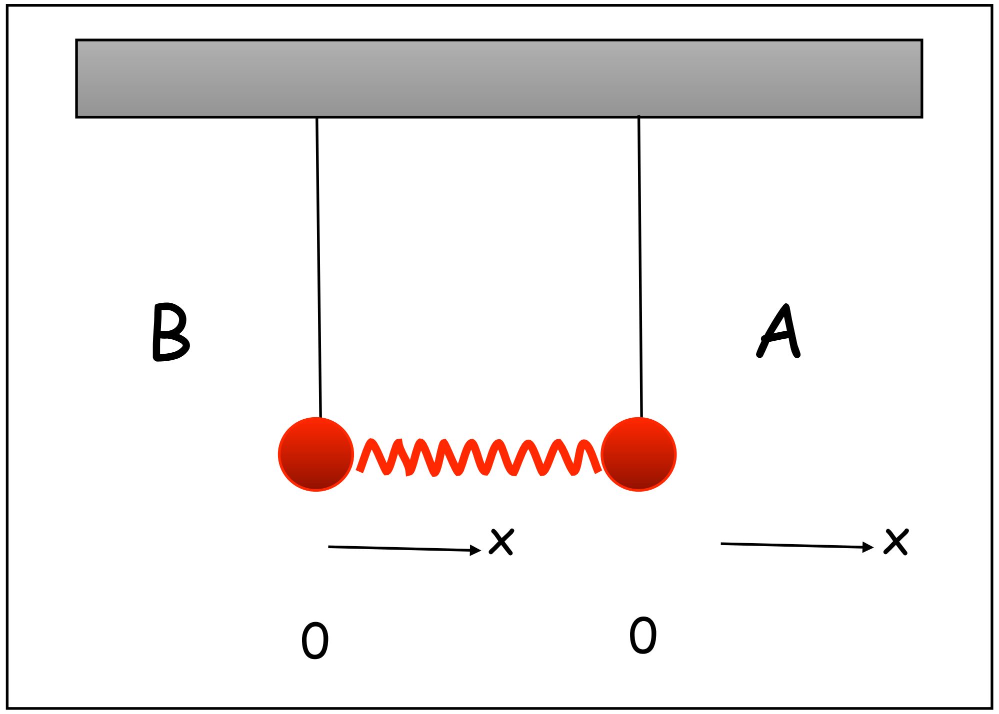
The equations of motion are

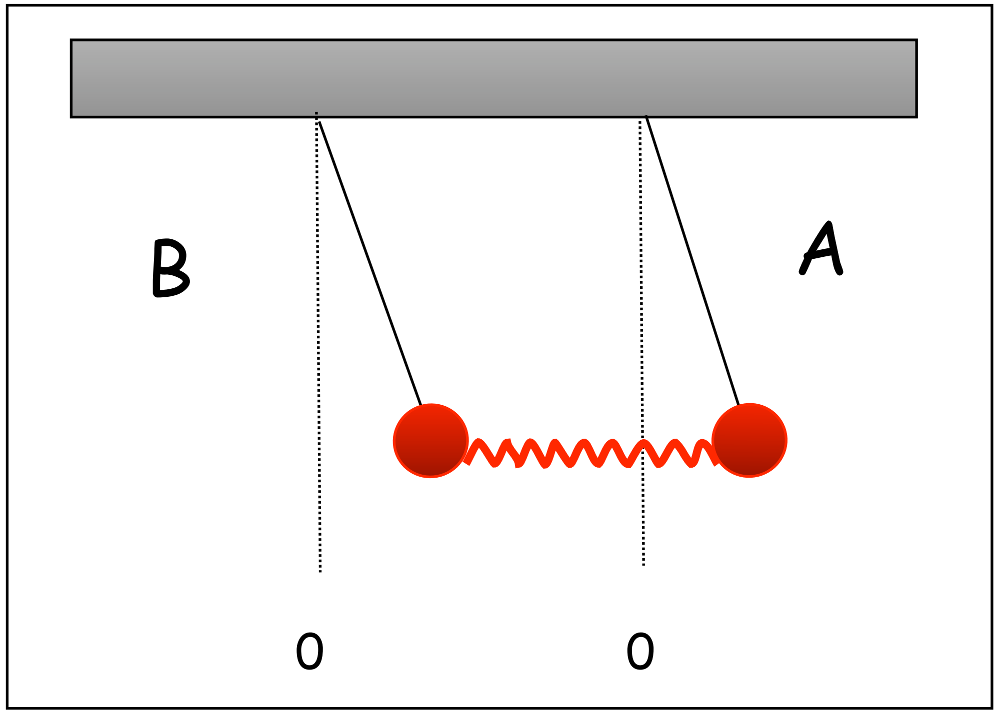
$$x_A = c\cos(\omega_o t)$$

$$x_B = c\cos(\omega_o t)$$

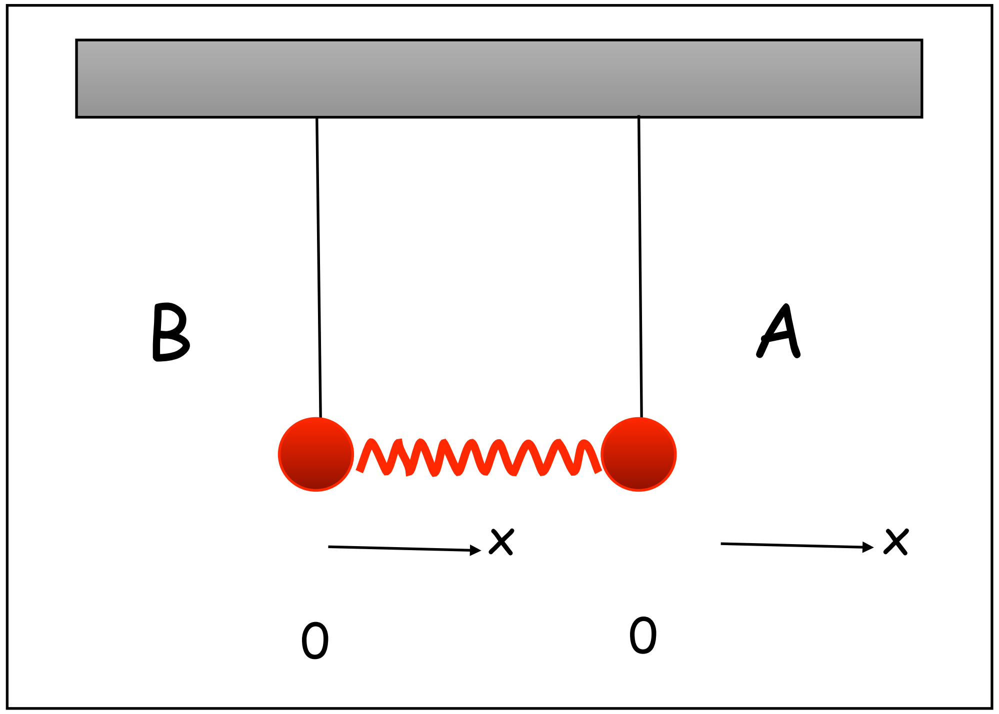
ie both masses vibrate at the same frequency and with the same amplitude.

This is one NORMAL MODE of a coupled system

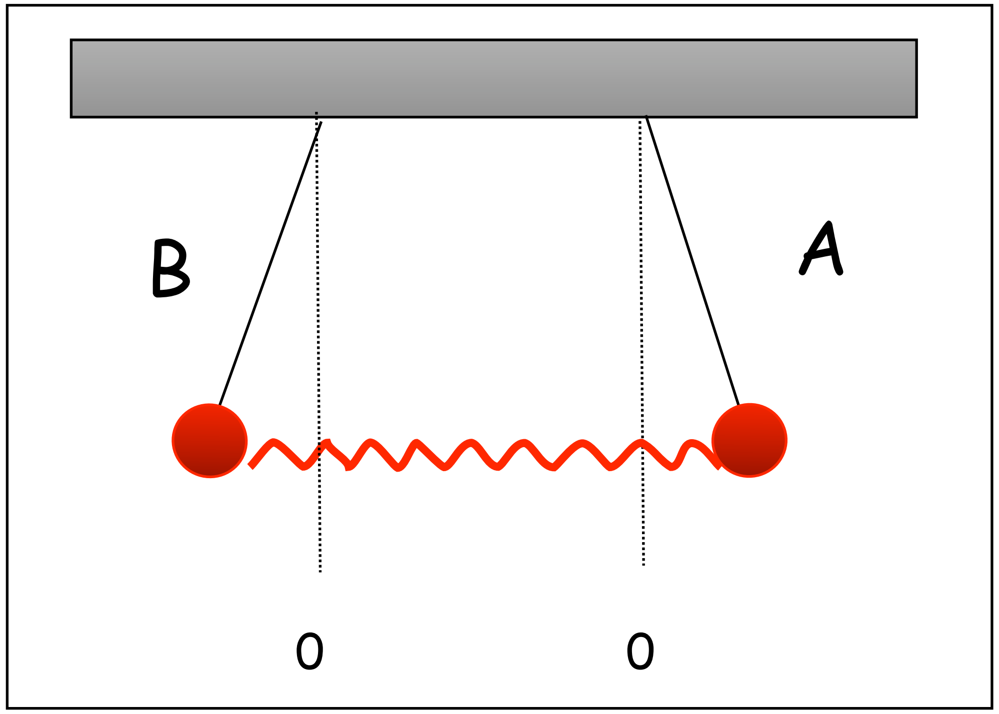




Move A and B to opposite sides by equal amounts then release.



The spring is first stretched then a half cycle later it will be compressed.

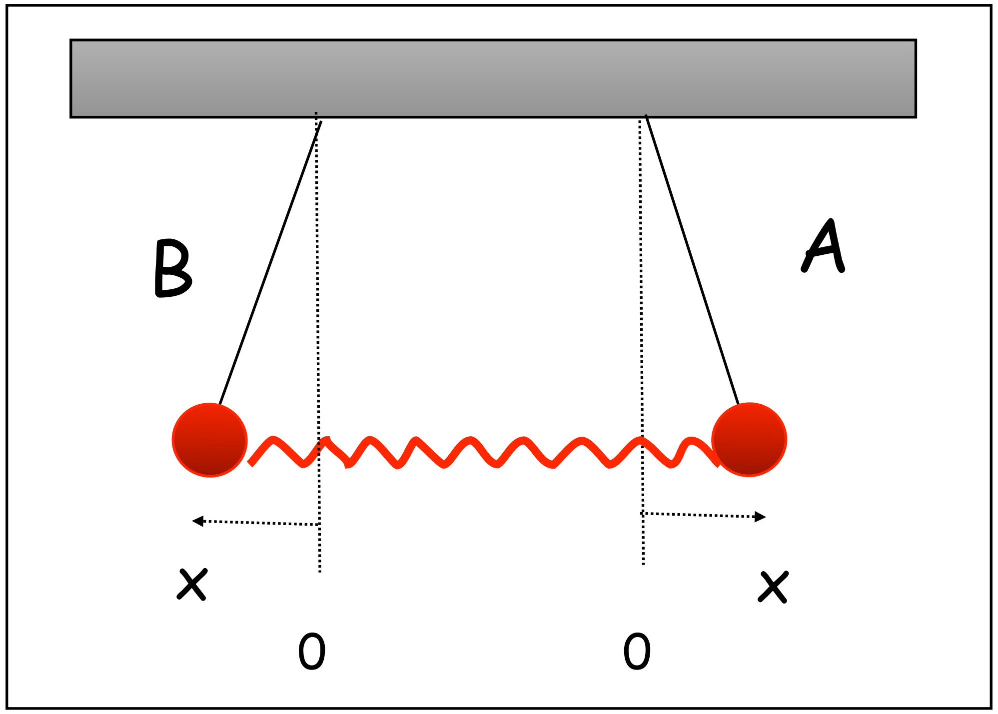


The motions of A and B will be mirror images.

This is the second NORMAL MODE of the system

Equation of motion for coupled oscillators

Consider the situation where both masses are free to move and both are displaced a small distance x
The spring is stretched by 2x and exerts a restoring force of 2kx on the masses



Equation of motion for mass A $$m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + 2kx_A = 0$$

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + 2\omega_c^2)x_A = 0$$

where $\omega_c^2 = k/m$

if $\sqrt{(\omega_o^2 + 2\omega_c^2)} = \omega'$ then $\omega' = \sqrt{\left(\frac{g}{l} + \frac{2k}{m}\right)}$

This has the solution $x_A = D \cos \omega' t$

Motion of B is the mirror image of A $x_B = -D \cos \omega' t$

Each pendulum oscillates with SHM.

The coupling spring has increased the restoring force and therefore increased the frequency of oscillation.

A and B are always 180° (π) out of phase.

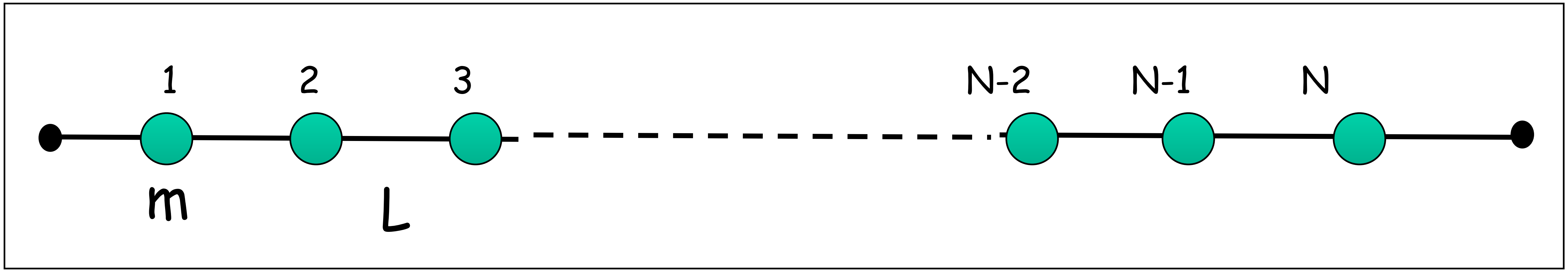
What is a **wave**? - 2

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can propagate, superpose and **stand**

Normal modes of the system

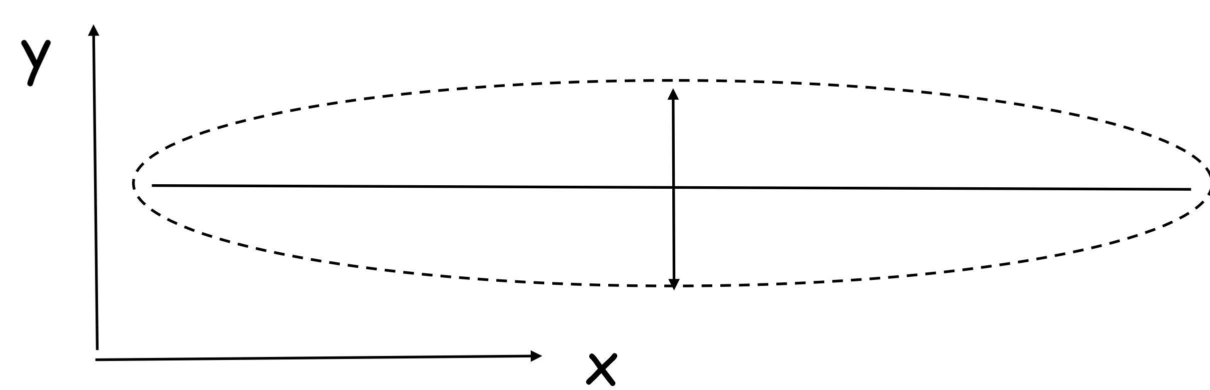
N coupled oscillators



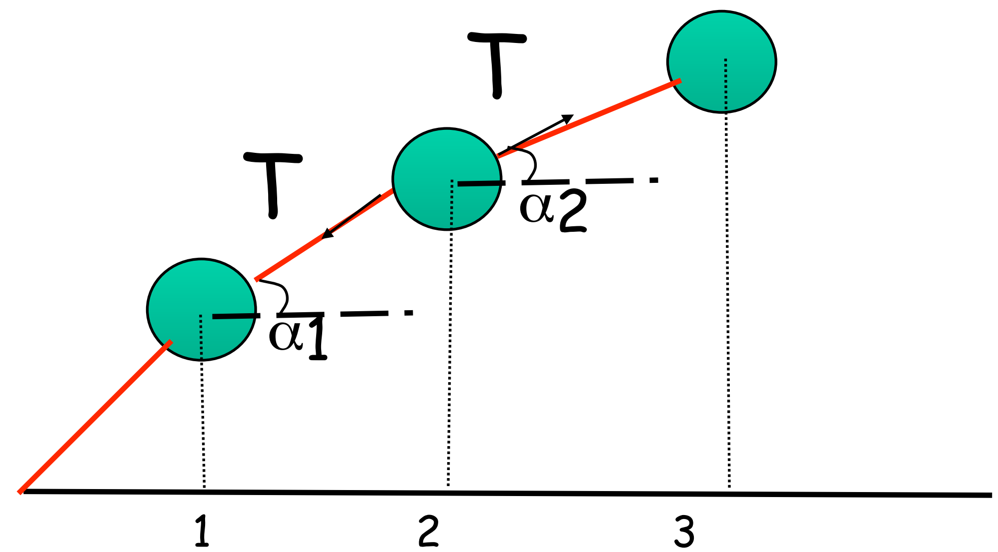
Consider a flexible elastic string to which is attached N identical particles, each mass m, equally spaced a distance L apart.

The ends of the string are fixed a distance L from mass 1 and mass N. The initial tension in the string is T.

Consider small transverse displacements of the masses



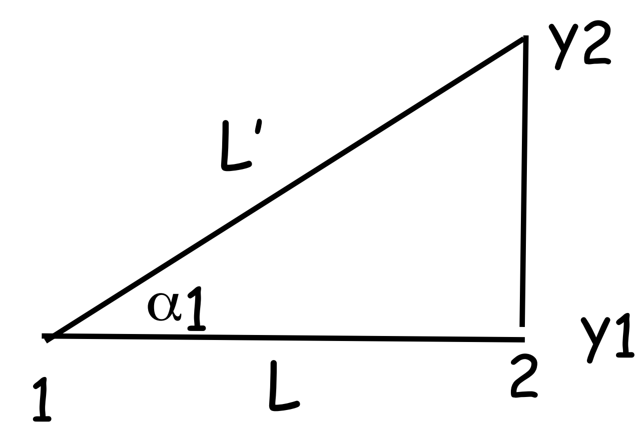
Suppose particle 1 is displaced to y_1, particle 2 to y_2 etc



Length of string between particles 1 and 2

$= \cos \alpha_1 = L / L'$

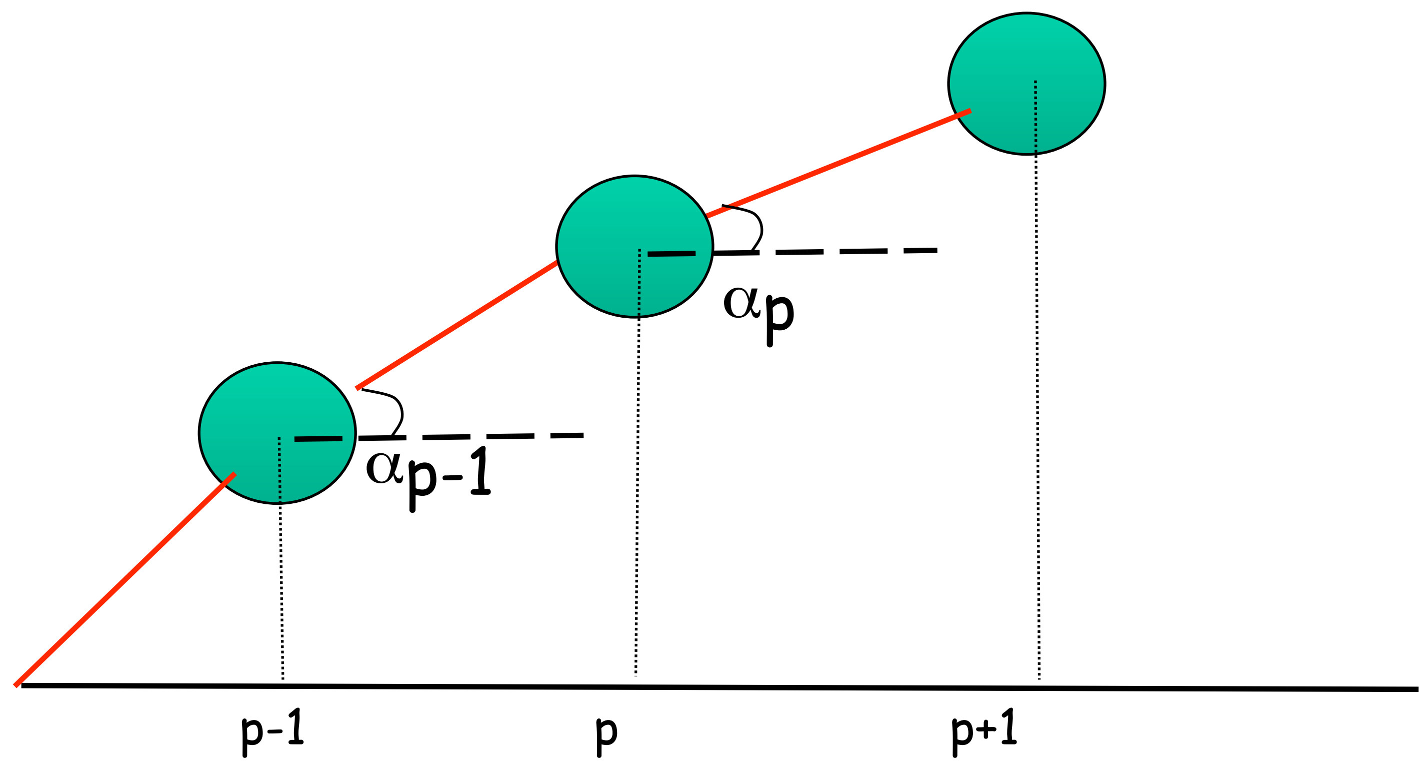
ie $\quad L' = L / \cos \alpha_1$



for small α $\cos \alpha \sim (1 - \alpha_1^2 / 2)$

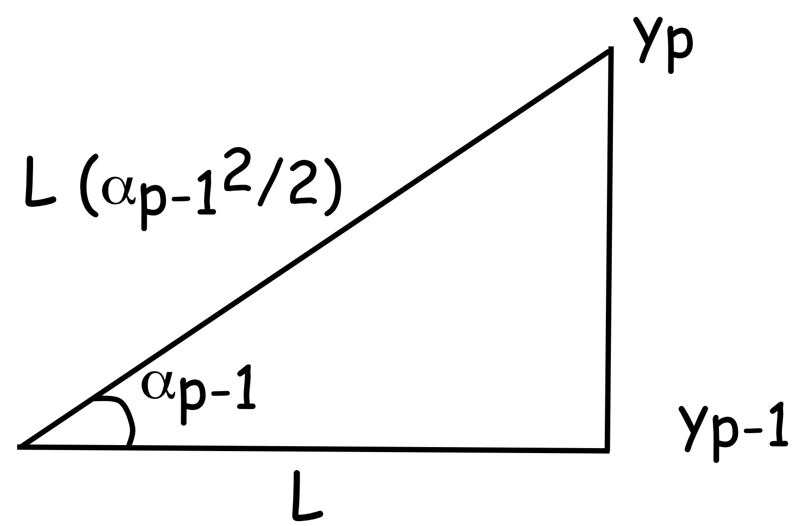
$L' \sim L (1 + \alpha_1^2 / 2)$ ie the increase in length $= L (\alpha_1^2 / 2)$

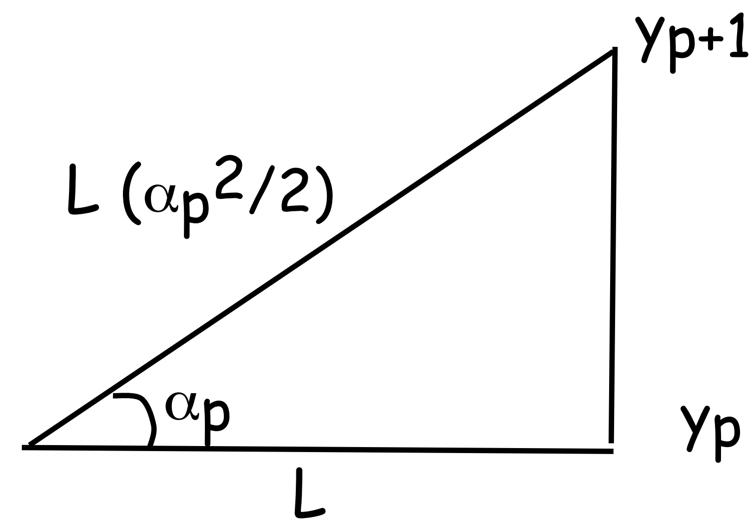
for small angles this increase is small and can be ignored consider masses p-1, p and p+1 at some point along string



for small displacements y (compared to L)

Resultant Force on p = $-T \sin \alpha_{p-1} + T \sin \alpha_p$





$$\sin\alpha_{p-1} = \frac{y_p - y_{p-1}}{L\,(1+\alpha_{p-1}^2/2)} \sim \frac{y_p - y_{p-1}}{L} \quad \text{for small } \alpha$$

$$\sin\alpha_{p} = \frac{y_{p+1} - y_{p}}{L\,(1+\alpha_{p}^2/2)} \sim \frac{y_{p+1} - y_{p}}{L} \quad \text{for small } \alpha$$

Force on particle $\quad F_p = -T\left(\frac{y_p - y_{p-1}}{L}\right) + T\left(\frac{y_{p+1} - y_p}{L}\right)$

$$F_p = -T\left(\frac{y_p - y_{p-1}}{L}\right) + T\left(\frac{y_{p+1} - y_p}{L}\right)$$

but $\quad F_p = m_p a_p$

$$m\frac{d^2 y_p}{dt^2} = -T\left(\frac{y_p - y_{p-1}}{L}\right) + T\left(\frac{y_{p+1} - y_p}{L}\right)$$

Substitute $T/mL = \omega_o{}^2$

$$\therefore \quad \frac{d^2 y_p}{dt^2} = -\omega_o^2\left(y_p - y_{p-1}\right) + \omega_o^2\left(y_{p+1} - y_p\right)$$

$$\text{or} \quad \frac{d^2 y_p}{dt^2} + 2\omega_o^2 y_p - \omega_o^2 (y_{p+1} + y_{p-1}) = 0$$

We can write a similar expression for all N particles

Therefore we have a set of N (coupled) differential equations one for each value of p from p=1 to p=N.

N.B. at fixed ends: $y_0 = 0$ and $y_{N+1} = 0$

Special case N=1

Generally $$\frac{d^2y_p}{dt^2} + 2\omega_o^2\, y_p - \omega_o^2\,(y_{p+1} - y_{p-1}) = 0$$

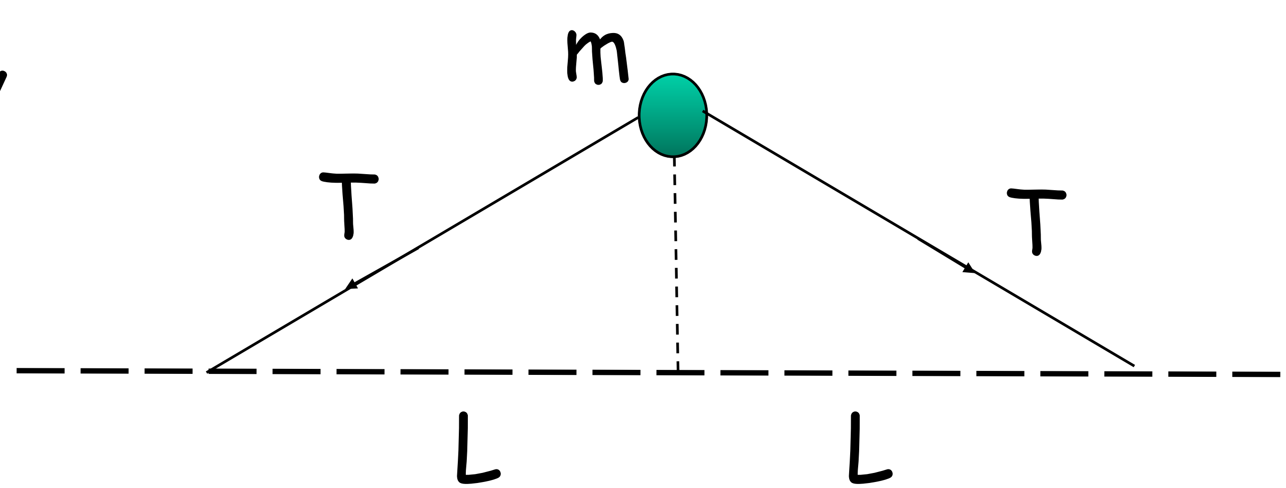
$$\frac{d^2y_1}{dt^2} + 2\omega_o^2\, y_1 - \omega_o^2\,(y_2 - y_0) = 0 \quad \text{for } N = 1$$

no particle =0

$$\frac{d^2y_1}{dt^2} + 2\omega_o^2\, y_1 = 0$$

This is transverse harmonic motion with angular frequency

$$2\omega_o^2 = \frac{2T}{mL}$$



Special case N=2

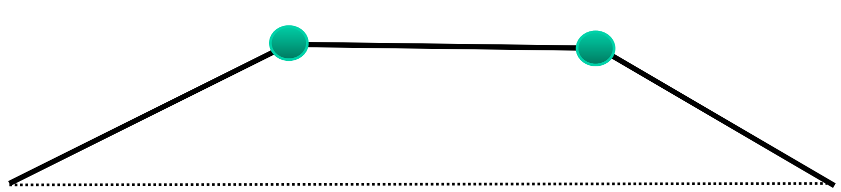
$$\frac{d^2y_1}{dt^2} + 2\omega_o^2 y_1 - \omega_o^2 y_2 = 0$$

$$\frac{d^2y_2}{dt^2} + 2\omega_o^2 y_2 - \omega_o^2 y_1 = 0$$

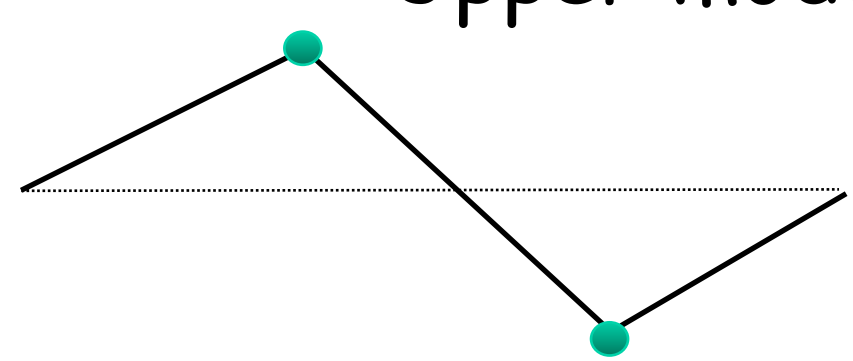
These are similar to the equations for coupled pendula but here we have the simplification that $\omega_o = \omega_c$

We get two normal modes of oscillation

Lower mode $\omega = \omega_o$

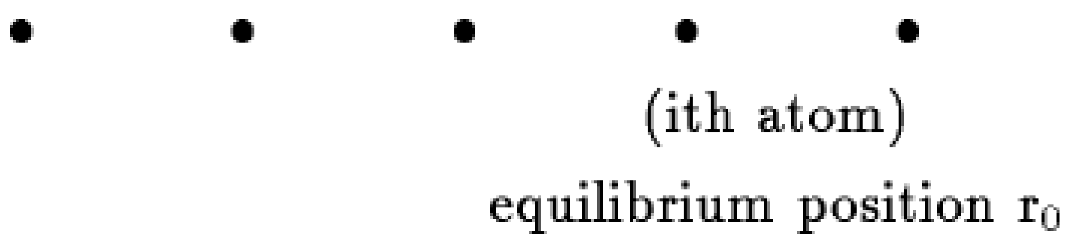


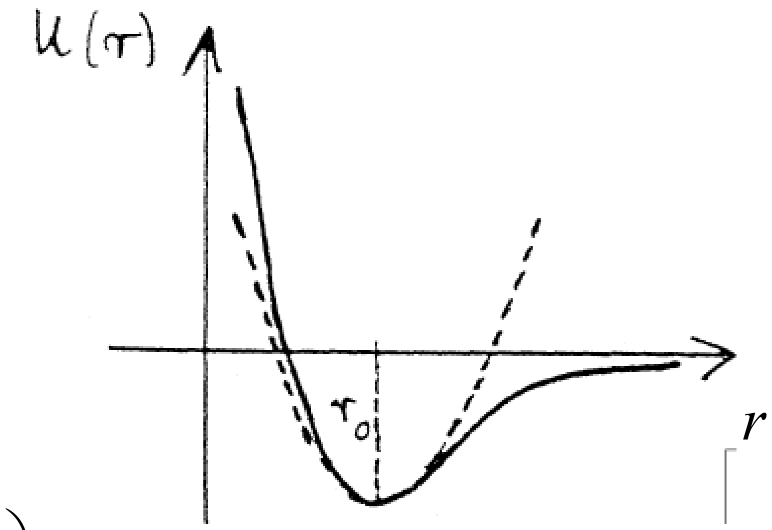
Upper mode $\omega = \sqrt{3}\omega_o$



Interatomic potential

Now we consider a monatomic 1-D lattice in the x-direction. The lattice atoms are very close to eqilibrium. Let us examine a single i-th atom and find the r_i potential as a function of displacement from equilibrium, $U(r_i)$.





We expand this potential into a Taylor's series:

$$U(r_i) = U(r_0) + (r_i - r_0)\left(\frac{dU}{dr_i}\right)_{r_0} + \frac{1}{2}(r_i - r_0)^2\left(\frac{d^2U}{dr_i^2}\right)_{r_0} + \frac{1}{6}(r_i - r_0)^3\left(\frac{d^3U}{dr_i^3}\right)_{r_0} + \dots$$

The first term of this expansion is just the equilibrium binding energy ($\equiv$ const). The second term is the slope of the potential at its minimum (= 0). The fourth and higher terms become increasingly smaller. We are therefore left with the third term as the only significant change in the potential energy for a small displacement $u = r_i - r_o$. This has the form

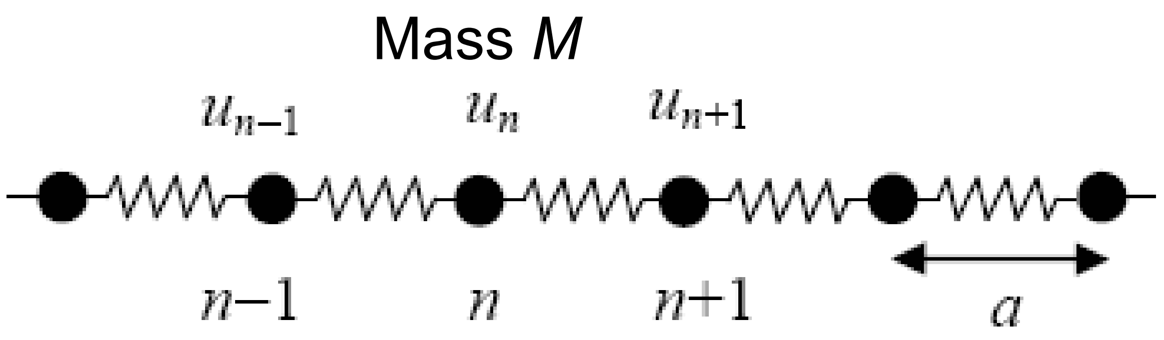
$$\Delta U = \tfrac{1}{2}Cu^2 \quad (C = d^2U/dr_i^2 \text{ at } r_i = r_0)$$

representing the *harmonic approximation*, since it is the same as the energy stored in a spring, or the potential energy of a harmonic oscillator. Our simple model of the dynamic crystal structure should therefore be a "ball and spring" model, with the lengths of the springs equivalent to the equilibrium separations of the ion cores.

Monatomic 1D lattice

Let us examine the simplest periodic system within the context of harmonic approximation ($F = dU/du = Cu$) - a one-dimensional crystal lattice, which is a sequence of masses m connected with springs of force constant C and separation a.

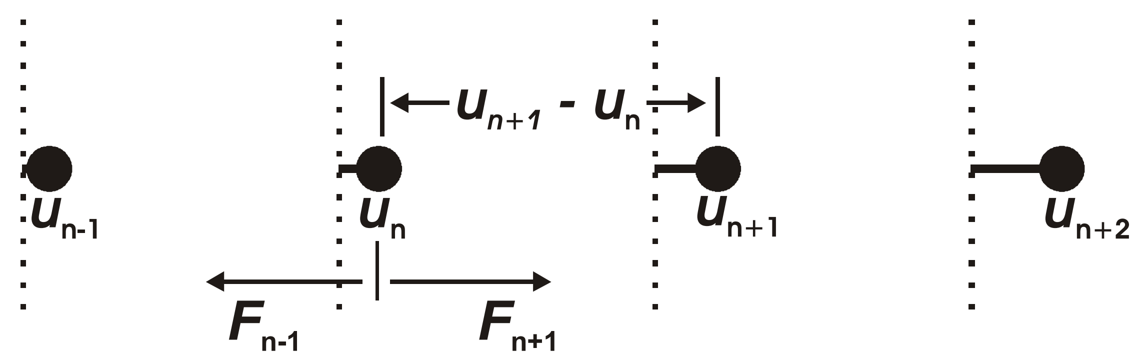
The collective motion of these springs will correspond to solutions of a wave equation. Note: by construction we can see that 3 types of wave motion are possible,
2 transverse, 1 longitudinal (or compressional)



How does the system appear with a longitudinal wave?:

The force exerted on the n-th atom in the lattice is given by

$F_n = F_{n+1,n} - F_{n-1,n} = C[(u_{n+1} - u_n) - (u_n - u_{n-1})]$.



Applying Newton's second law to the motion of the n-th atom we obtain

$$M\frac{d^2u_n}{dt^2} = F_n = -C(2u_n - u_{n+1} - u_{n-1})$$

Note that we neglected hereby the interaction of the n-th atom with all but its nearest neighbors. A similar equation should be written for each atom in the lattice, resulting in N coupled differential equations, which should be solved simultaneously (N - total number of atoms in the lattice). In addition the boundary conditions applied to end atoms in the lattice should be taken into account.