SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Body waves & seismic rays

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Heterogeneities

.. What happens if we have heterogeneities ?

Depending on the kind of reflection part or all of the signal is reflected or transmitted.

- What happens at a free surface?
- Can a P wave be converted in an S wave or vice versa?
- How big are the amplitudes of the reflected waves?

Impedance

- Any medium through which waves propagates will present an impedance to those waves.
- If medium is lossless or possesses no dissipative mechanism ,the impedance is real and can be determined by the energy storing parameters, inertia and elasticity.
- Presence of loss mechanism will introduce a complex term.
- Impedance presented by a string to a traveling wave propagating on it is called transverse impedance.

$$\text{transverse impedance} = \frac{\text{transverse force}}{\text{transverse velocity}}$$

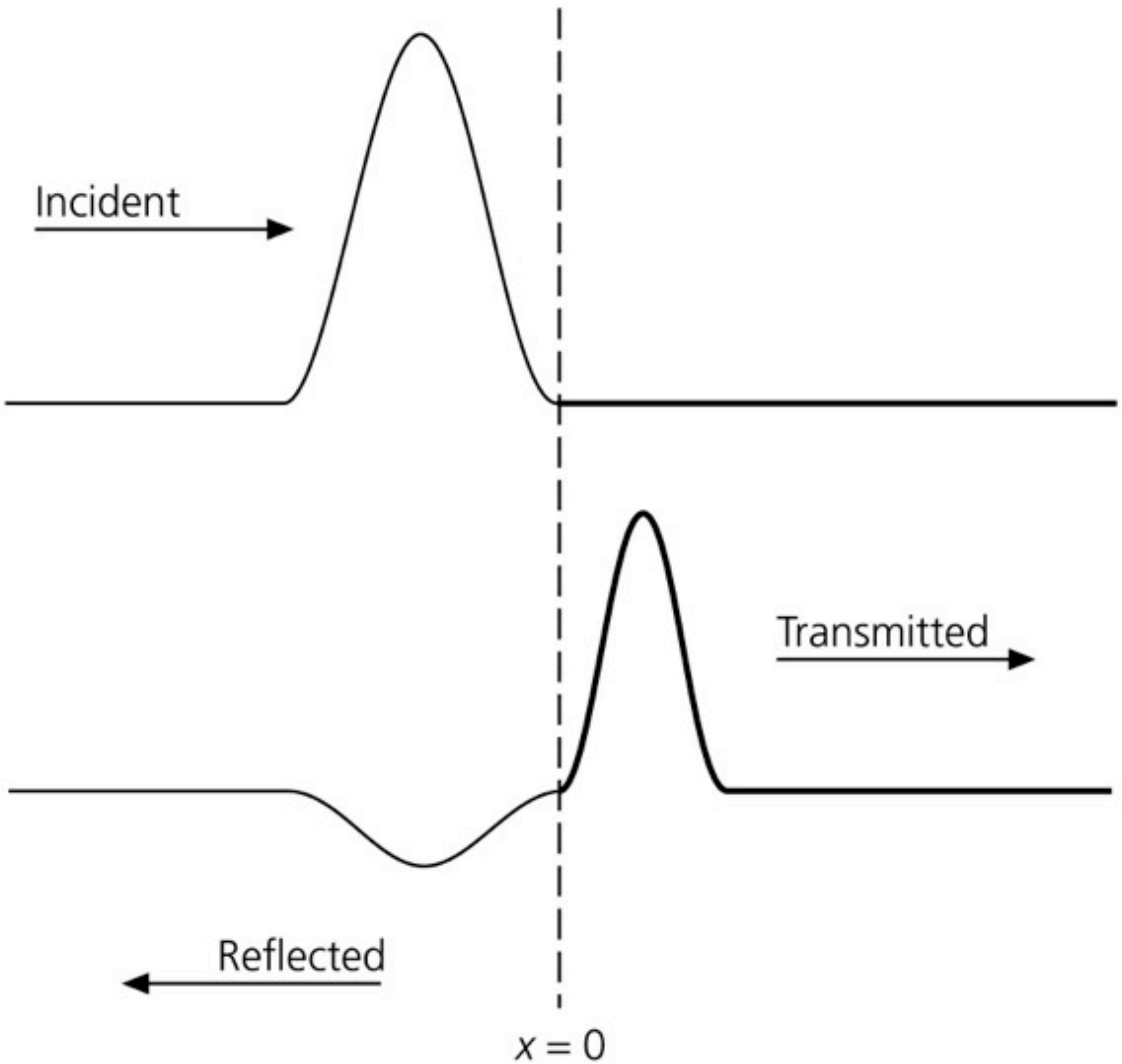
$$F_T \approx -T\tan\theta = -T\left(\frac{\partial y}{\partial x}\right) = T\frac{\omega}{v}y \quad v_T \approx \omega y$$

$$Z = \frac{T}{v} = \sqrt{T\rho} = \rho v$$

$$\text{acoustic impedance} = \frac{\text{pressure}}{\text{sound flux}} = \rho v$$

Heterogeneous string

Figure 2.2-5: Transmitted and reflected wave pulses.



Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

Displacement continuity

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Right side:

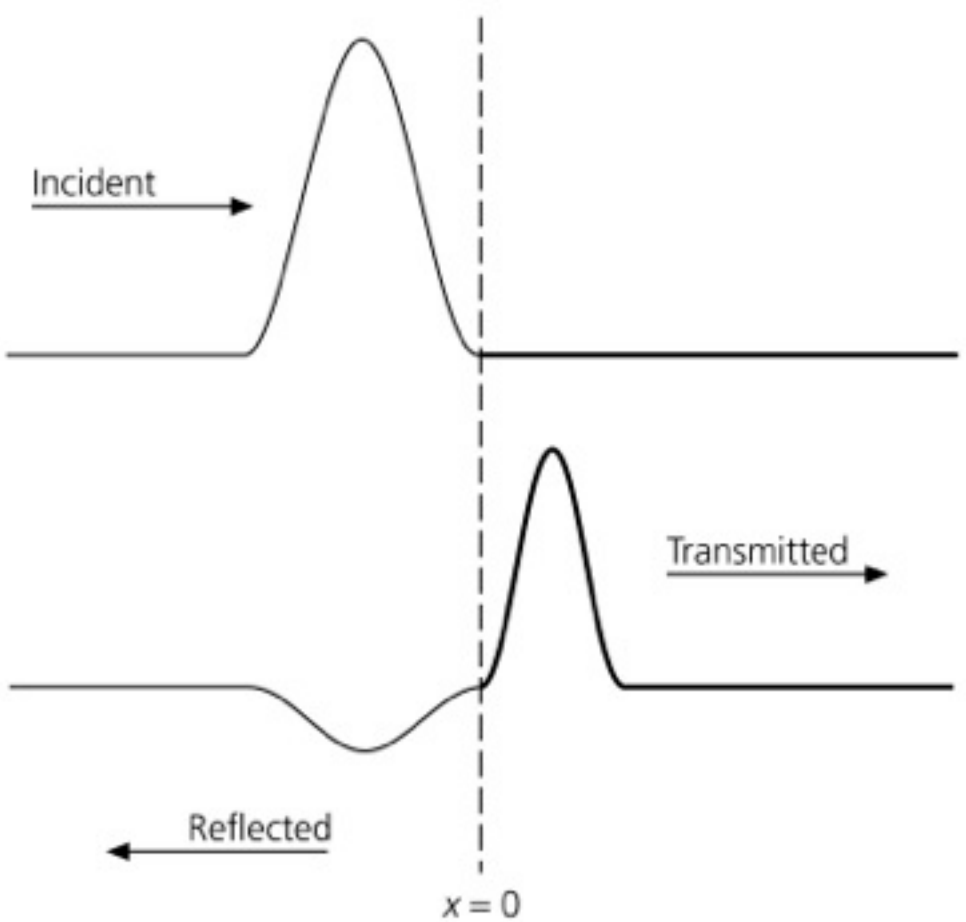
$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

$$u_1(0, t) = u_2(0, t)$$

$$Ae^{i\omega t} + Be^{i\omega t} = Ce^{i\omega t}$$

$$A + B = C$$

Figure 2.2-5: Transmitted and reflected wave pulses.

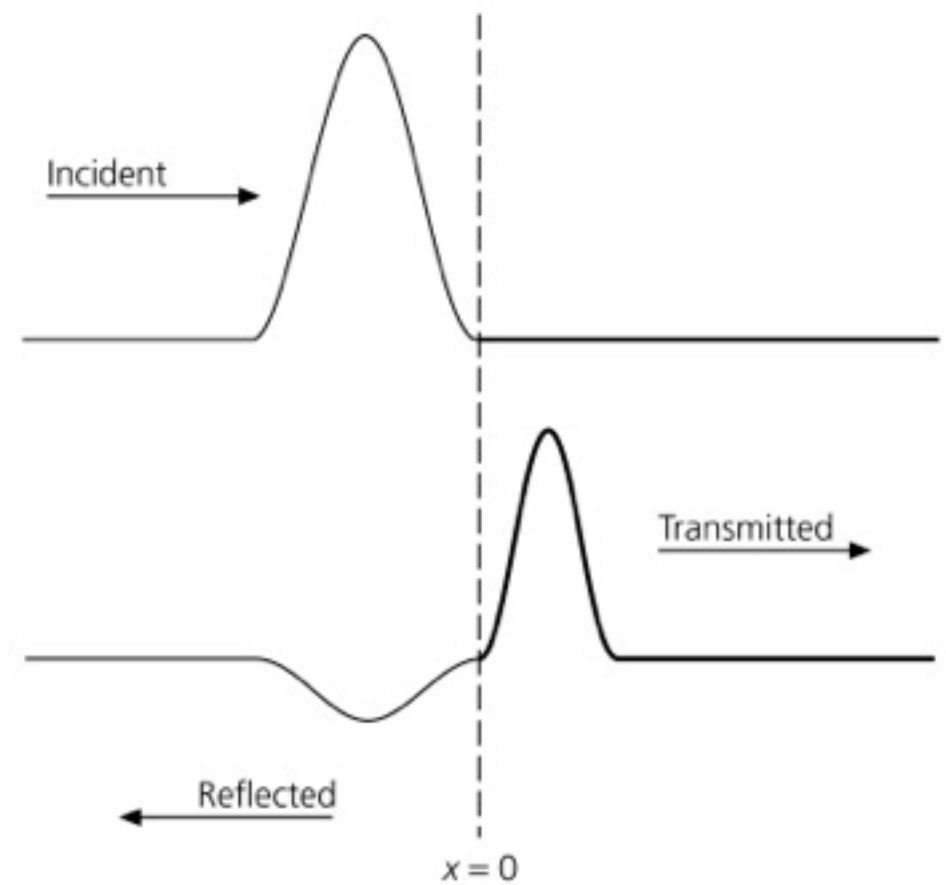


Force continuity

Left side:

$$u_1(x, t) = Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)}$$

Figure 2.2-5: Transmitted and reflected wave pulses.



Right side:

$$u_2(x, t) = Ce^{i(\omega t - k_2 x)}$$

$$\boxed{\tau \frac{\partial u_1(0, t)}{\partial x} = \tau \frac{\partial u_2(0, t)}{\partial x}}$$

$$\tau k_1(A - B) = \tau k_2 C$$

Because the velocities on the two sides are $v_i = (\tau/\rho_i)^{1/2}$ and $k_i = \omega/v_i$,

$$\rho_1 v_1 (A - B) = \rho_2 v_2 C$$

$$A + B = C$$

$$\rho_1 v_1 \, (A - B) = \rho_2 v_2 C$$

Reflection coefficient:

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Transmission coefficient:

$$T_{12} = \frac{C}{A} = \frac{2\,\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2}$$

$$R_{12} = - R_{21} \qquad T_{12} + T_{21} = 2$$

$$R_{12} = \frac{B}{A} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2}$$

Fixed end?

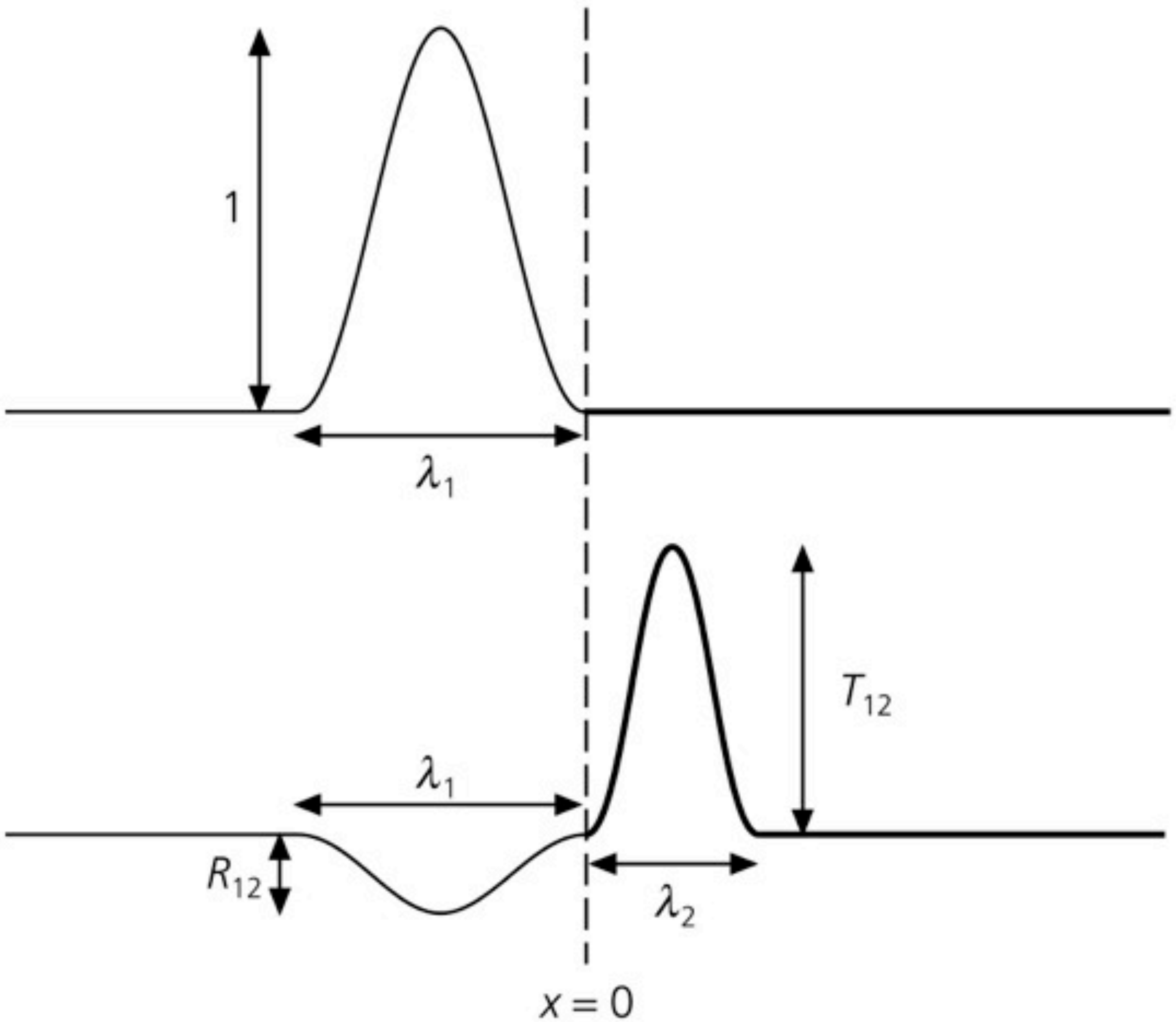
$$R_{fixed} = \frac{\rho_1 v_1 - \infty}{\rho_1 v_1 + \infty} = -1$$

Free end?

$$R_{fixed} = \frac{\rho_1 v_1 - 0}{\rho_1 v_1 + 0} = 1$$

Polarity?

Figure 2.2-7: Reflected and transmitted amplitudes.



$$\omega = v_1 k_1 = v_2 k_2 = v_1 2\pi/\lambda_1 = v_2 2\pi/\lambda_2$$

Kinetic energy

Kinetic energy:

$$KE = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

because the mass of the spring is $m = \rho dx$

Averaged over one wavelength, with $u(x, t) = A\cos(\omega t - kx)$:

$$KE = \frac{\rho}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{\rho\ A^2\ \omega^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

Identity:

$$\int_0^\lambda \sin^2(\omega t - kx)dx = \lambda/2$$

$$KE = A^2\omega^2\rho/4$$

Potential energy

Potential energy:

strain:

$$e = \frac{(dx^2 + du^2)^{1/2} - dx}{dx} = \left[1 + \left(\frac{du}{dx}\right)^2\right]^{1/2} - 1 = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

(using the Taylor series approximation $(1 + a^2)^{1/2} \approx 1 + a^2/2$ for small a)

$$PE = \int_0^L e\tau\, dx = \frac{\tau}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$PE = \frac{\tau}{2\lambda}\int_0^\lambda \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\tau\ A^2\ k^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx$$

$$PE = \tau A^2 k^2/4 = A^2 \omega^2 \rho/4$$

Total energy

$$KE = A^2 \omega^2 \rho/4$$

$$PE = A^2 \omega^2 \rho/4$$

Total energy:

$$E = PE + KE = A^2 \omega^2 \rho/2$$

Energy flux:

$$\dot{\mathbf{E}} = A^2 \omega^2 \rho \mathrm{v}/2$$

$$\dot{\mathbf{E}}_R + \dot{\mathbf{E}}_T = R_{12}^2 \omega^2 \rho_1 \mathrm{v}_1/2 + T_{12}^2 \omega^2 \rho_2 \mathrm{v}_2/2$$

$$= (\omega^2/2)\,[R_{12}^2 \mathrm{v}_1 \rho_1 + T_{12}^2 \mathrm{v}_2 \rho_2] = \omega^2 \rho_1 \mathrm{v}_1/2 = \dot{\mathbf{E}}_I$$

What happens when the material parameters change at a discontinuity interface?

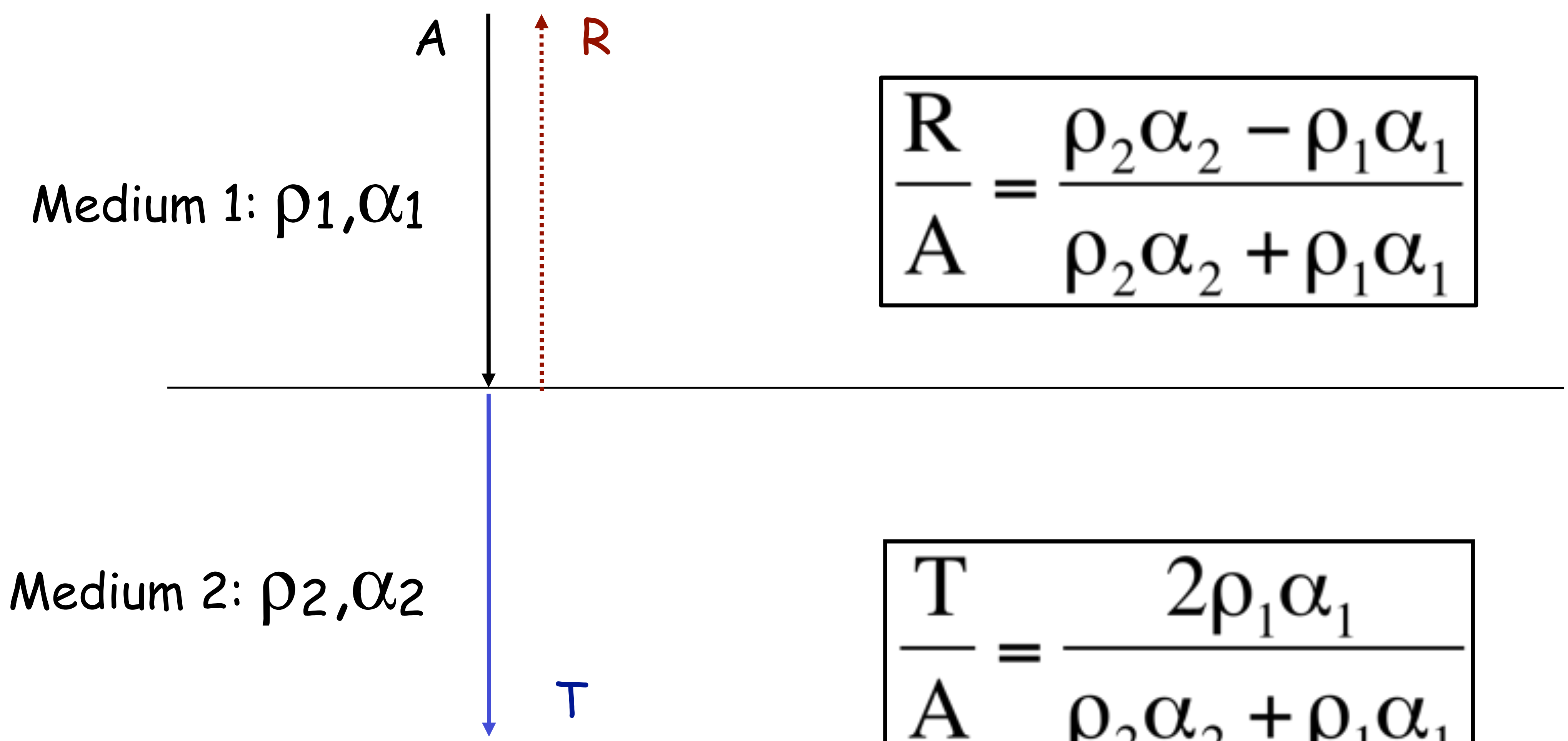
Continuity of displacement and traction fields is required

http://www.walter-fendt.de/ph14e/huygenspr.htm

Kinematic (displacement continuity) *gives* **Snell's law**, but how much is reflected, how much transmitted?

Reflection & Transmission coefficients

Let's take the most simple example: P-waves with **normal** incidence on a material interface. Dynamic conditions give:

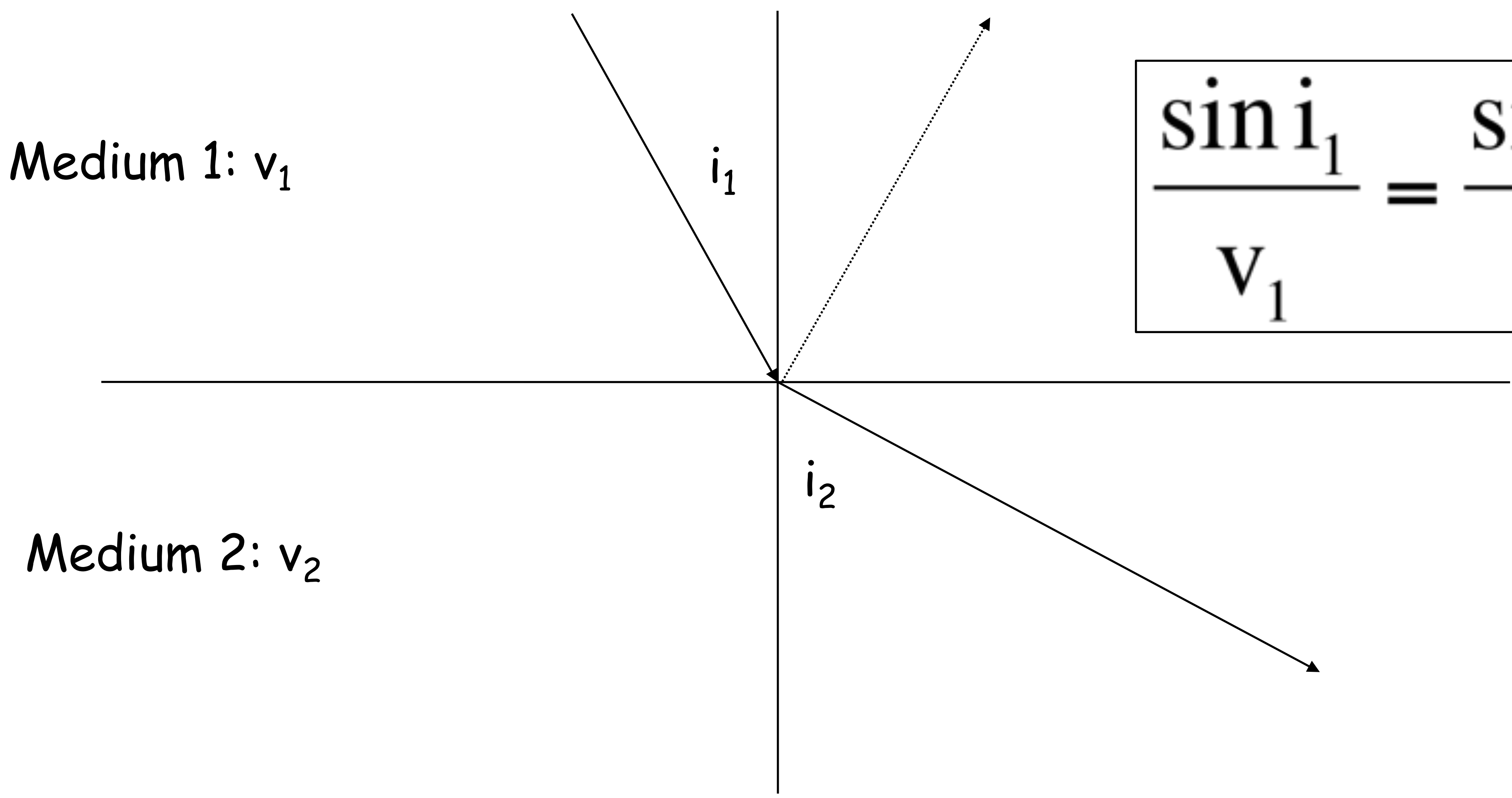


$$\frac{R}{A} = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}$$

$$\frac{T}{A} = \frac{2\rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}$$

At oblique angles conversions from S-P, P-S have to be considered.

What happens at a plane material discontinuity?

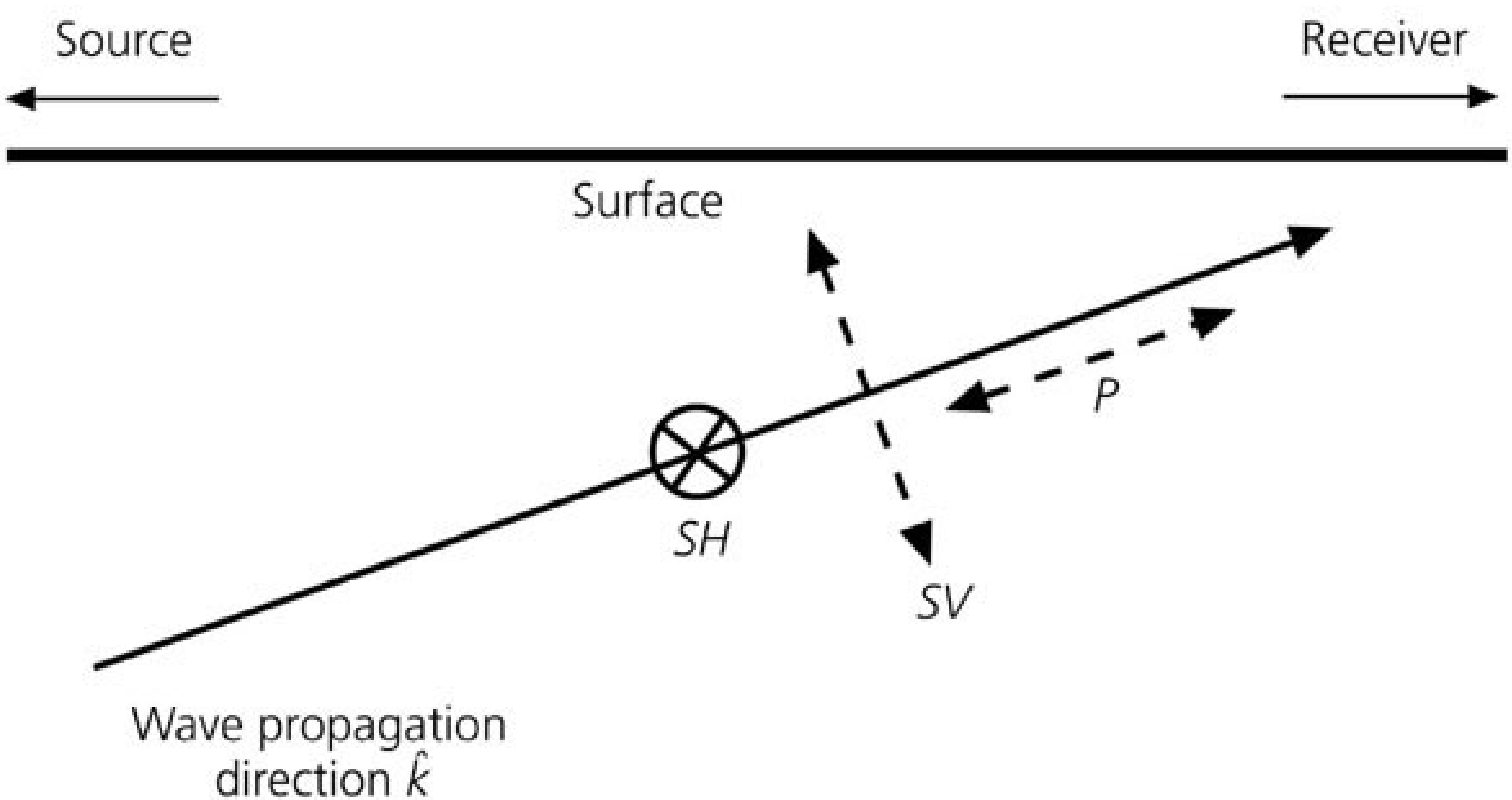


$$\frac{\sin i_1}{v_1} = \frac{\sin i_2}{v_2}$$

A special case is the **free surface** condition, where the surface tractions are zero.

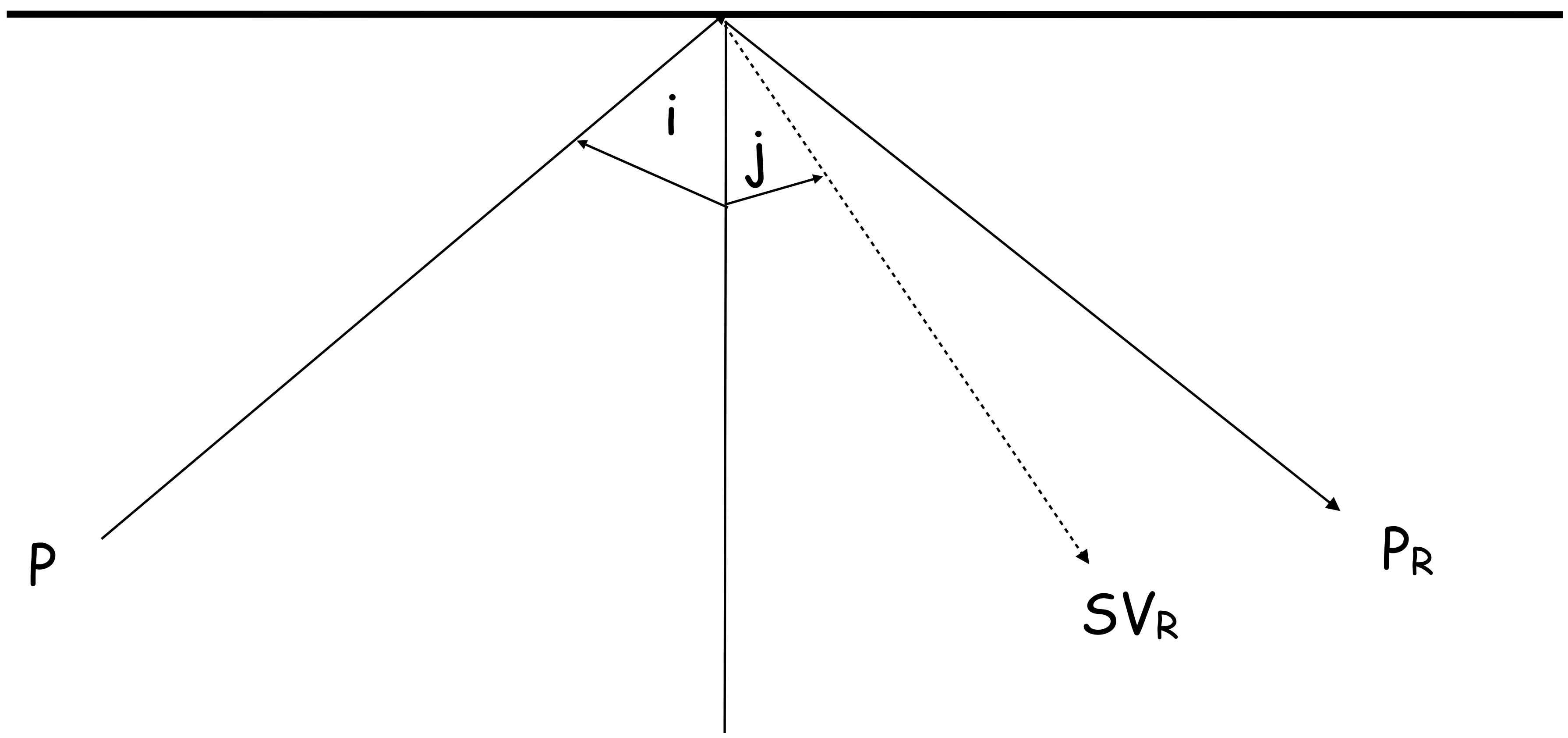
Free surface: P-SV-SH

Figure 2.4-4: Displacements for *P*, *SV*, and *SH*.



Case 1: Reflections at a free surface

A P wave is incident at the free surface ...



In general (also for an S incident wave) the reflected amplitudes can be described by the scattering matrix S

$$S = \begin{pmatrix} P_u P_d & S_u P_d \\ P_u S_d & S_u S_d \end{pmatrix}$$

Reflection and Transmission – Ansatz

How can we calculate in general the amount of energy that is transmitted or reflected at a material discontinuity?

We know that in homogeneous media the displacement can be described by the corresponding potentials

$$\mathbf{u} = \nabla \Phi + \nabla \times \mathbf{\Psi}$$

in 2-D (i.e. the wavefield does not depend on y coordinate) this gives:

$$u_x = \partial_x \Phi + 0 - \partial_z \Psi_y$$

$$u_y = 0 + \partial_z \Psi_x - \partial_x \Psi_z$$

$$u_z = \partial_z \Phi + \partial_x \Psi_y - 0$$

and an incoming P wave has the form (a_j indicate the direction cosines):

$$\Phi = A_0 \exp\{i[(k_j x_j - \omega t)]\} = A_0 \exp\left\{i\left[\frac{\omega}{\alpha}(a_j x_j - \alpha t)\right]\right\}$$

Free surface: apparent velocity

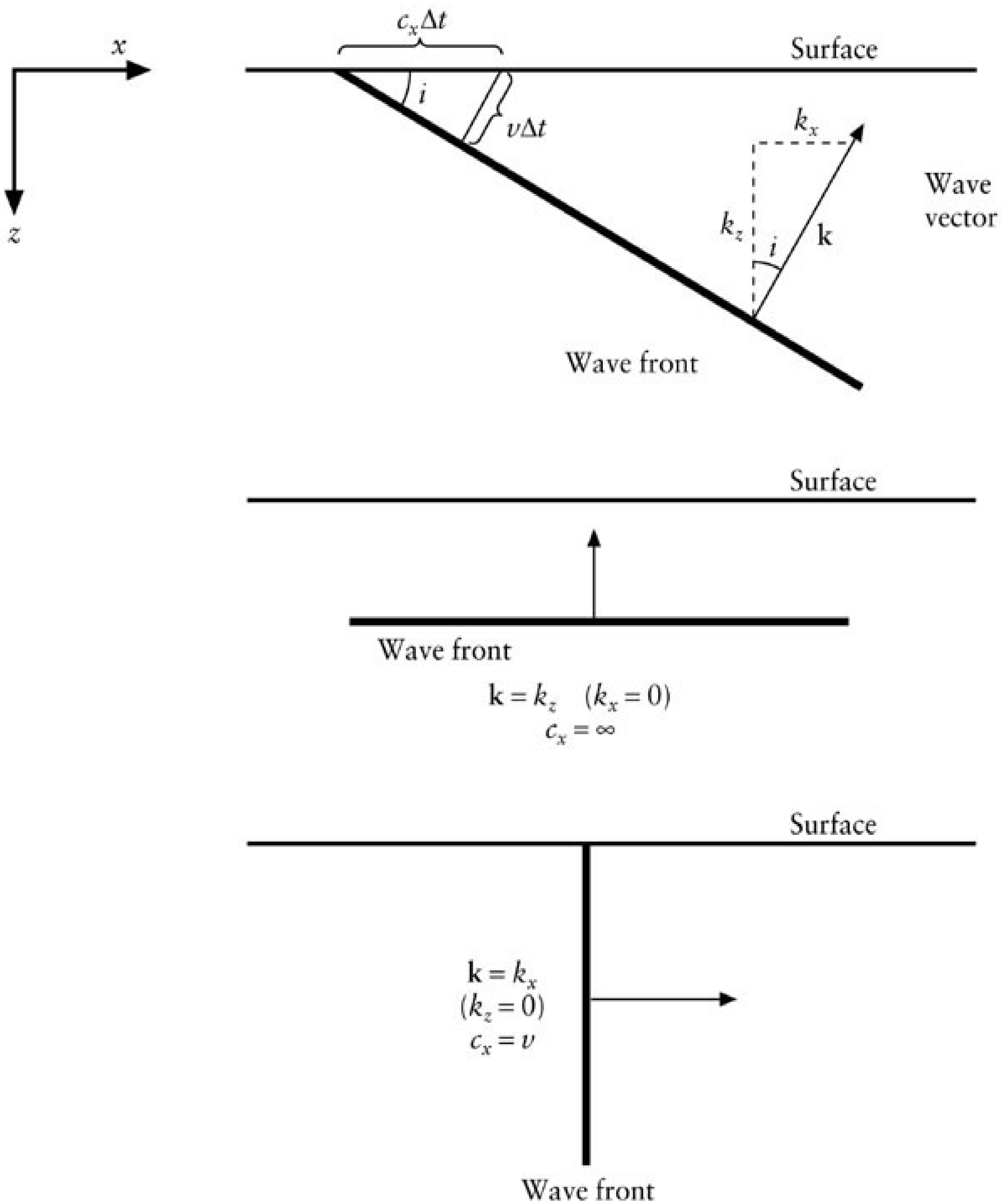
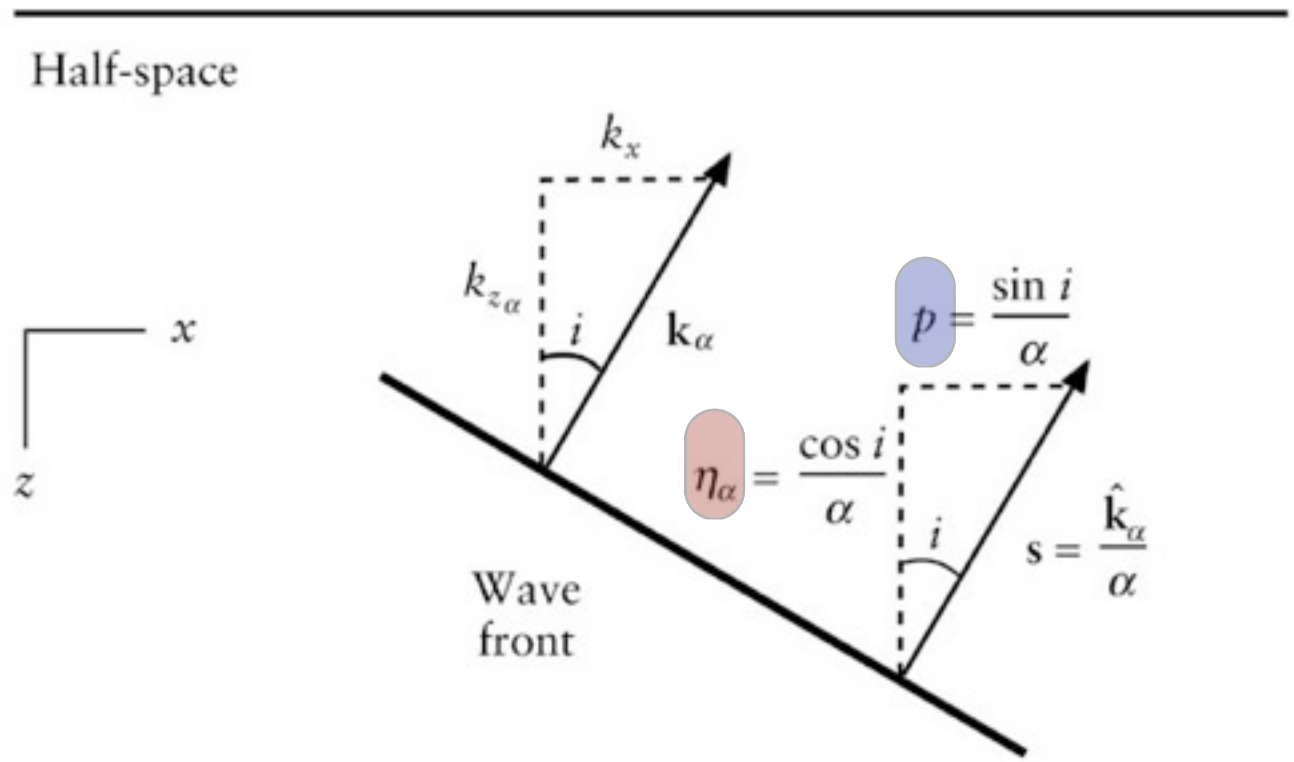


Figure 2.5-9: Definition of the slowness vector.



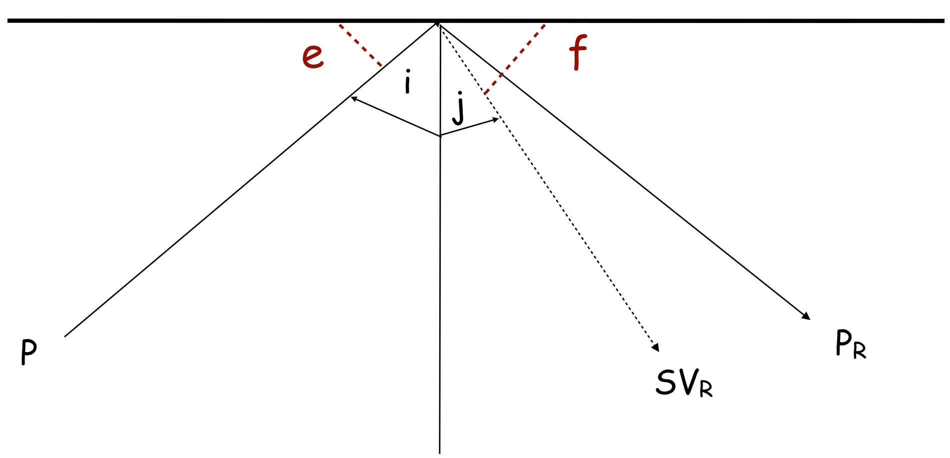
$$\Phi = A_0 e^{i(k_j x_j - \omega t)} =$$
$$= A_0 e^{i\omega(px - \eta z - t)} =$$
$$= A_0 e^{i\frac{\omega}{\alpha}(a_j x_j - \alpha t)} =$$
$$= A_0 e^{i(k_x x - k_x r_\alpha z - \omega t)}$$

$$\omega p = k_x ; \omega\eta = k_z = \omega \frac{\sqrt{1-\sin^2 i}}{\alpha} = \omega p \sqrt{\left(\frac{c_x}{\alpha}\right)^2 - 1} = \omega p r_\alpha$$

... here a_i are the components of the vector normal to the wavefront : a_i=(sin i, 0, -cos i)=(cos e, 0, -sin e), where e is the angle between surface and ray direction, so that for the **free surface**

$$\Phi = A_0 \exp[ik(x - z\tan e - ct)] + A\exp[ik(x + z\tan e - ct)]$$

$$\Psi = B\exp[ik'(x + z\tan f - c't)]$$



where

$$c = \frac{\alpha}{\cos e} = \frac{\alpha}{\sin i} \qquad c' = \frac{\beta}{\cos f}$$

$$k = \frac{\omega}{\alpha}\cos e = \frac{\omega}{\alpha}\sin i = \frac{\omega}{c} \qquad k' = \frac{\omega}{\beta}\cos f$$

what we know is that z=0 is a free surface, i.e.

$$\sigma_{xz}\big|_{z=0} = 0$$

$$\sigma_{zz}\big|_{z=0} = 0$$

... putting the equations for the potentials (displacements) into these equations leads to a relation between incident and reflected (transmitted) amplitudes

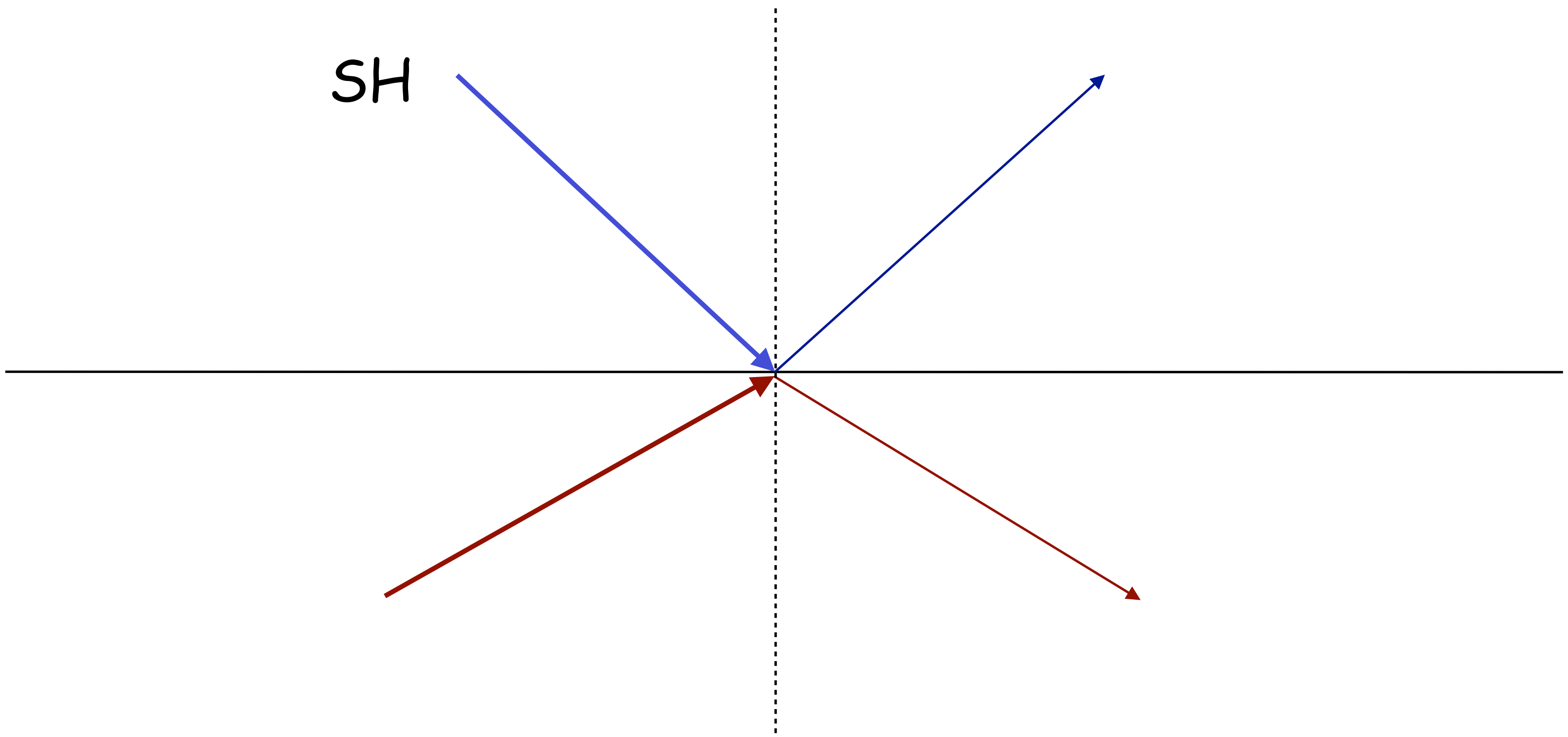
$$R_{PP} = \frac{A}{A_0} = \frac{4\tan e \tan f - (1 - \tan^2 f)^2}{4\tan e \tan f + (1 - \tan^2 f)^2}$$

$$R_{PS_V} = \frac{B}{A_0} = \frac{4\tan e - (1 - \tan^2 f)}{4\tan e \tan f + (1 - \tan^2 f)^2}$$

These are the reflection coefficients for a plane P wave incident on a free surface, and reflected P and SV waves.

Case 2: SH waves

For layered media SH waves are completely decoupled from P and SV waves

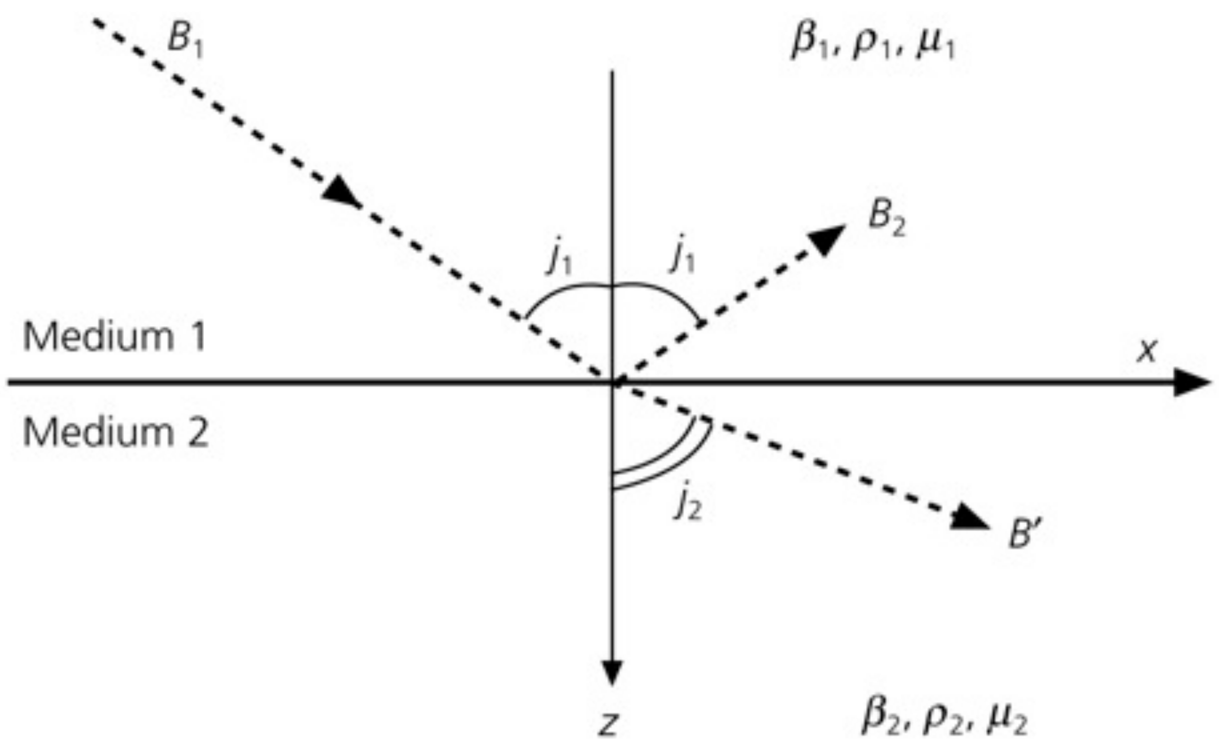


There is no conversion:
only SH waves are reflected or transmitted

$$S = \begin{pmatrix} S_u S_d & S_u S_d \\ S_u S_d & S_u S_d \end{pmatrix}$$

Case 2: SH waves

Figure 2.6-2: *SH* wave incident on a solid-solid boundary.



In medium 1: $u_y^-(x, z, t) = B_1 \exp(i(\omega t - k_x x - k_x r_{\beta_1} z)) + B_2 \exp(i(\omega t - k_x x + k_x r_{\beta_1} z))$

In medium 2: $u_y^+(x, z, t) = B' \exp(i(\omega t - k_x x - k_x r_{\beta_2} z))$

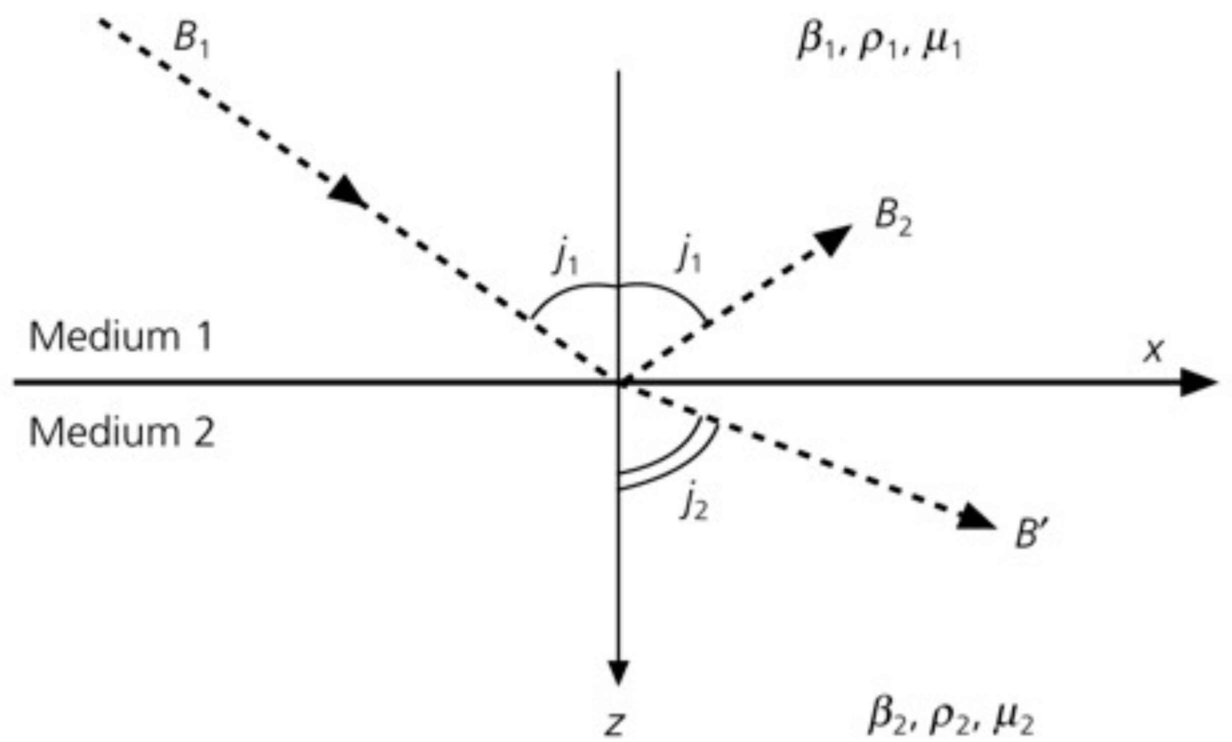
Boundary condition: continuity of displacement: $u_y^-(x, 0, t) = u_y^+(x, 0, t)$

$$(B_1 + B_2) \exp(i(\omega t - k_x x)) = B' \exp(i(\omega t - k_x x))$$

$$\boxed{B_1 + B_2 = B'}$$

Case 2: SH waves

Figure 2.6-2: *SH* wave incident on a solid-solid boundary.



Boundary condition: traction σ_{yz} is continuous: $\sigma_{yz} = 2\mu e_{yz} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \mu\left(\frac{\partial u_y}{\partial z}\right)$.

(in this case, u_x and u_z are zero, so $\sigma_{xz} = \sigma_{zz} = 0$):

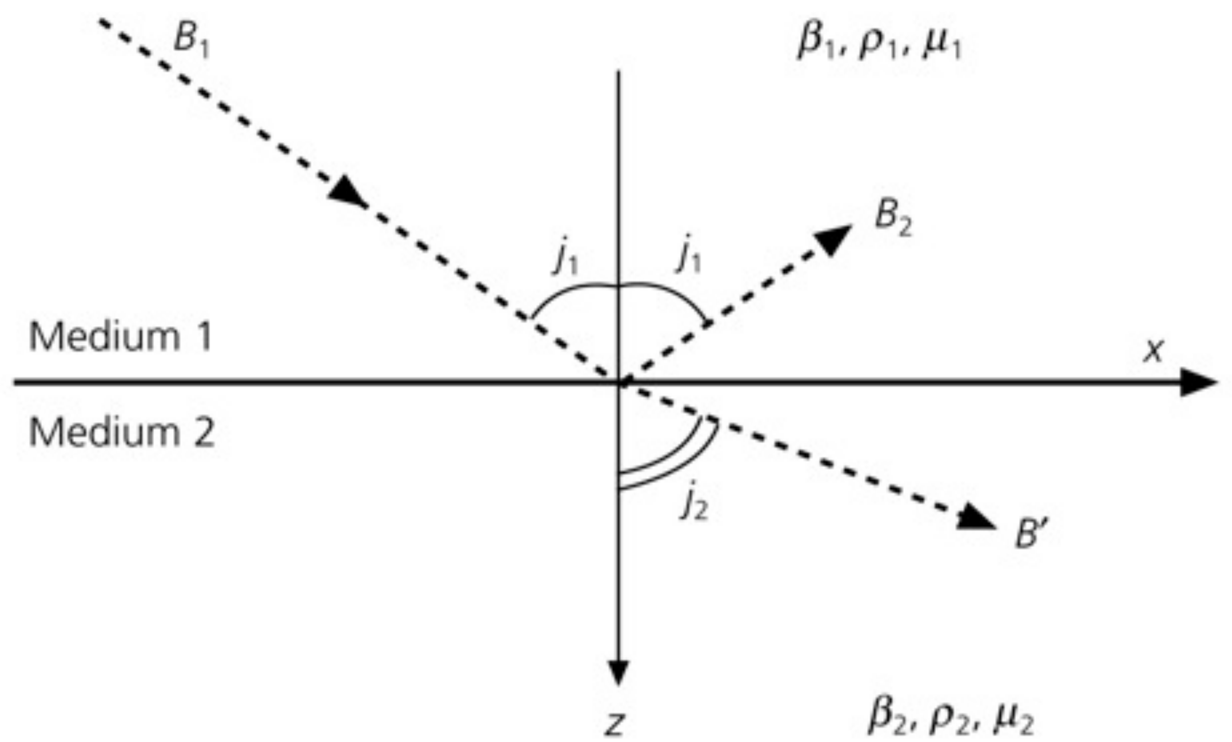
$$\sigma_{yz}^{-}(x, 0, t) = \sigma_{yz}^{+}(x, 0, t)$$

$$\mu_1 i k_x r_{\beta_1}(B_2 - B_1)\exp(i(\omega t - k_x x)) = -\mu_2 i k_x r_{\beta_2} B' \exp(i(\omega t - k_x x))$$

$$\boxed{(B_1 - B_2) = B'(\mu_2 r_{\beta_2})/(\mu_1 r_{\beta_1})}$$

Case 2: SH waves

Figure 2.6-2: *SH* wave incident on a solid-solid boundary.

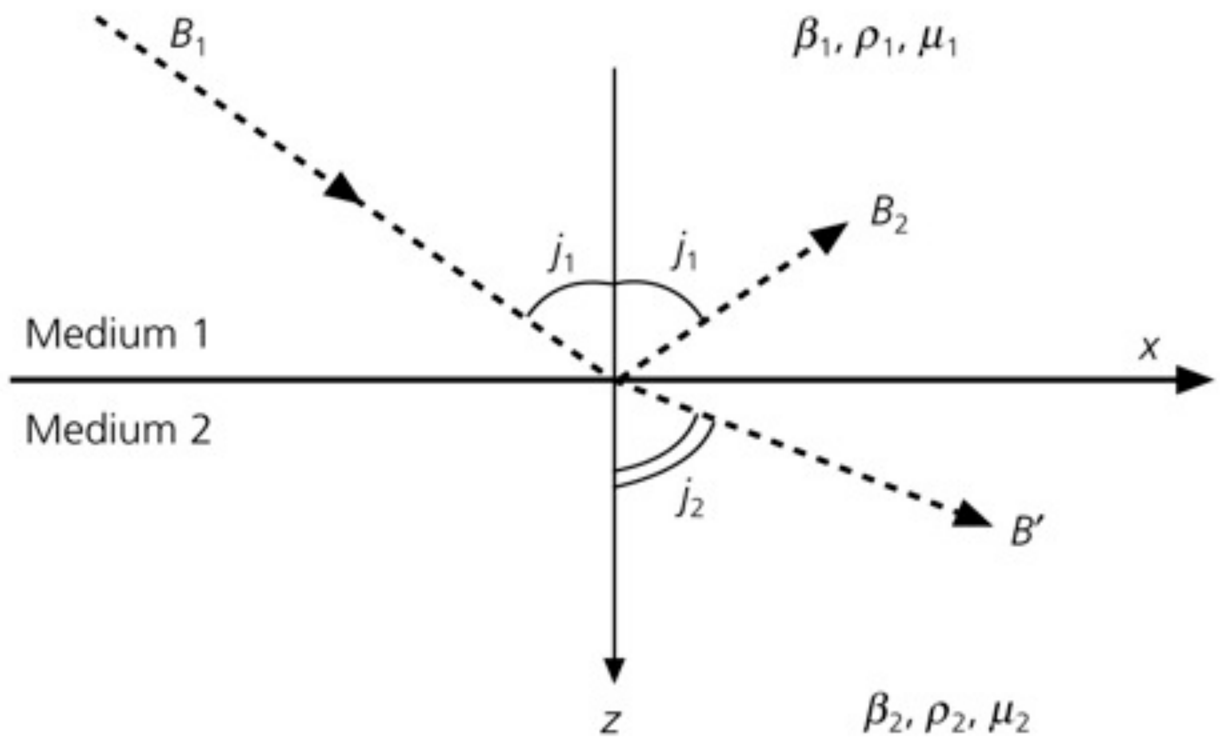


$$B_1 + B_2 = B' \qquad (B_1 - B_2) = B'(\mu_2 r_{\beta_2})/(\mu_1 r_{\beta_1})$$

$$T_{12} = \frac{B'}{B_1} = \frac{2\mu_1 r_{\beta_1}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}} \qquad R_{12} = \frac{B_2}{B_1} = \frac{\mu_1 r_{\beta_1} - \mu_2 r_{\beta_2}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}}$$

Case 2: SH waves

Figure 2.6-2: *SH* wave incident on a solid-solid boundary.



Using $r_{\beta_i} = c_x \cos$.

$$T_{12} = \frac{2\rho_1\beta_1 \cos j_1}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2} \qquad R_{12} = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

$$R_{12} = -R_{21} \qquad T_{12} + T_{21} = 2 \qquad 1 + R_{12} = T_{12}$$

$R_{12} = 1$ at surface and CMB.

At vertical incidence ($j_1 = j_2 = 0$): $T_{12} = \dfrac{2\rho_1\beta_1}{\rho_1\beta_1 + \rho_2\beta_2}$ $\qquad R_{12} = \dfrac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 + \rho_2\beta_2}$

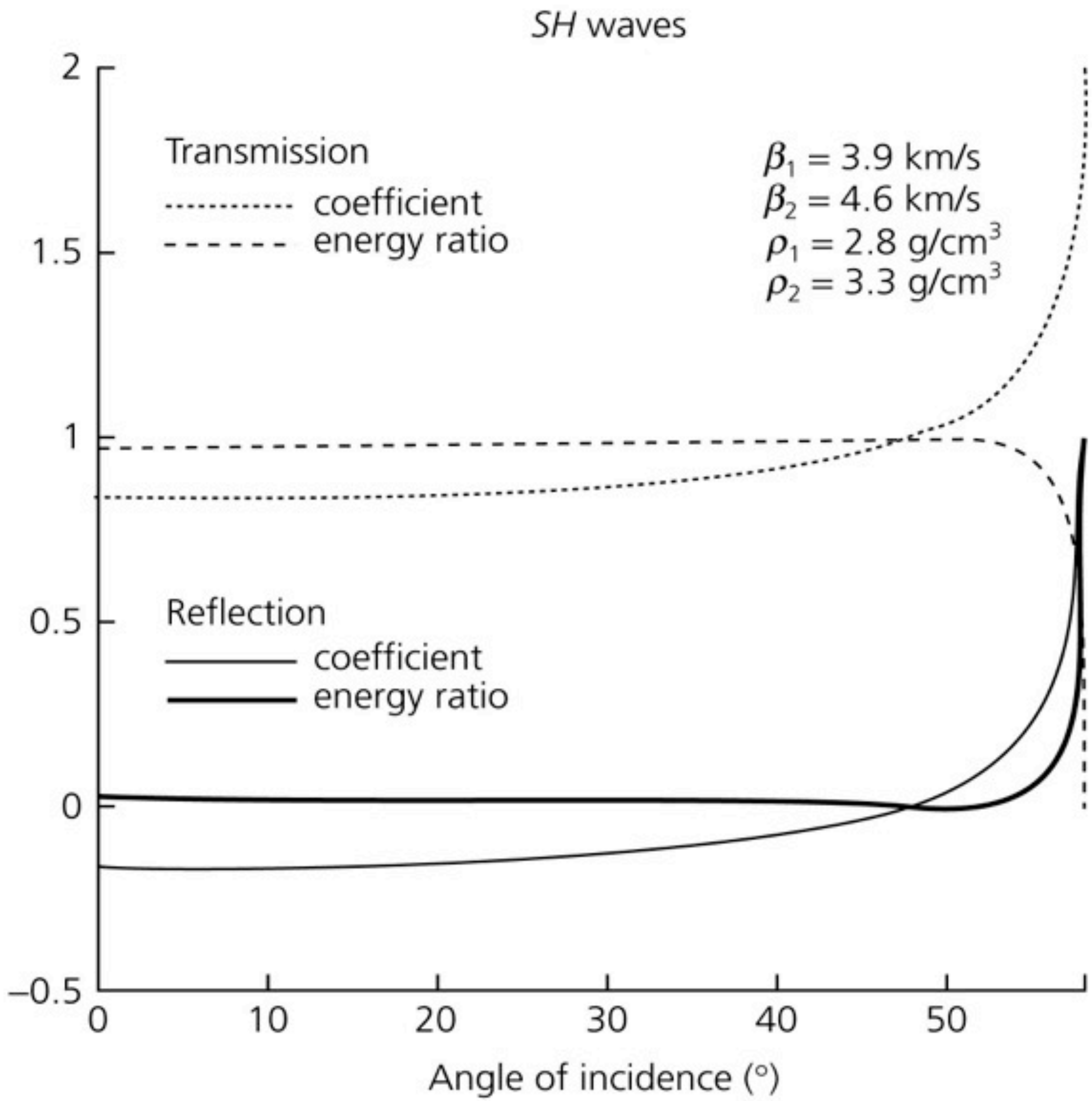
Case 2: SH waves

$$\frac{\dot{\mathbf{E}}_R}{\dot{\mathbf{E}}_I} = R_{12}^2$$

$$\frac{\dot{\mathbf{E}}_T}{\dot{\mathbf{E}}_I} = T_{12}^2 \; \frac{\rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1}$$

For example,
if $R_{12} = 0.1$,
then the energy ratio is
$\dot{\mathbf{E}}_R / \dot{\mathbf{E}}_I = 0.01$.

Figure 2.6-4: Reflection and transmission coefficients for incident *SH* waves.



Case 2: SH waves

At the critical angle, $c_x = \dfrac{\beta_1}{\sin j_1} = \dfrac{\beta_2}{\sin j_2} = \dfrac{\beta_2}{1} = \beta_2$

For angles i_1 that are GREATER than the critical angle, we have the unusual situation that

$c_x = \dfrac{\beta_1}{\sin j_1} < \beta_2$ If $c_x < \beta_2$, then $r_{\beta_2} = (c_x^2/\beta_2^2 - 1)^{1/2}$ becomes an imaginary number!!

This means that the transmitted wave $u_y(x, z, t) = B' \exp(i(\omega t - k_x x - k_x r_{\beta_2} z))$ has a *real* exponent!

Pick the negative sign of the square root of -1 (Why?) to define $r_{\beta_2} = -i r^*_{\beta_2}$ $r^*_{\beta_2} = (1 - c_x^2/\beta_2^2)^{1/2}$

so that the z term in the displacement, $\exp(-ik_x r_{\beta_2} z) = \exp(-k_x r^*_{\beta_2} z)$, decays exponentially away from the interface in medium 2 as $z \to \infty$.

The transmitted wave becomes an *evanescent* or *inhomogeneous* wave "trapped" near the interface.

Case 2: SH waves

$$\exp(-ik_x r_{\beta_2} z) = \exp(-k_x r^*_{\beta_2} z)$$

$$R_{12} = \frac{\mu_1 r_{\beta_1} + i\mu_2 r^*_{\beta_2}}{\mu_1 r_{\beta_1} - i\mu_2 r^*_{\beta_2}}$$

This a complex number divided by its conjugate, so the magnitude of the reflection coefficient is one, but there is a phase shift of 2ε:

$$R_{12} = e^{i2\varepsilon} \qquad \varepsilon = \tan^{-1}\frac{\mu_2 r^*_{\beta_2}}{\mu_1 r_{\beta_1}}$$

At critical incidence,

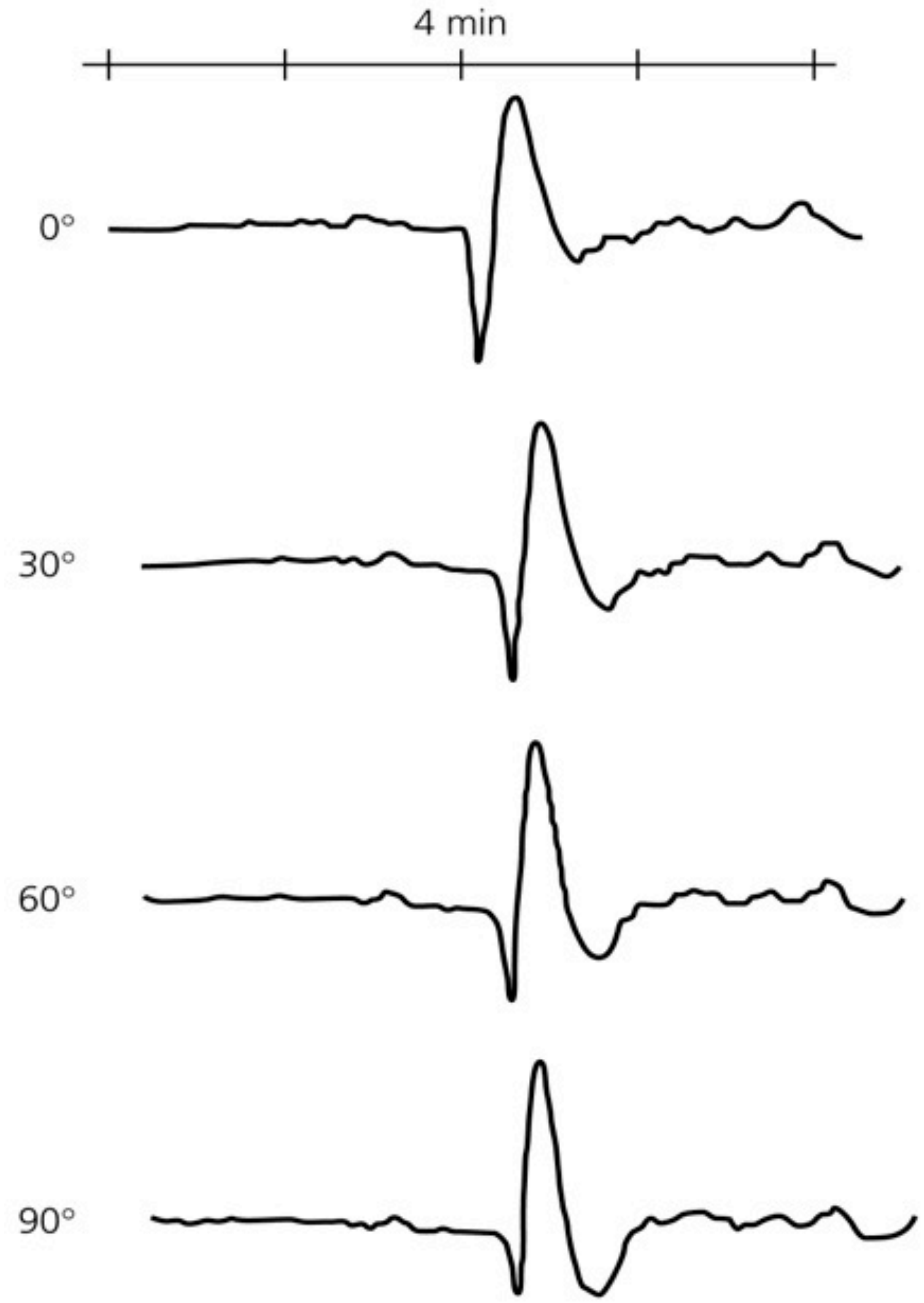
$c_x = \beta_2$, so $r^*_{\beta_2} = 0$ and $\varepsilon = 0°$

As the angle of incidence increases beyond critical, ε increases.

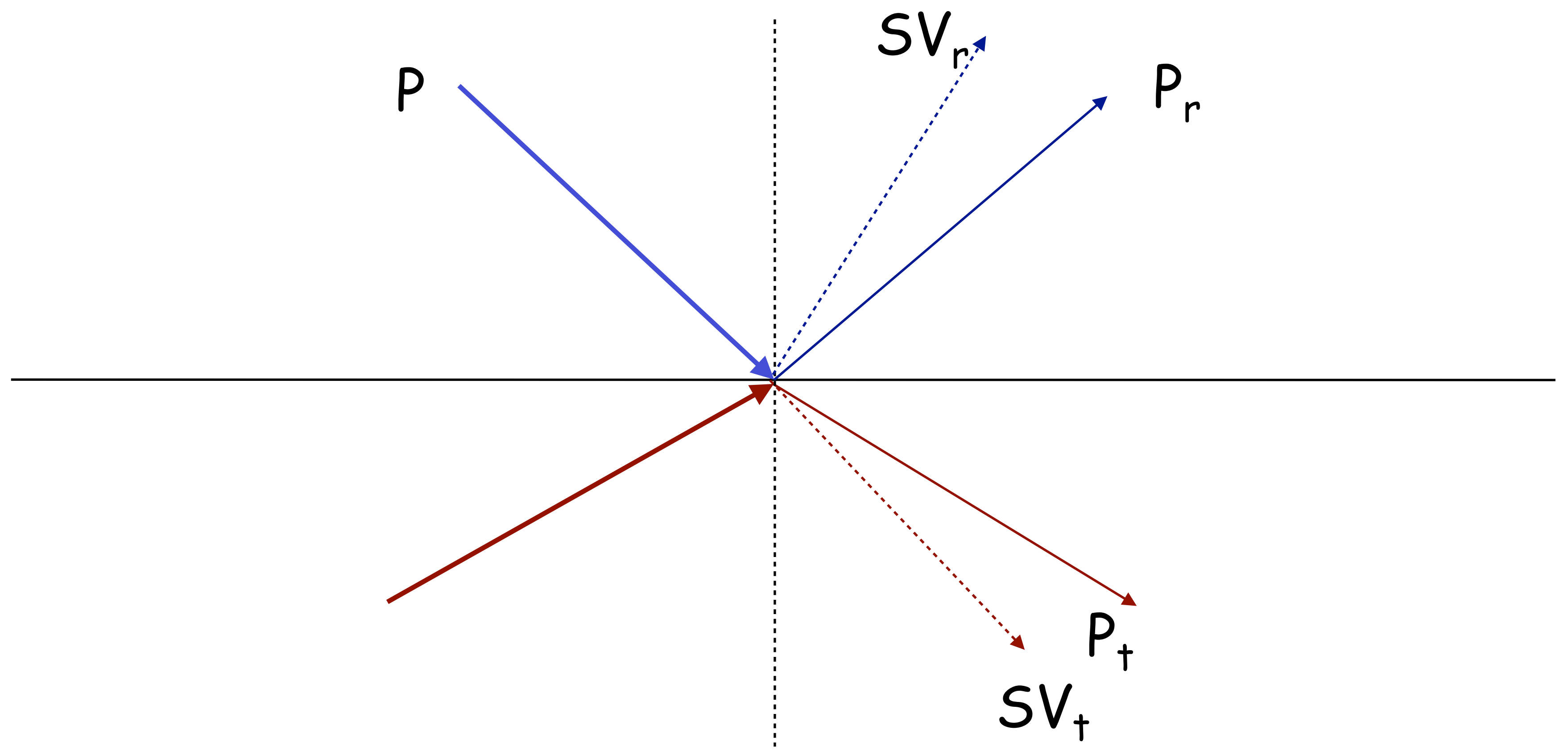
At grazing incidence, $j_1 = 90°$, we have

$c_x = \beta_1$, $r_{\beta_1} = 0$ and $\varepsilon = 90°$

Figure 2.6-5: Effect of phase shifts on a seismic waveform.

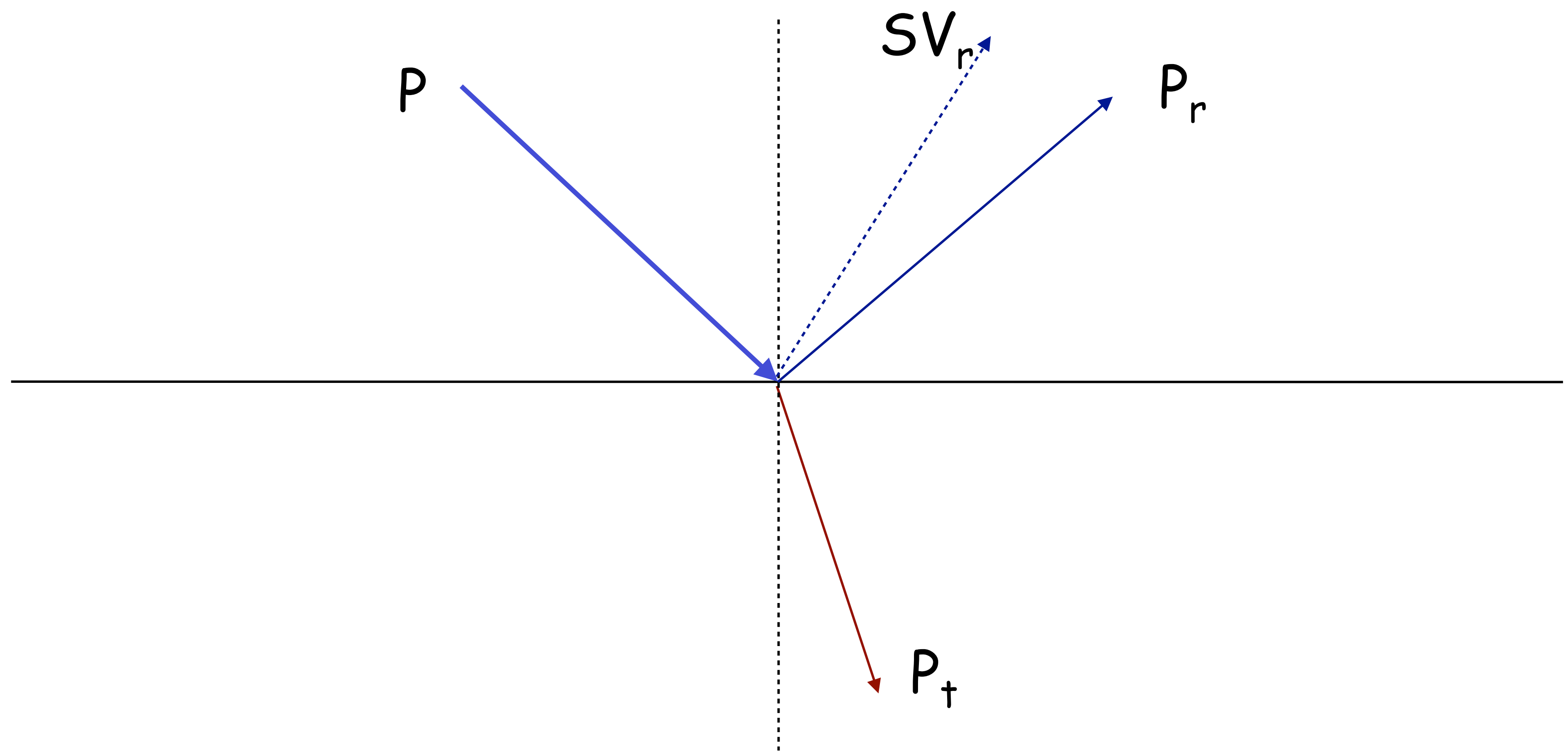


Case 3: Solid-solid interface



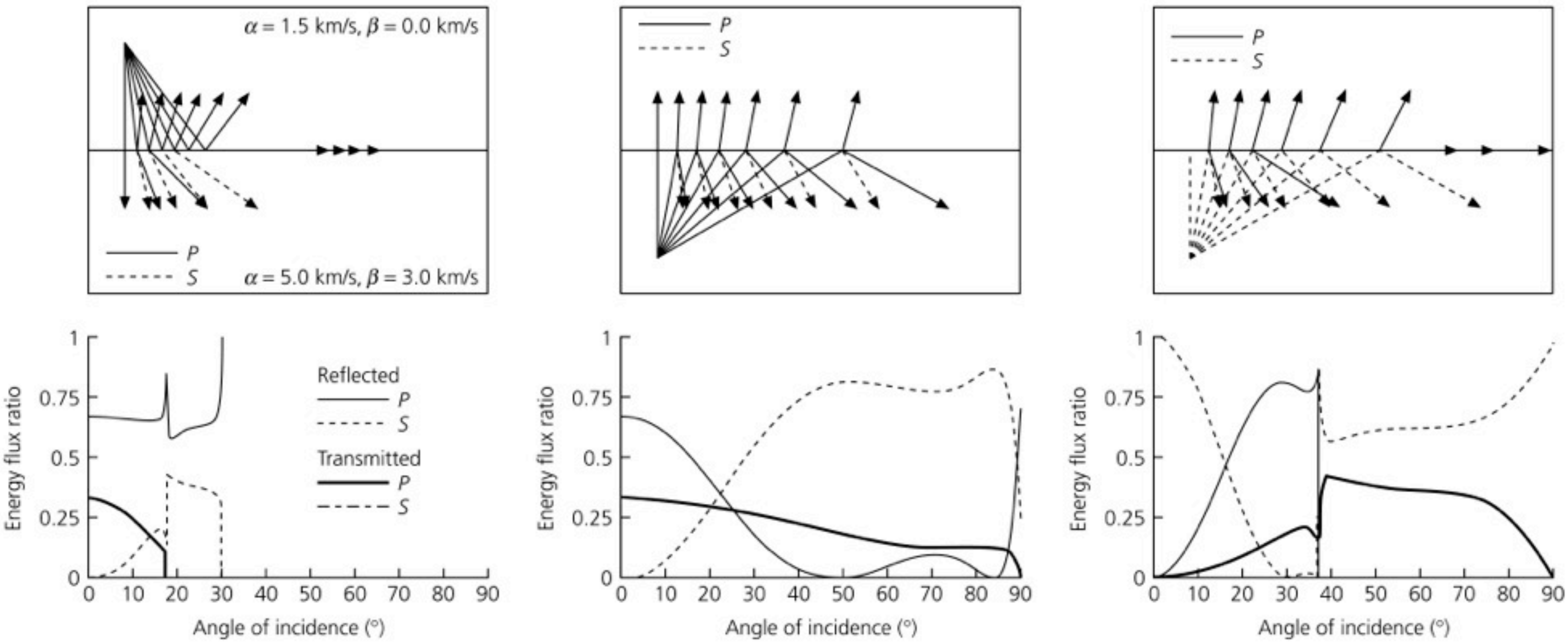
To account for all possible reflections and transmissions we need 16 coefficients, described by a 4x4 scattering matrix.

Case 4: Solid-Fluid interface

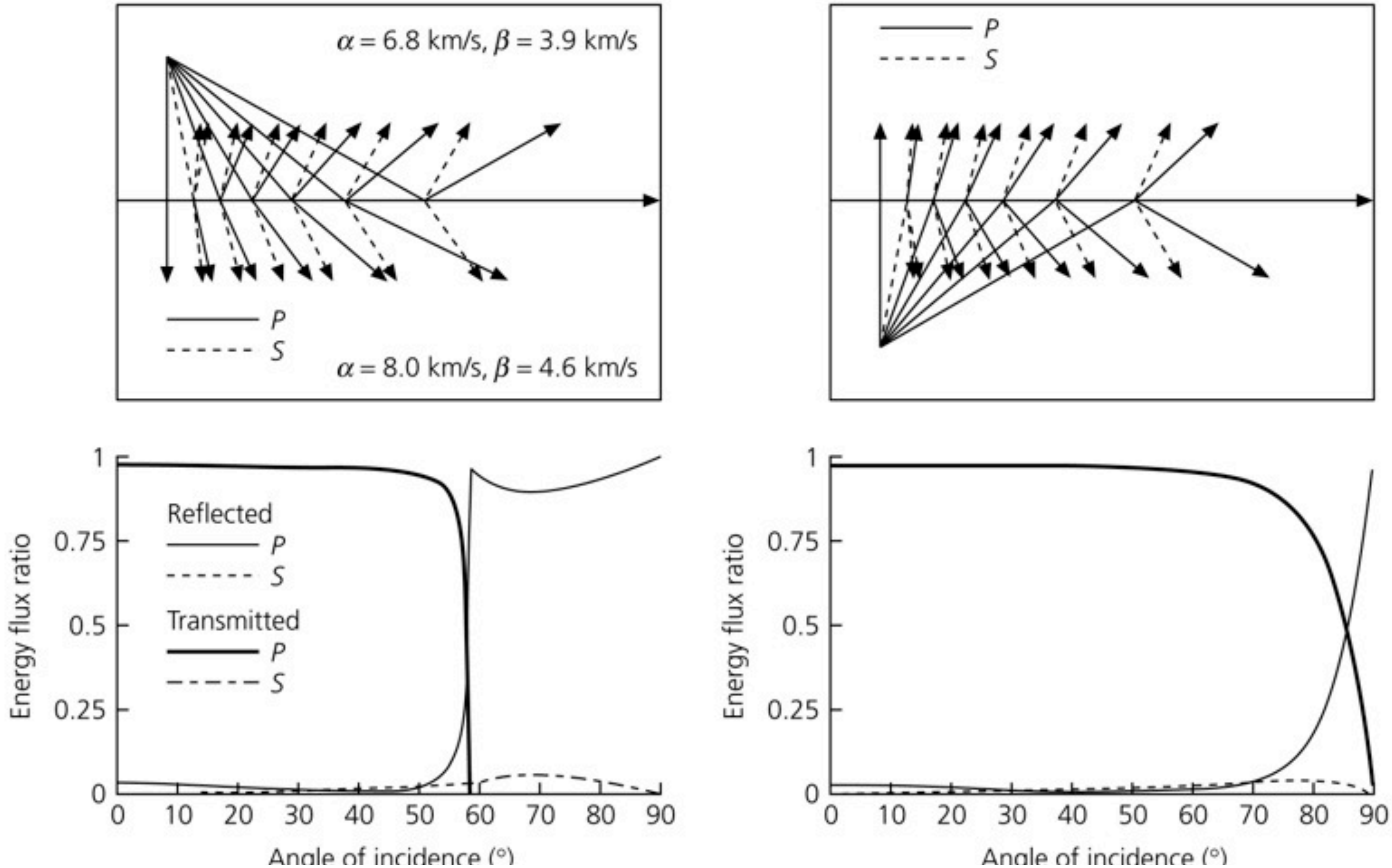


At a solid-fluid interface there is no conversion to SV in the lower medium.

Ocean-Crust interface



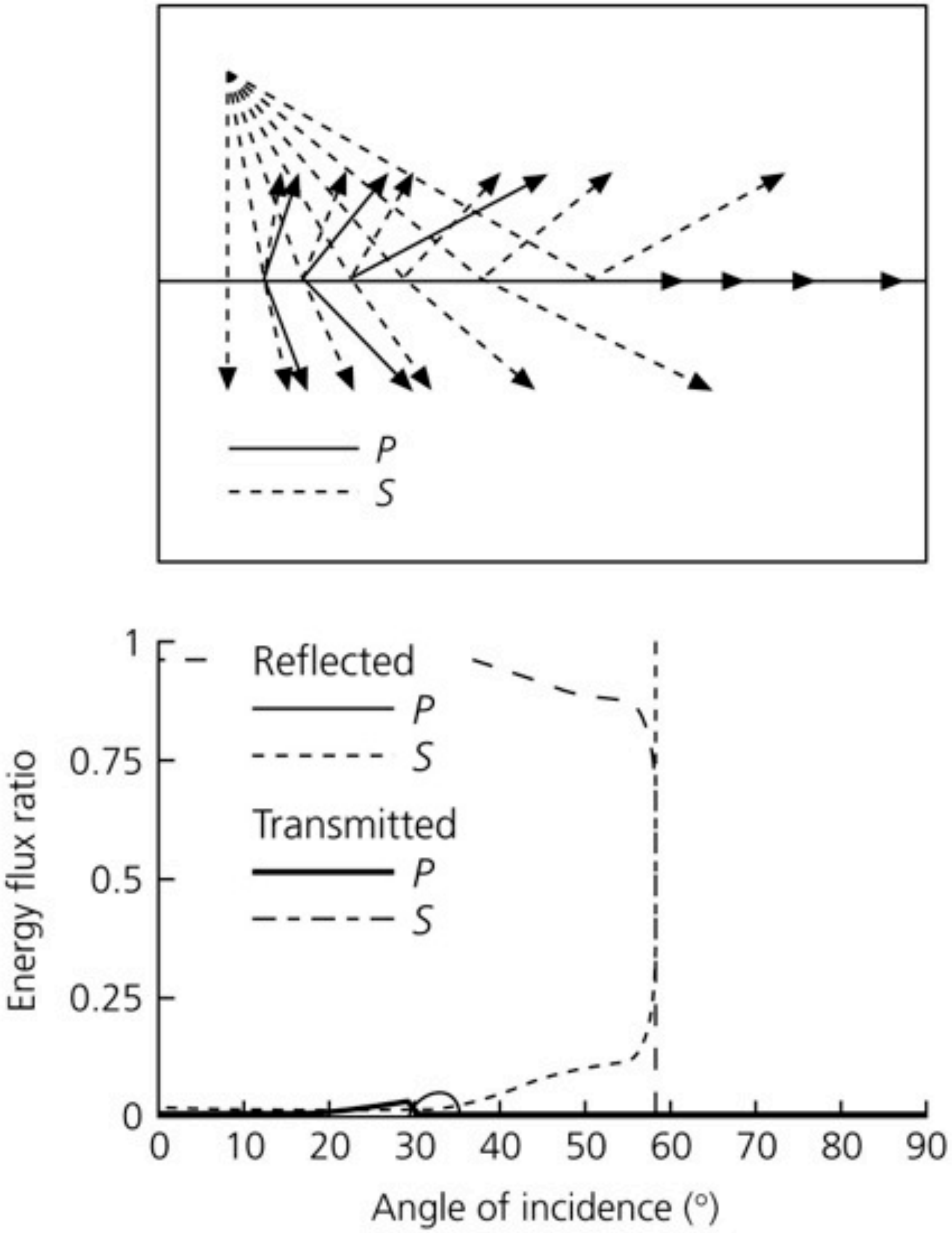
Crust-Mantle interface

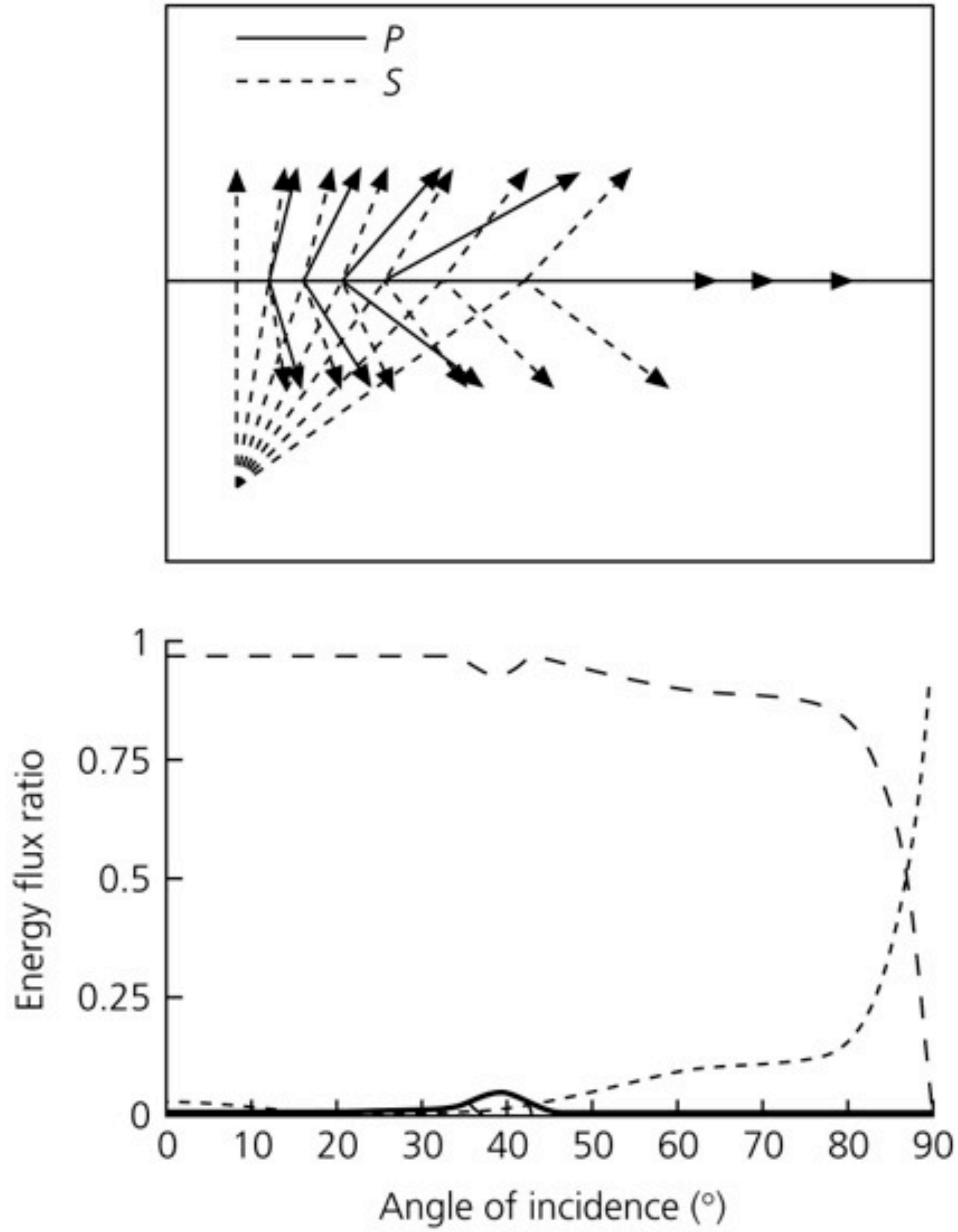


For a P wave vertically incident from above, the impedances are $\rho_1\alpha_1 = 19.0$ and $\rho_2\alpha_2 = 26.4$

The reflection and transmission coefficients are $R_{12} = -0.16$ and $T_{12} = 0.84$

The energy flux ratios are $\frac{\dot{\mathbf{E}}_R}{\dot{\mathbf{E}}_I} = R_{12}^2 = 0.03$ $\quad \frac{\dot{\mathbf{E}}_T}{\dot{\mathbf{E}}_I} = T_{12}^2 \frac{\rho_2\alpha_2}{\rho_1\alpha_1} = 0.97$


P
S
Reflected
P
S
Transmitted
P
S
Energy flux ratio
Angle of incidence (°)


P
S
Energy flux ratio
Angle of incidence (°)

Body Waves and Ray Theory

Ray theory: basic principles

Wavefronts, Huygens principle, Fermat's principle, Snell's Law

Rays in layered media

Travel times in a layered Earth, continuous depth models,
Travel time diagrams, shadow zones,

Travel times in a spherical Earth

Seismic phases in the Earth, nomenclature, travel-time curves for teleseismic phases

Basic principles

- **Ray definition**

  Rays are defined as the normals to the wavefront and thus point in the direction of propagation.

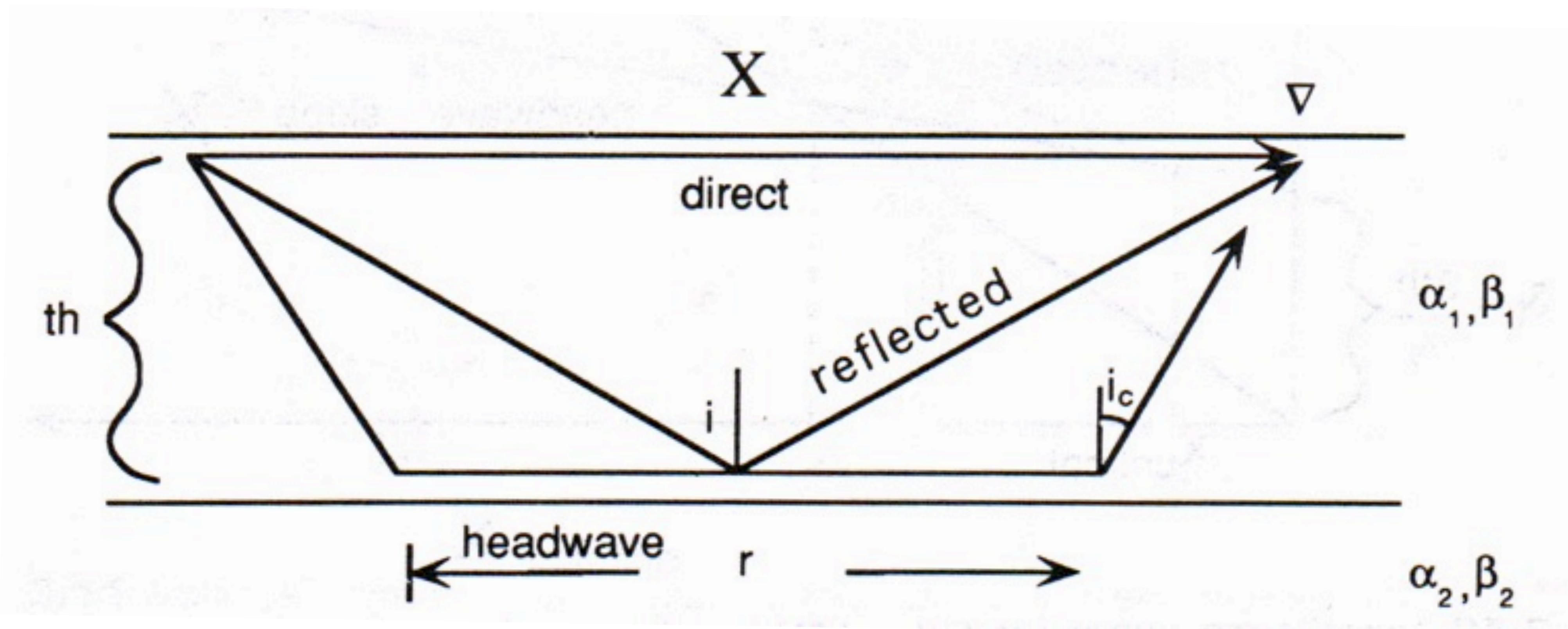
- **Rays in smoothly varying or not too complex media**

  Rays corresponding to P or S waves behave much as light does in materials with varying index of refraction: rays bend, focus, defocus, get diffracted, birefringence et.

- **Ray theory is a high-frequency approximation**

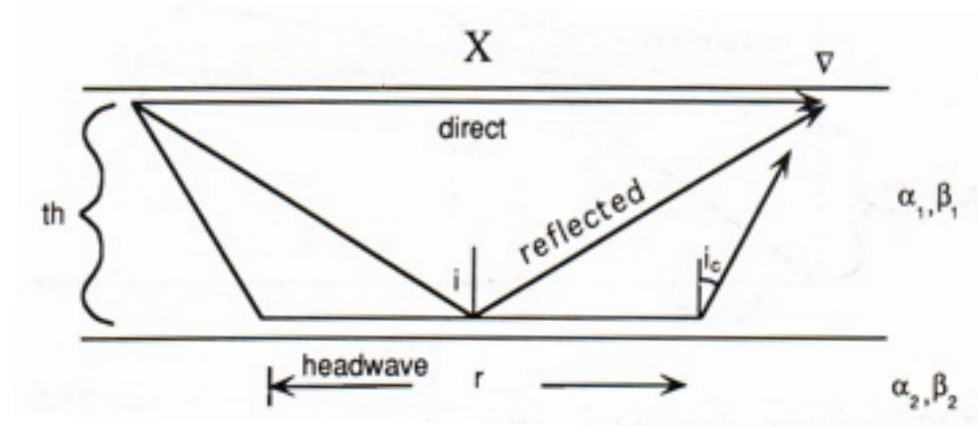
  This statement is the same as saying that the medium (apart from sharp discontinuities, which can be handled) must vary smoothly compared to the wavelength.

Much information can be learned by analysing recorded seismic signals in terms of layered structured (e.g. crust and Moho).
We need to be able to predict the arrival times of reflected and refracted signals …



Travel Times in Layered Media

Let us calculate the arrival times for reflected and refracted waves as a function of layer depth, h, and velocities α_i, i denoting the i-th layer:



We find that the travel time for the reflection is:

$$T_{refl} = \frac{2h}{\alpha_1 \cos i} = \frac{2\sqrt{h^2 + X^2/4}}{\alpha_1}$$

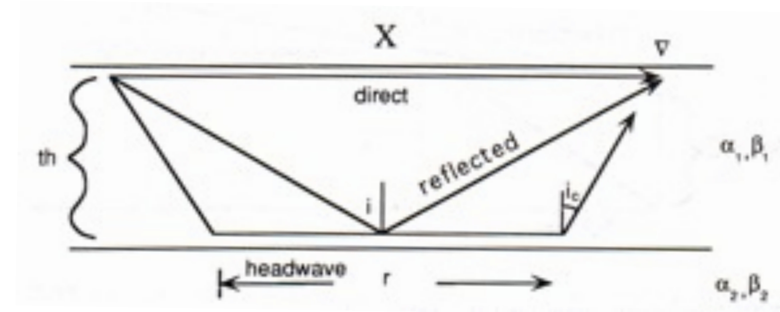
And for the the refraction:

$$T_{refr} = \frac{2h}{\alpha_1 \cos i_c} + \frac{r}{\alpha_2}$$

$$r = X - 2h\tan i_c$$

where i_c is the critical angle:

$$\frac{\sin(i_1)}{\alpha_1} = \frac{\sin(r_2)}{\alpha_2} \Rightarrow i_c = \arcsin\left(\frac{\alpha_1}{\alpha_2}\right)$$



Thus the refracted wave arrival is

$$T_{refr} = \frac{2h}{\alpha_1 \cos i_c} + \frac{1}{\alpha_2}\left(X - \frac{2h\alpha_1}{\alpha_2 \cos i_c}\right)$$

where we have made use of Snell's Law.

We can rewrite this using

$$\frac{1}{\alpha_2} = \frac{\sin i_c}{\alpha_1} = p$$

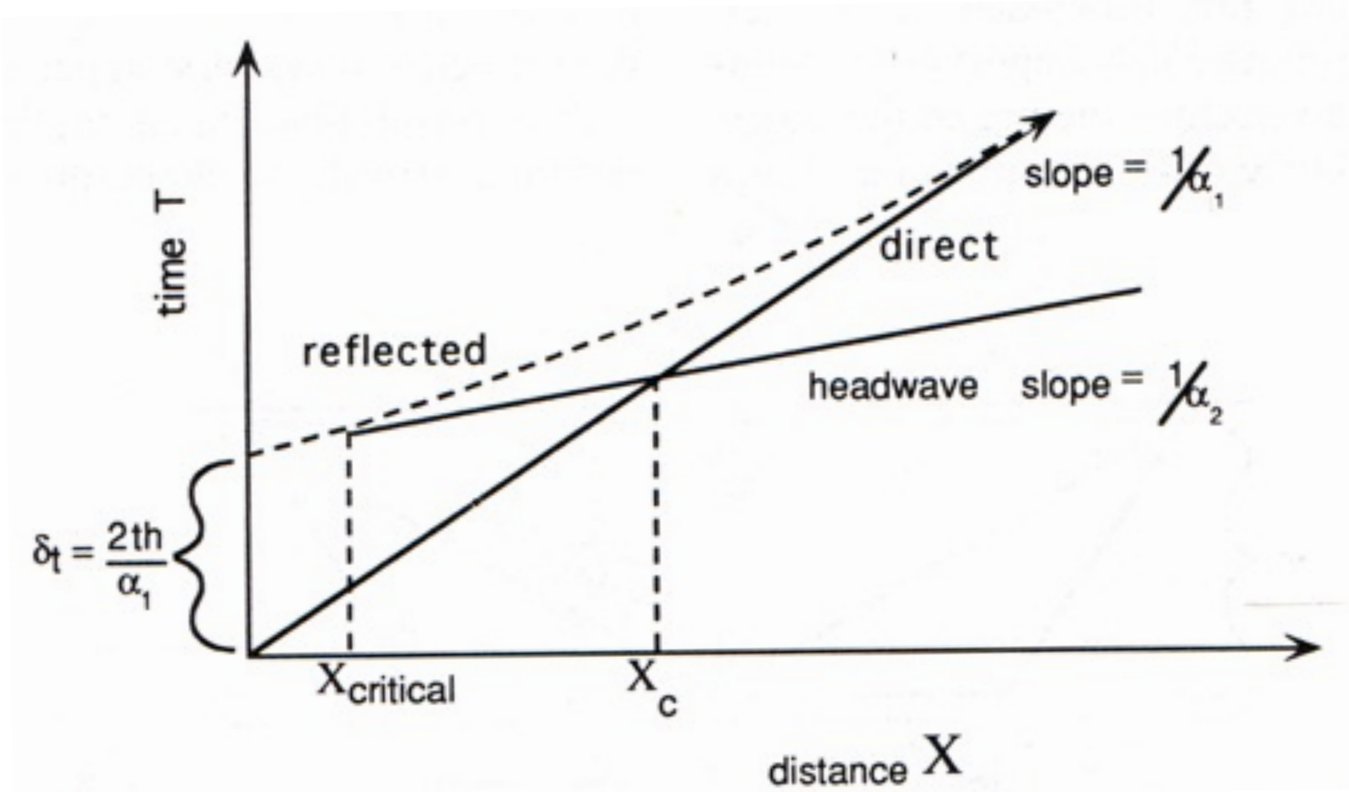
$$\cos i_c = (1-\sin^2 i_c)^{1/2} = (1-p^2\alpha_1^2)^{1/2} = \alpha_1\left(\frac{1}{\alpha_1^2} - p^2\right)^{1/2} = \alpha_1 \eta_1$$

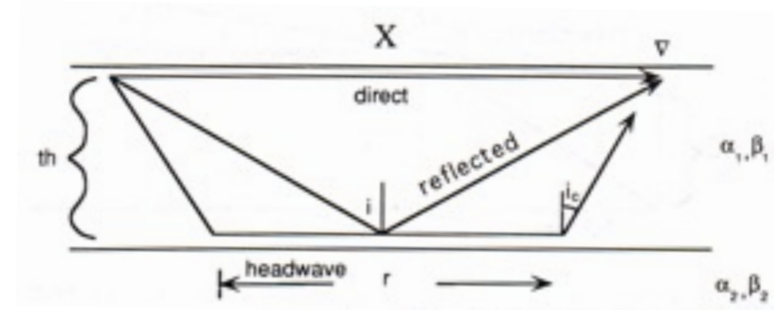
to obtain

$$T_{refr} = Xp + 2h\eta_1$$

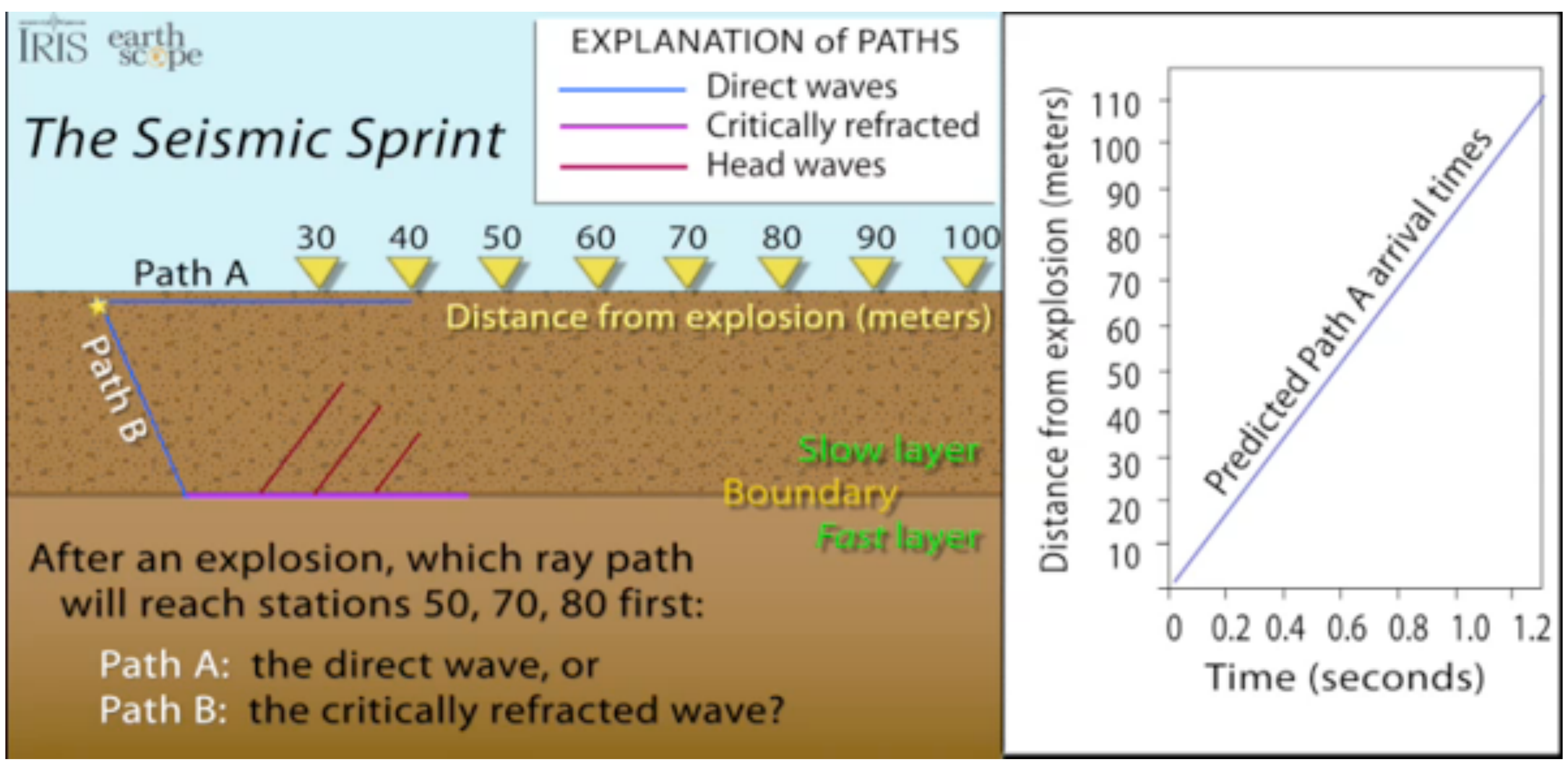
Which is very useful as we have separated the result into a vertical and horizontal term.

Travel time curves





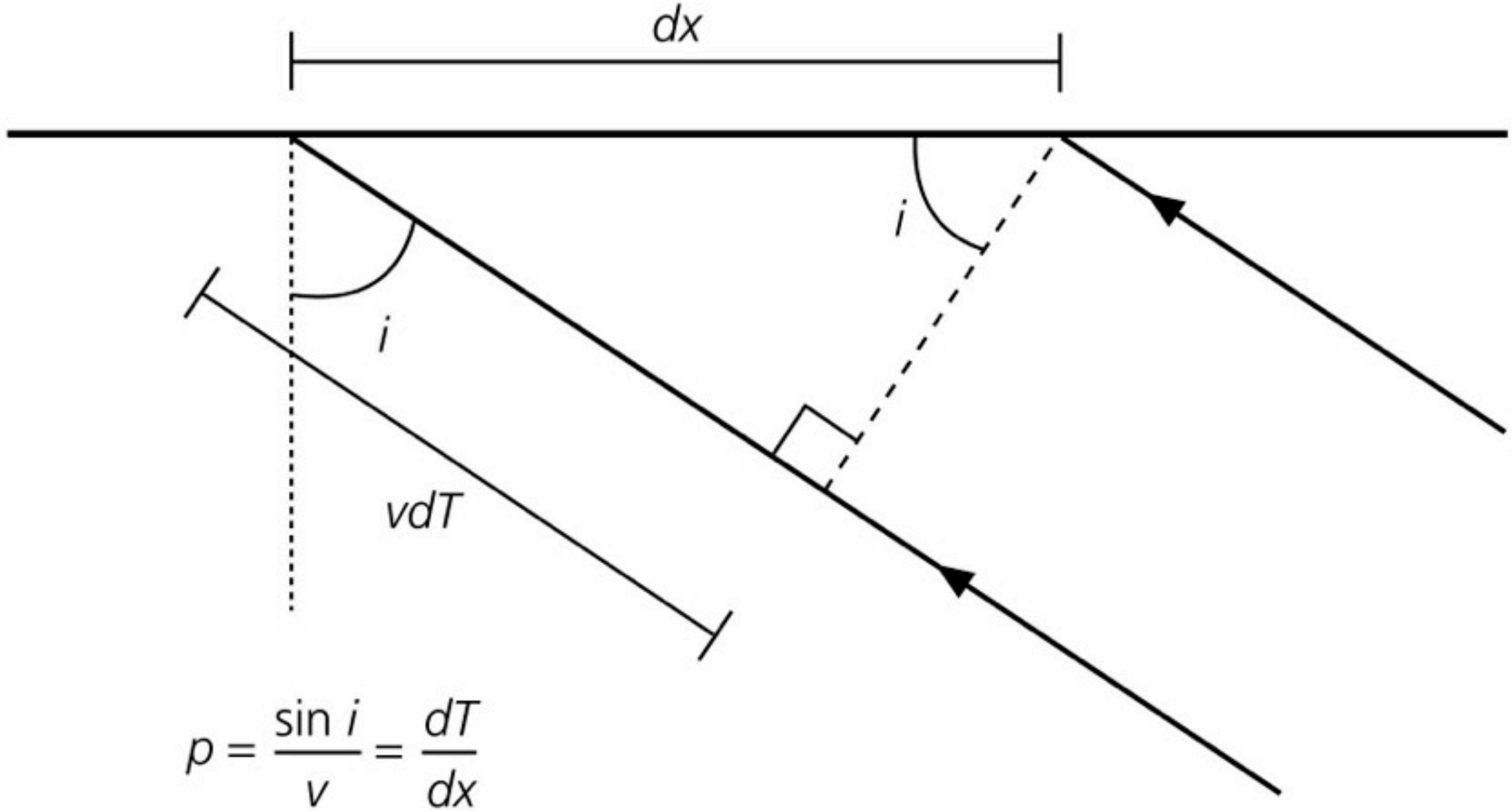
What can we determine if we have recorded the following travel time curves?



http://www.iris.edu/hq/programs/education_and_outreach/animations/13

Ray parameter

Figure 3.3-2: Cartoon demonstration of ray parameter.

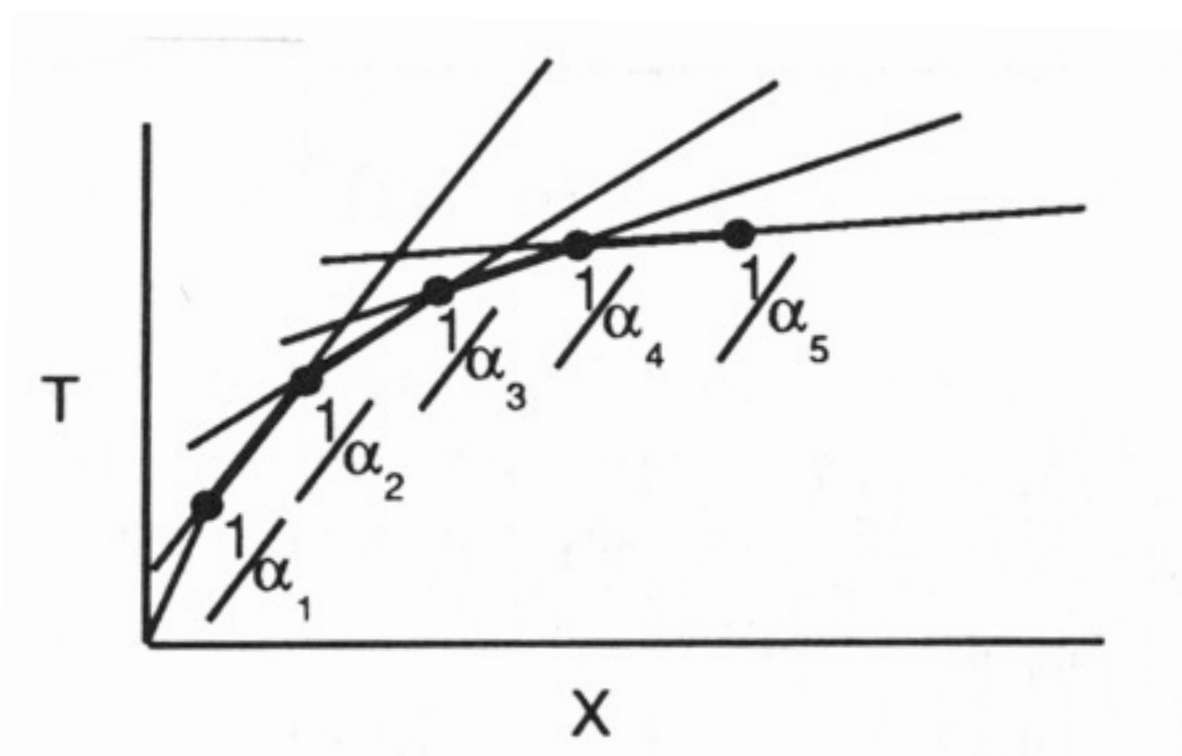


$$p_d = \frac{\sin(90^\circ)}{\alpha_1} = \frac{1}{\alpha_1} \quad p_r = \frac{\sin(i_c)}{\alpha_1} = \frac{1}{\alpha_2} \quad p_R = \frac{\sin(i)}{\alpha_1}$$

Generalization to many layers

The previous relation for the travel times easily generalizes to many layers:

$$T_{refr} = Xp + \sum_{i=1}^{n} 2h_i\eta_i$$



Travel time curve for a finely layered Earth. The first arrival is comprised of short segments of the head wave curves for each layer.

This naturally generalizes to infinite layers i.e. to a continuous depth model.

Total horizontal distance

$$x(p) = 2\sum_{j=0}^{n} x_j = 2\sum_{j=0}^{n} h_j \tan i_j$$

in a total time

$$T(p) = 2\sum_{j=0}^{n} \Delta T_j = 2\sum_{j=0}^{n} \frac{h_j}{\mathrm{v}_j \cos i_j}$$

For multiple layers,
multiple hyperbolas:

$$T(x)_{n+1}^2 = x^2/\bar{\mathrm{V}}_n^2 + t_n^2$$

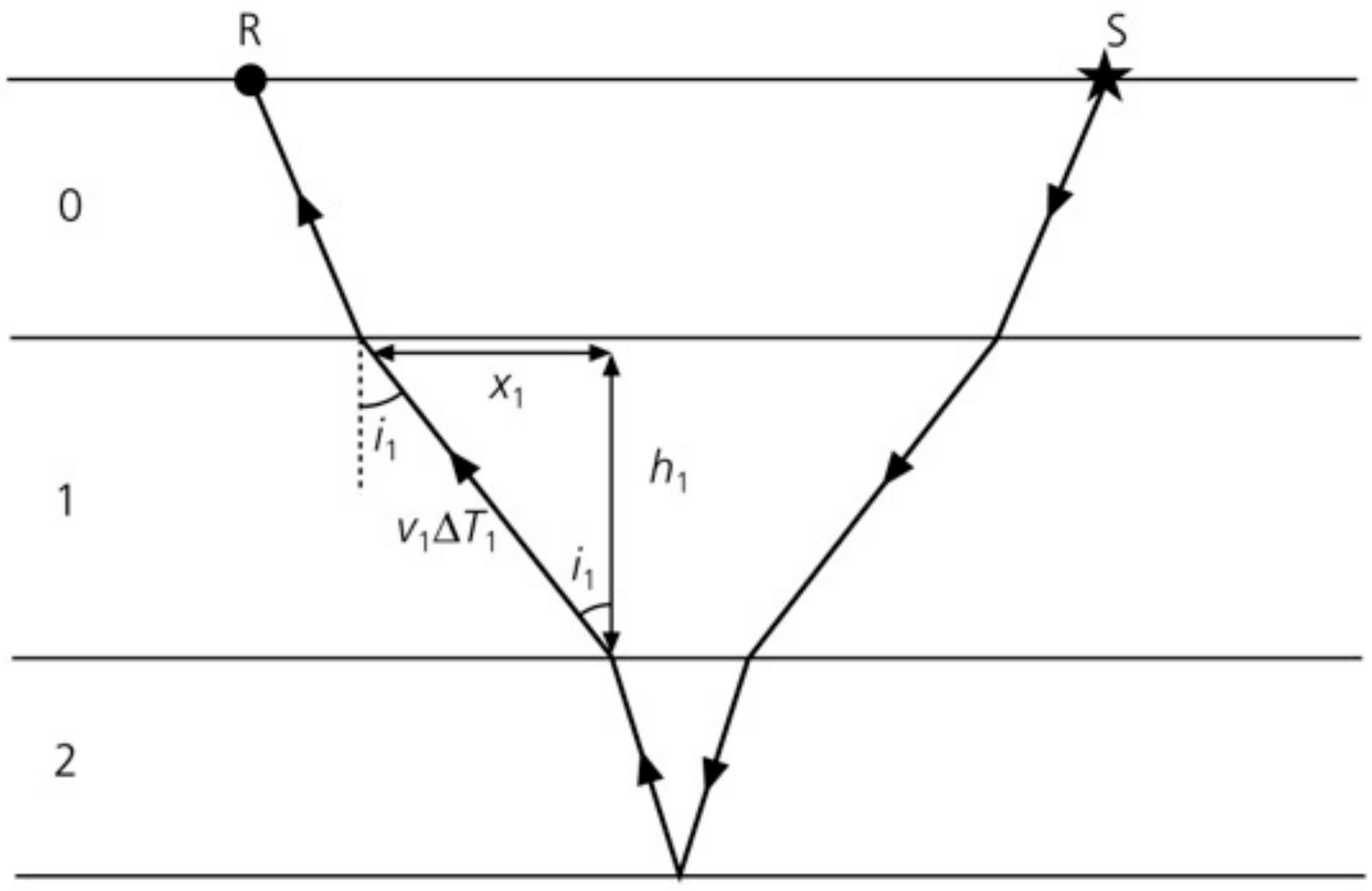
where t_n is the vertical
2-way travel time:

$$t_n = 2\sum_{j=0}^{n} \Delta t_j = 2\sum_{j=0}^{n} (h_j/\mathrm{v}_j)$$

and

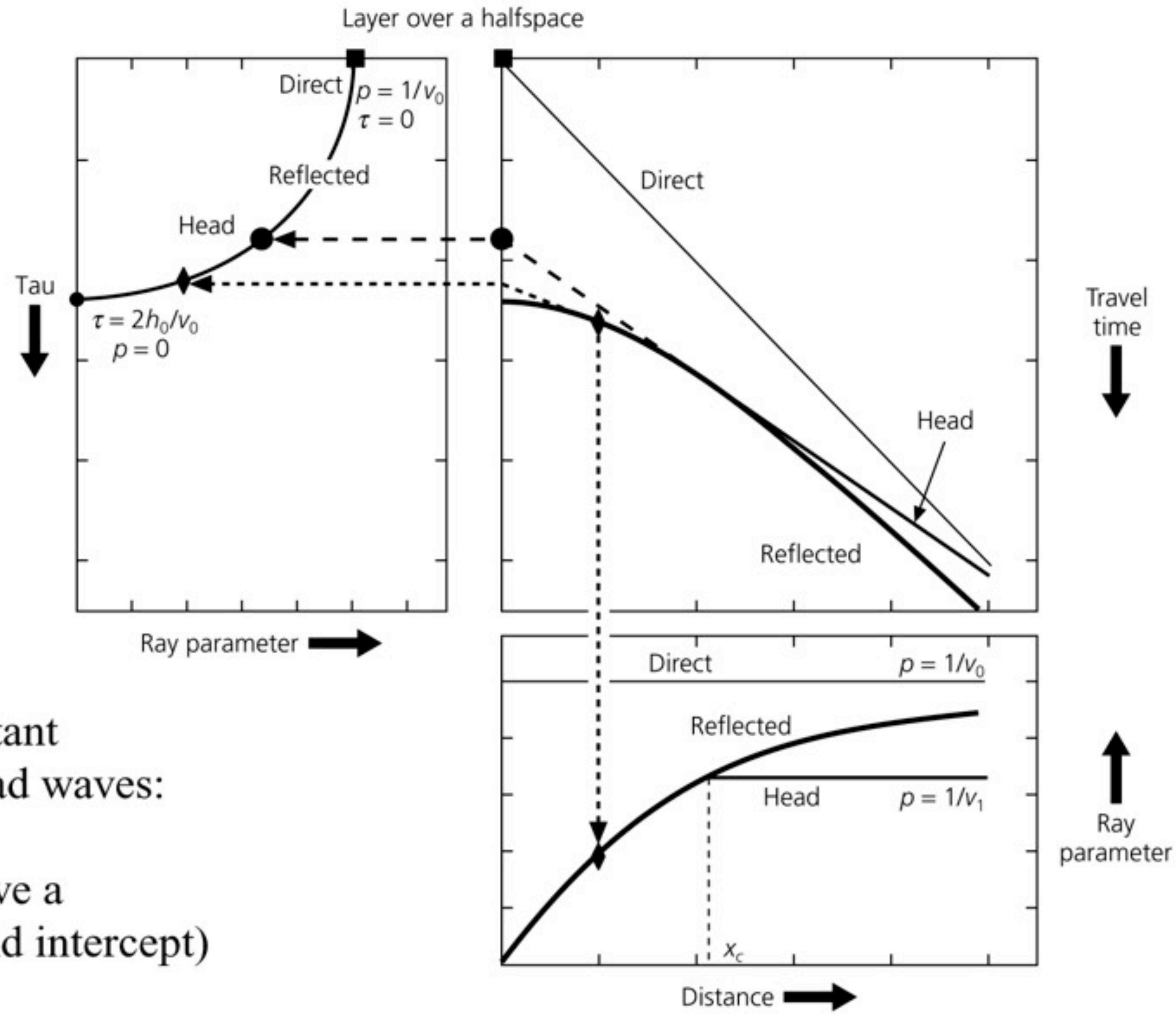
$$x = 2\sum_{j=0}^{n} x_j = 2\frac{\sin i_0}{\mathrm{v}_0}\sum_{j=0}^{n} \mathrm{v}_j^2 \Delta T_j \ .$$

Figure 3.3-3: Ray path through multilayered structure.



Tau-p plane

Figure 3.3-8: Relation between tau-p, and travel time curves.

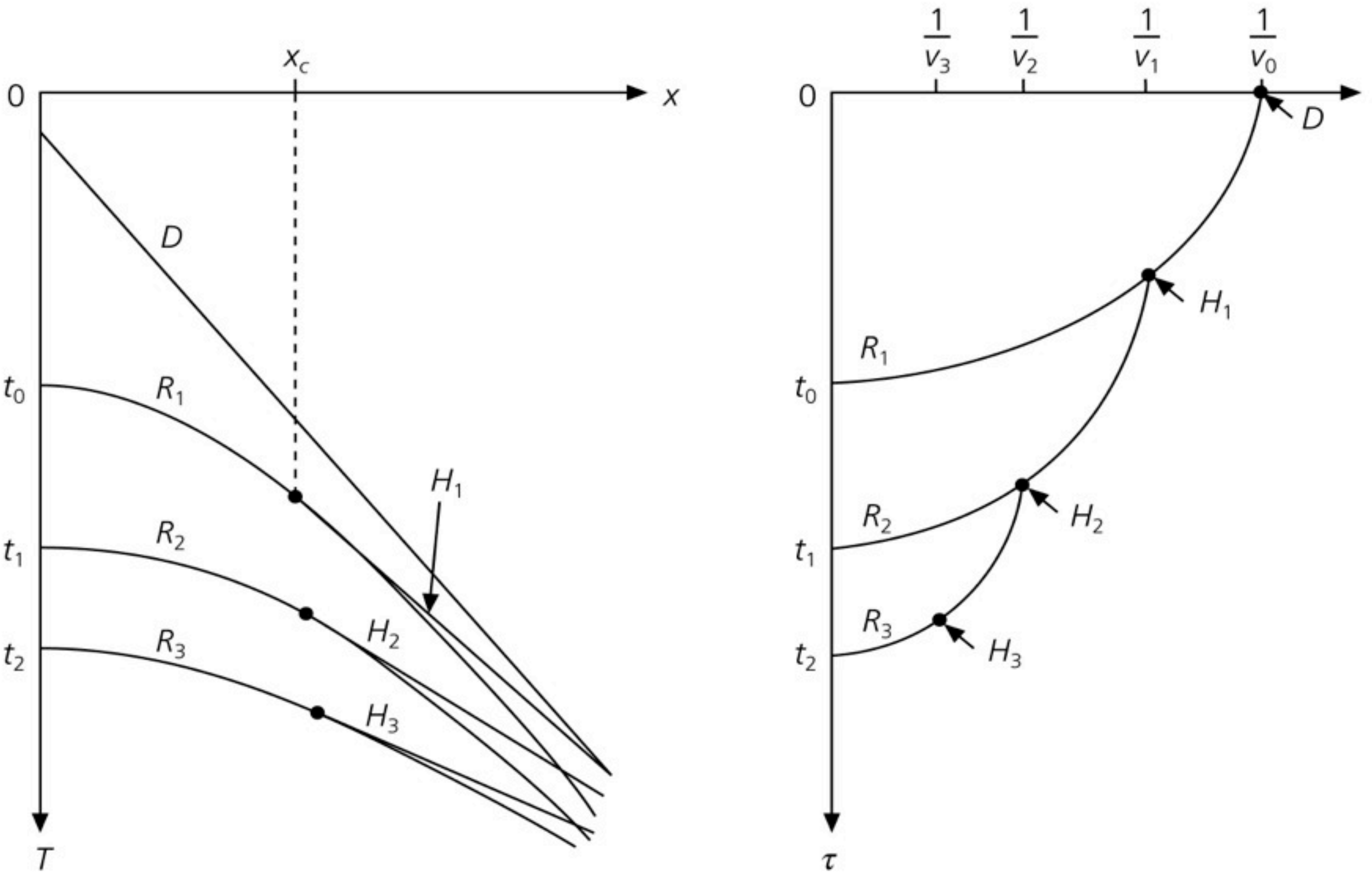


τ and p are constant for direct and head waves:

(straight lines have a constant slope and intercept)

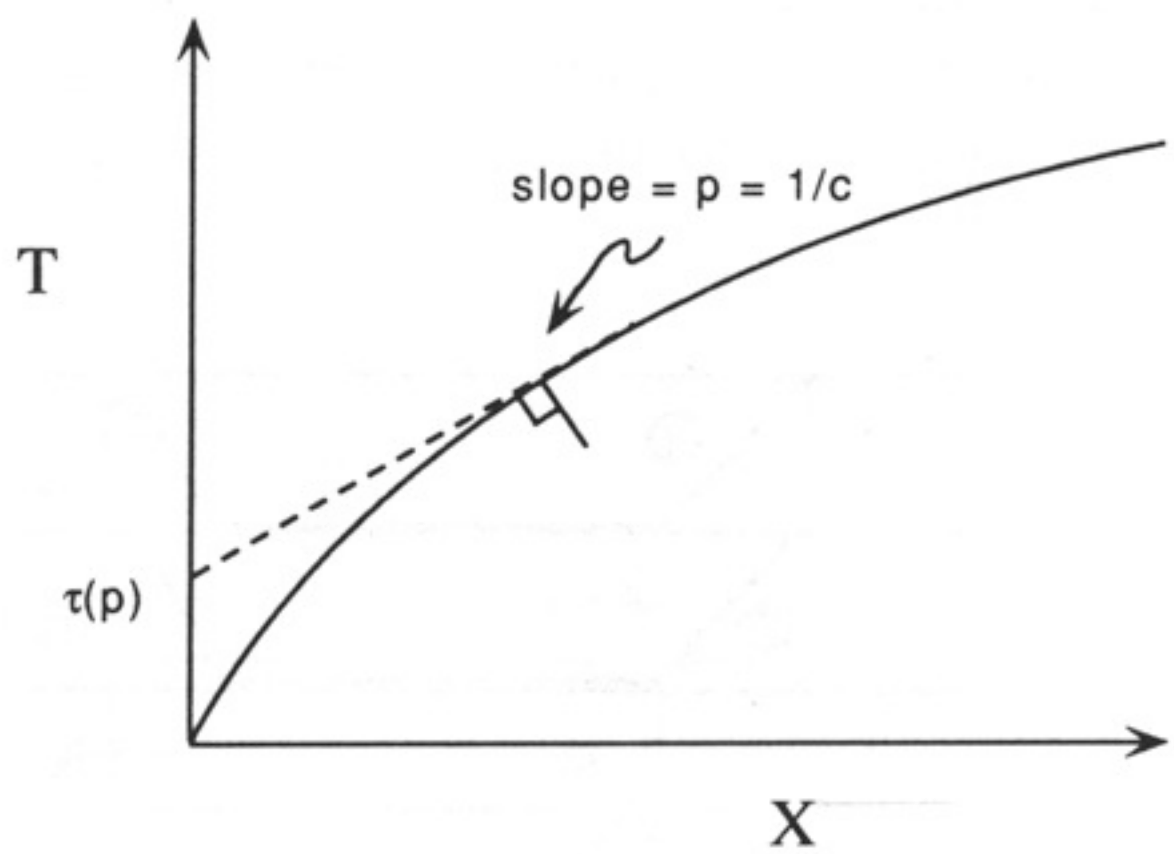
Multiple layers

Figure 3.3-9: Tau-P and travel time curves for multiple layers.



Travel Times for Continuous Media

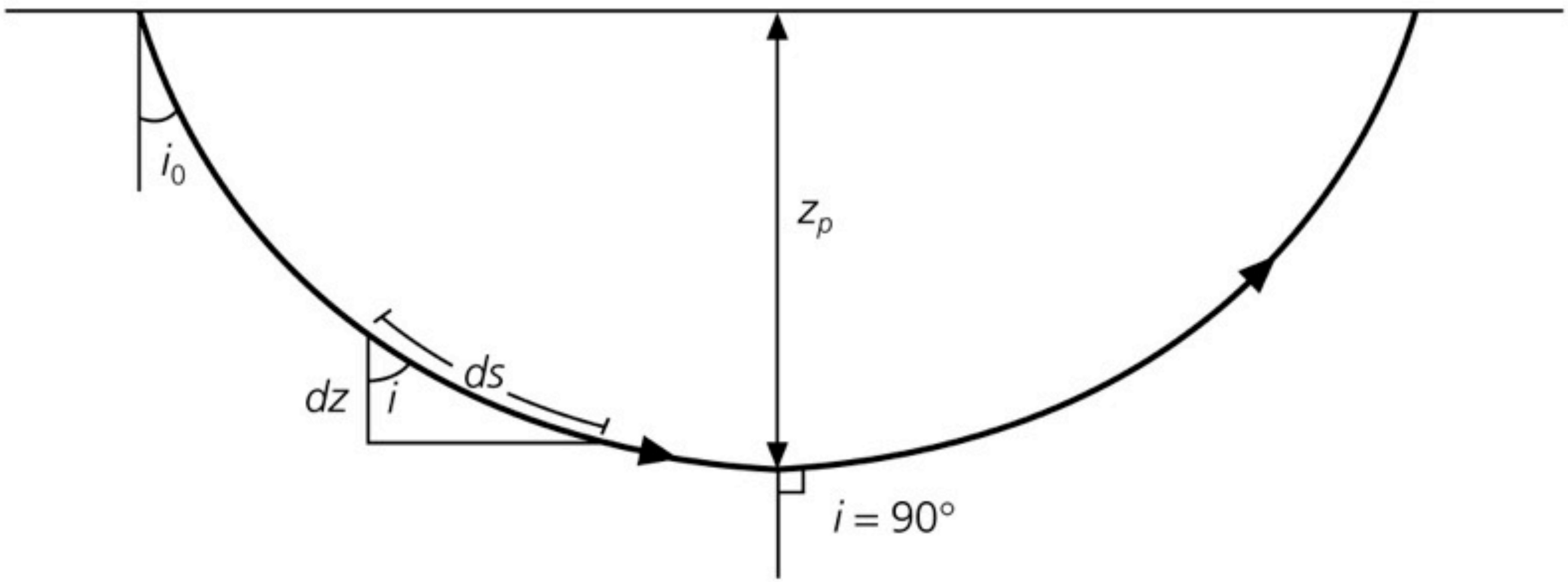
We now let the number of layers go to infinity and the thickness to zero. Then the summation is replaced by integration.



Now we can generalize the concept of intercept time τ of the tangent to the travel time curve and the slope p.

Continuously varying media

Figure 3.3-6: Ray path in a medium with smoothly increasing velocities.

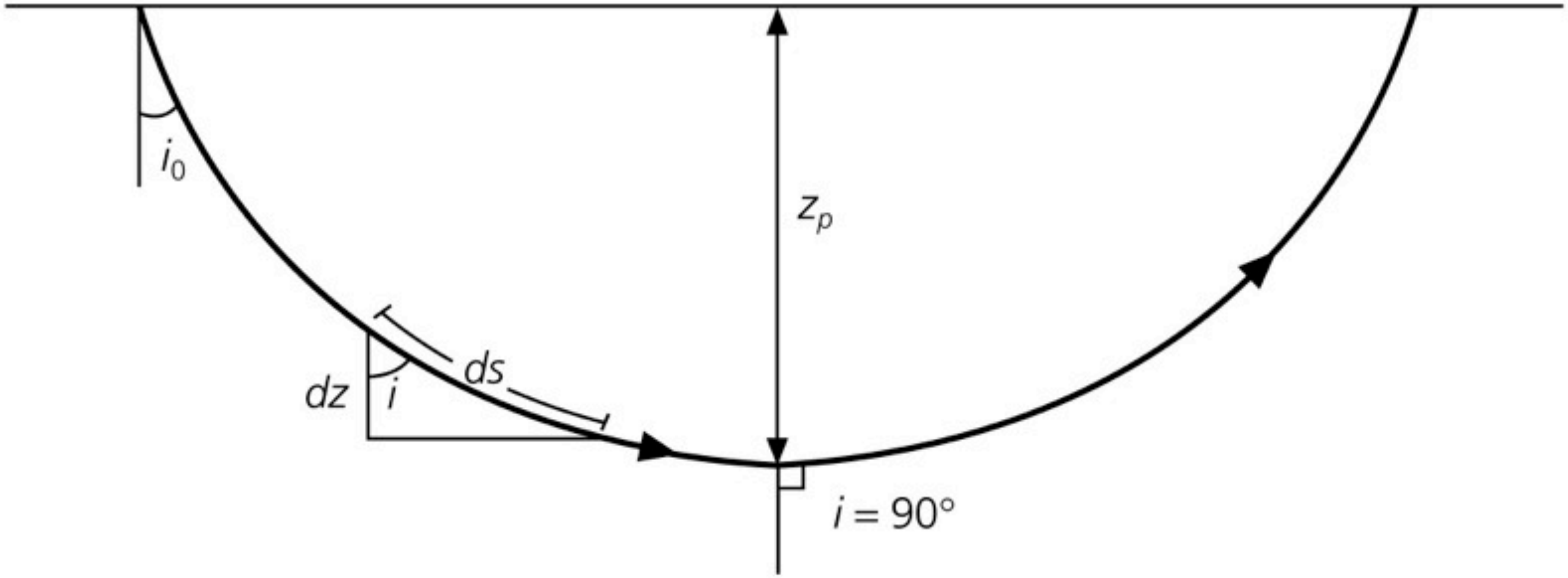


$$p = \sin i / \mathrm{v}(z) \qquad \tan i = \frac{dx}{dz} \qquad dx = dz \tan i$$

$$x(p) = 2\int dx = 2\int_0^{z_p} \tan i \; dz = 2p\int_0^{z_p} \left(\frac{1}{\mathrm{v}^2(z)} - p^2\right)^{-1/2} dz$$

(using $\sin i = p\mathrm{v}(z)$ and $\cos i = (1 - \sin^2 i)^{1/2} = (1 - p^2\mathrm{v}^2(z))^{1/2}$)

Figure 3.3-6: Ray path in a medium with smoothly increasing velocities.

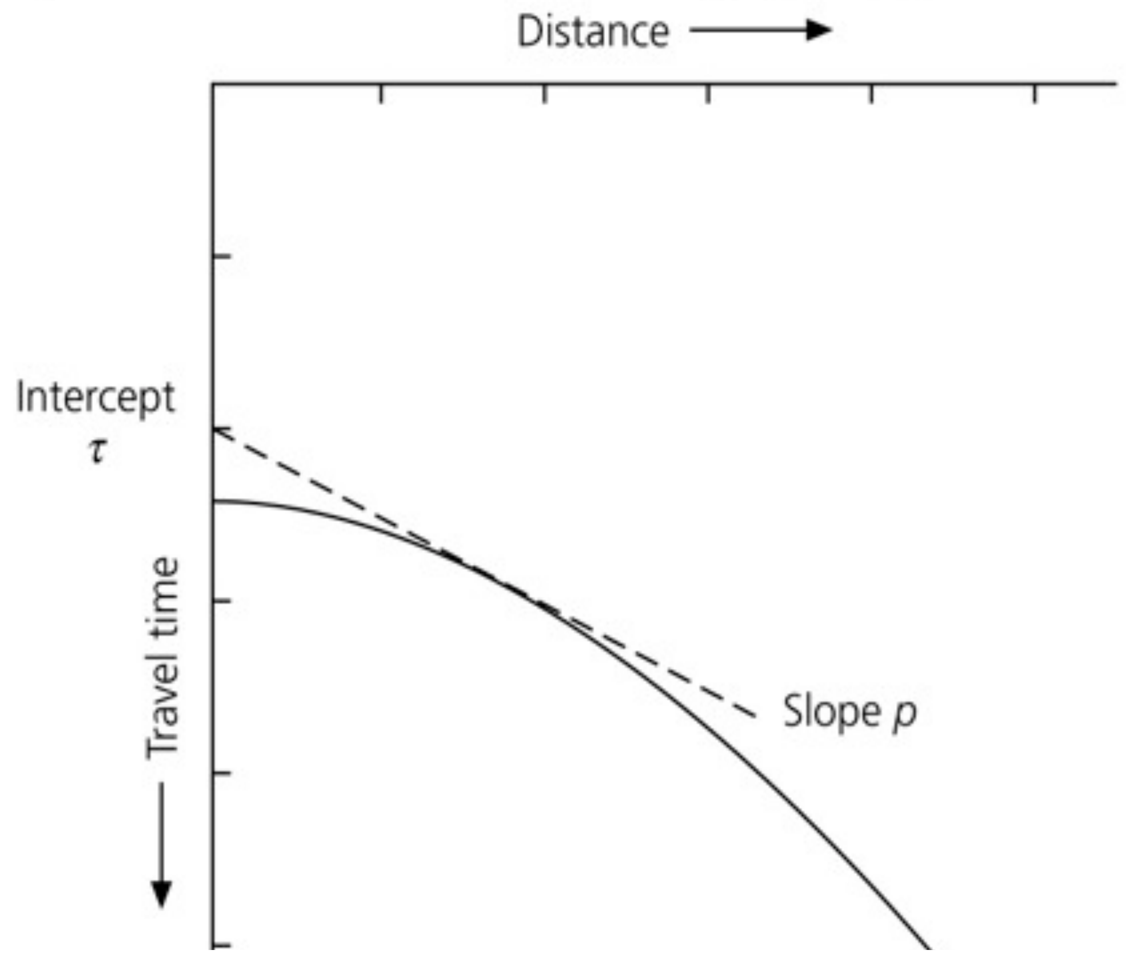


Write in terms of the slowness, $u(z) = 1/\mathrm{v}(z)$:

$$x(p) = 2p \int_0^{z_p} \frac{dz}{(u^2(z) - p^2)^{1/2}} \quad \text{and} \quad T(p) = \, = 2 \int_0^{z_p} \frac{u^2(z)dz}{(u^2(z) - p^2)^{1/2}}$$

Valid everywhere except at the exact bottom, where $u(z)$ equals p.

Figure 3.3-7: Relation between travel time curve, tau, and ray parameter.



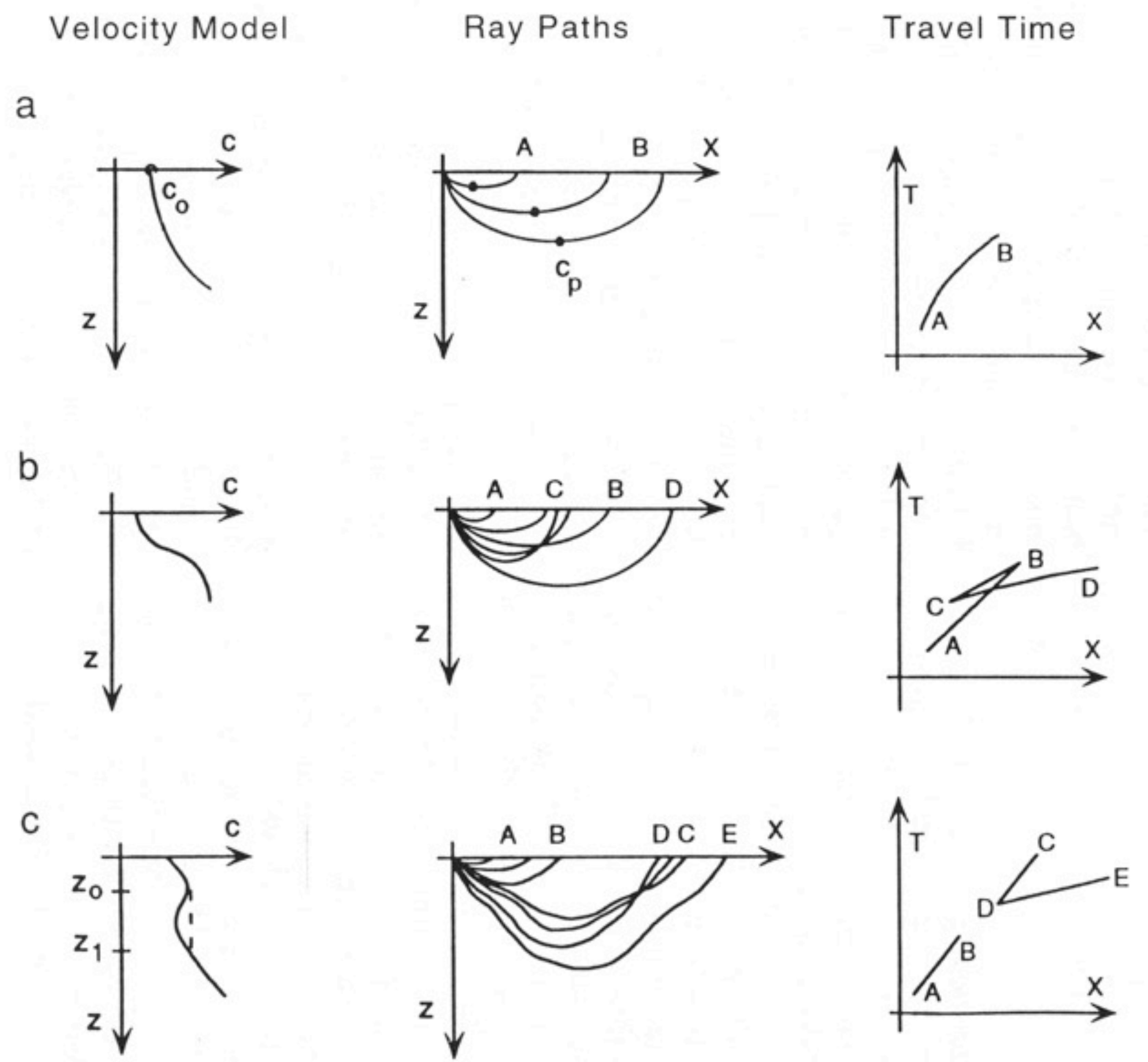
Define τ as a function of p for the reflected wave: $T(x) = px + \tau(p)$

$$\tau(p) = 2\sum_{j=0}^{n} \eta_j h_j = 2\sum_{j=0}^{n} (1/v_j^2 - p^2)^{1/2} h_j = 2\sum_{j=0}^{n} (u_j^2 - p^2)^{1/2} h_j$$

$\tau(p) = T(p) - px(p)$ and so $\dfrac{d\tau}{dp} = \dfrac{dT}{dp} - p\dfrac{dx}{dp} - x(p) = \dfrac{dT}{dx}\dfrac{dx}{dp} - p\dfrac{dx}{dp} - x(p) = -x(p)$

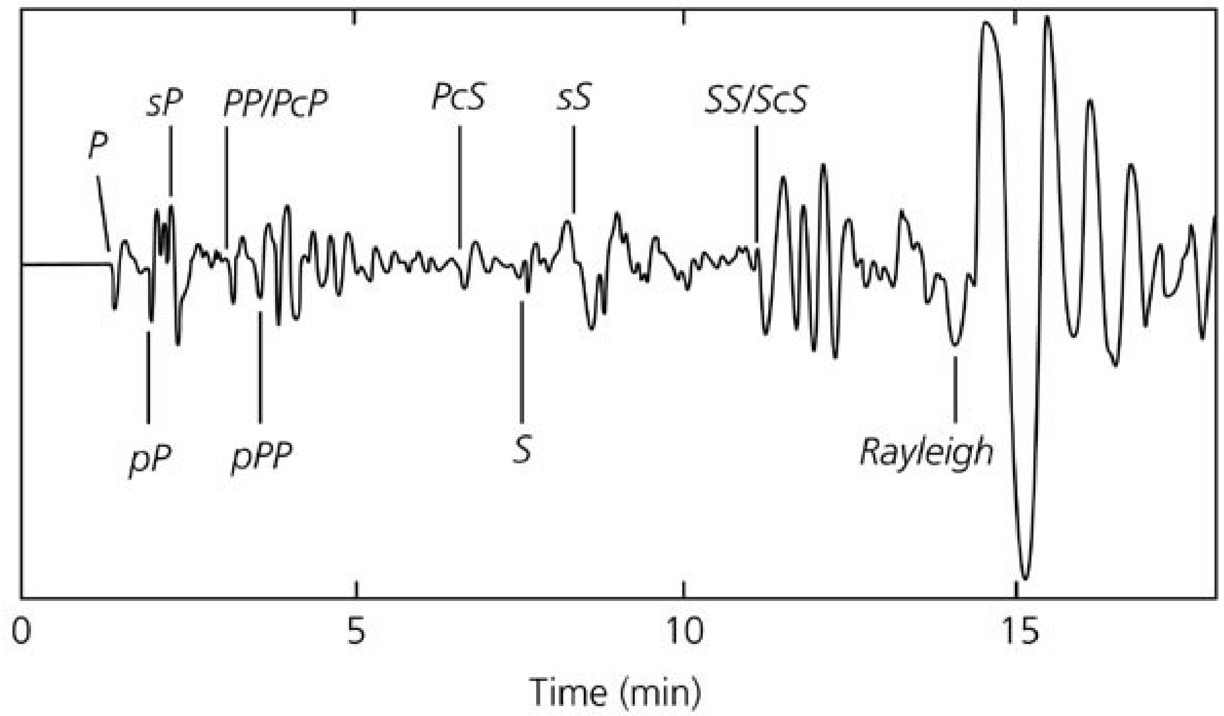
Just as p is the slope of the travel time curve, $T(x)$, the distance, x, is minus the slope of the $\tau(p)$ curve.

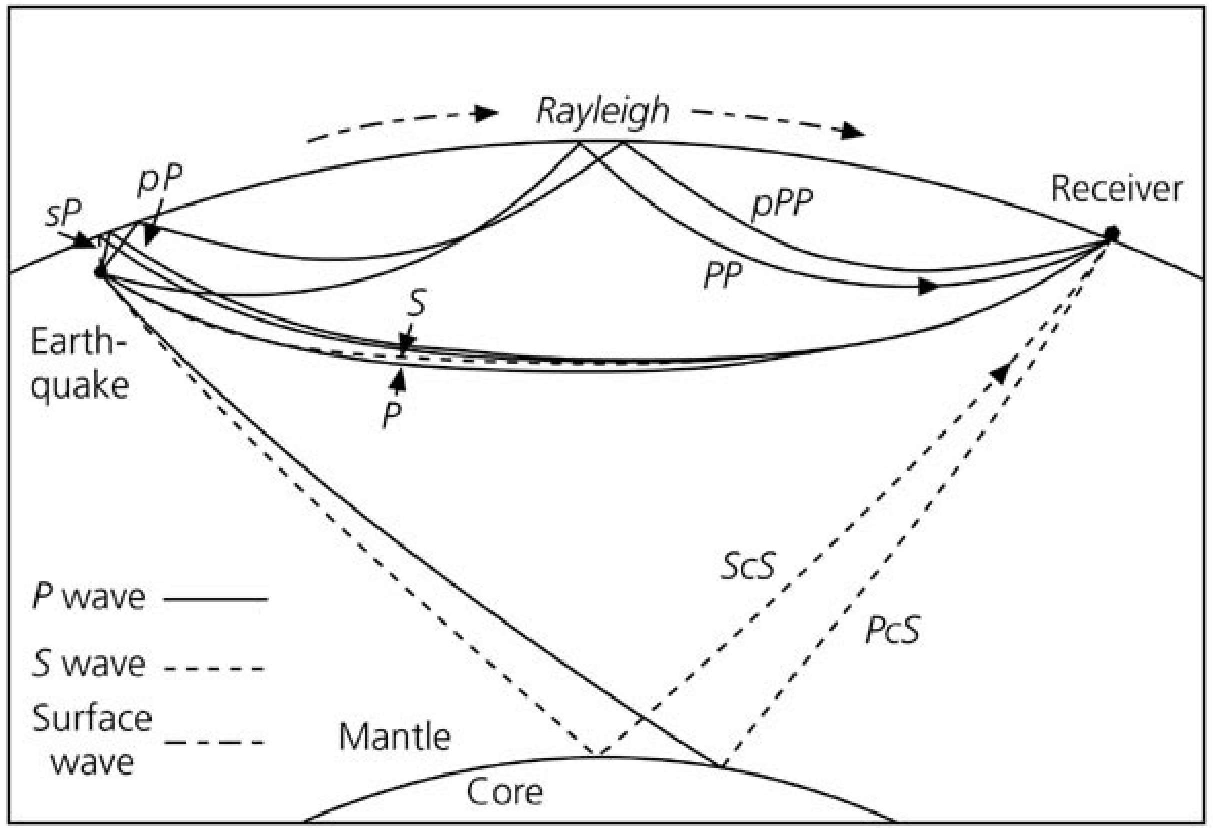
Travel Times: Examples



Ray paths inside the Earth

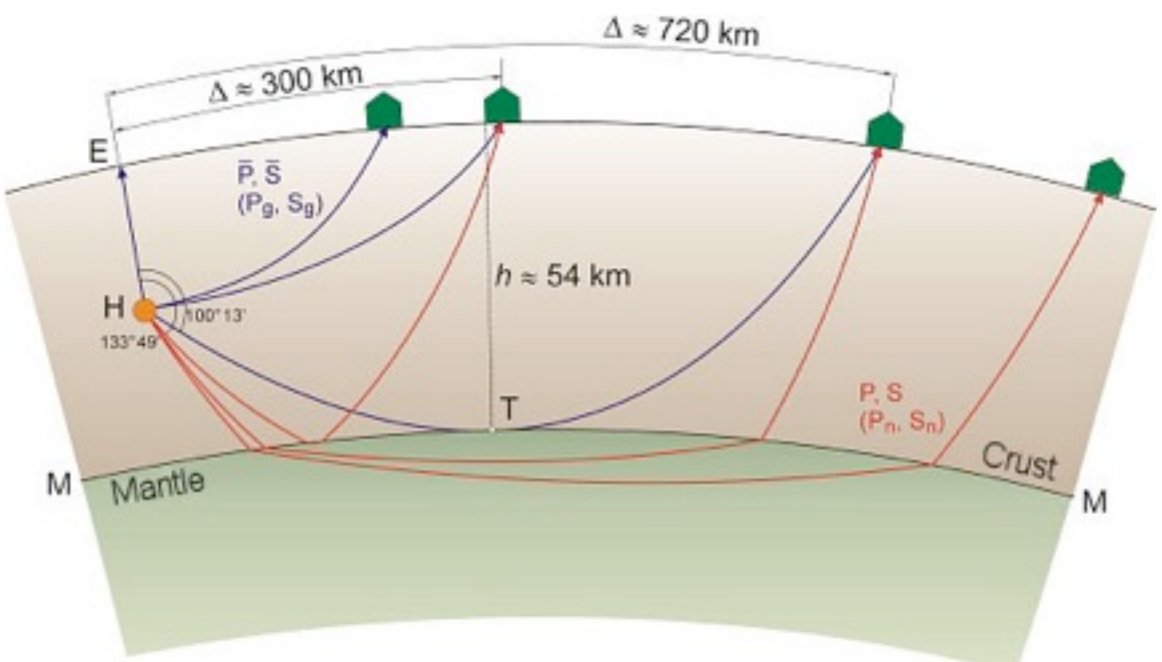
Figure 3.5-2: Selection of body phases and their ray paths.

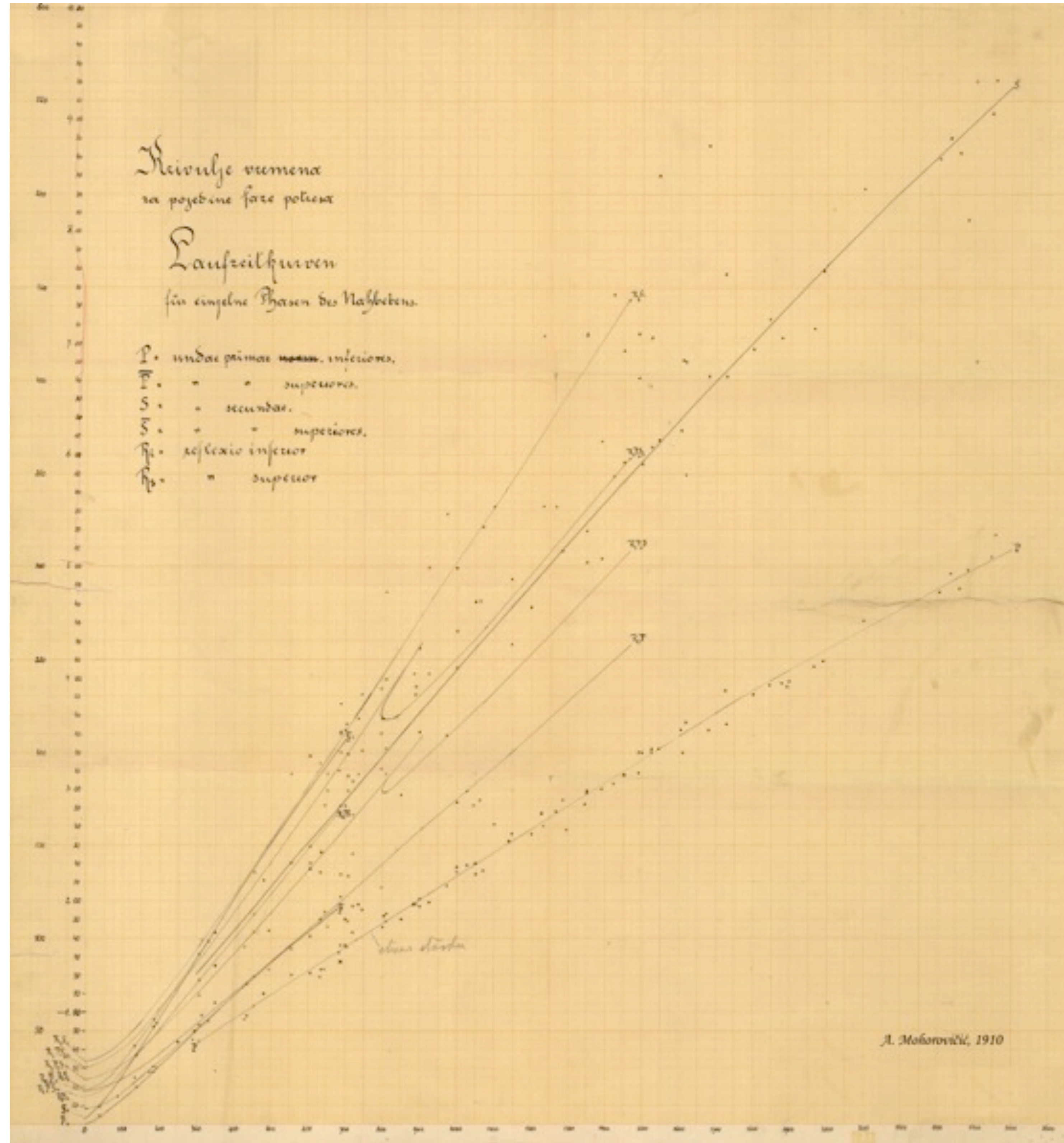




Andrija MOHOROVIČIĆ

Godišnje izvješće zagrebačkog meteorološkog opservatorija za godinu 1909. Godina IX, dio IV. - polovina 1. Potres od 8. X. 1909





http://www.gfz.hr/sobe-en/discontinuity.htm

The Inverse Problem

It seems that now we have the means to predict arrival times T_{pre} at a given the travel distance of a ray with a given emergence angle (ray parameter) and given structure. This is also termed a **forward (or direct) problem**.

We have recorded a set of travel times, T_{obs}, and we want to determine the structure of the Earth. Thus, what we really want is to solve the **inverse problem**.

In a very general sense we are looking for an Earth model that **minimizes** the difference between a theoretical prediction and the observed data:

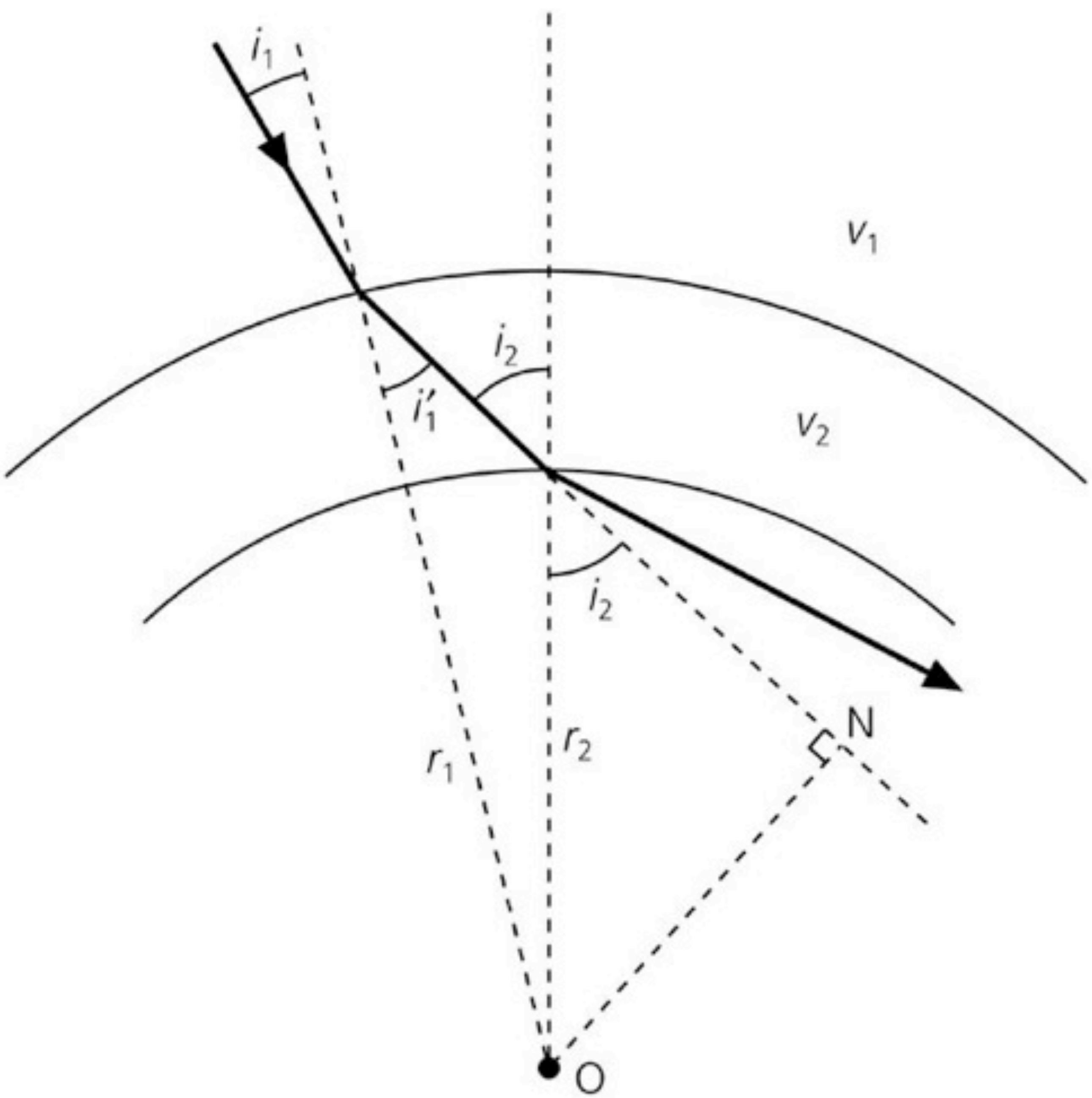
$$\sum \left(T_{obs} - T_{pre}(m) \right)$$

where m is an Earth model.

Rays in a Spherical Earth

How can we generalize these results to a spherical Earth which should allow us to invert observed travel times and find its internal velocity structure?

Figure 3.4-1: Geometry of Snell's law for a spherical earth.



Snell's Law applies in the same way:

$$\frac{\sin i_1}{v_1} = \frac{\sin i'_1}{v_2}$$

From the figure it follows

$$\frac{r_1 \sin i_1}{v_1} = \frac{r_1 \sin i'_1}{v_2} = \frac{r_2 \sin i_2}{v_2}$$

which is a general equation along the raypath (i.e. it is constant)

Ray Parameter in a Spherical Earth

... thus the ray parameter in a spherical Earth is defined as :

$$\frac{r \sin i}{v} = p$$

Note that the units (s/rad or s/deg) are different than the corresponding ray parameter for a flat Earth model.

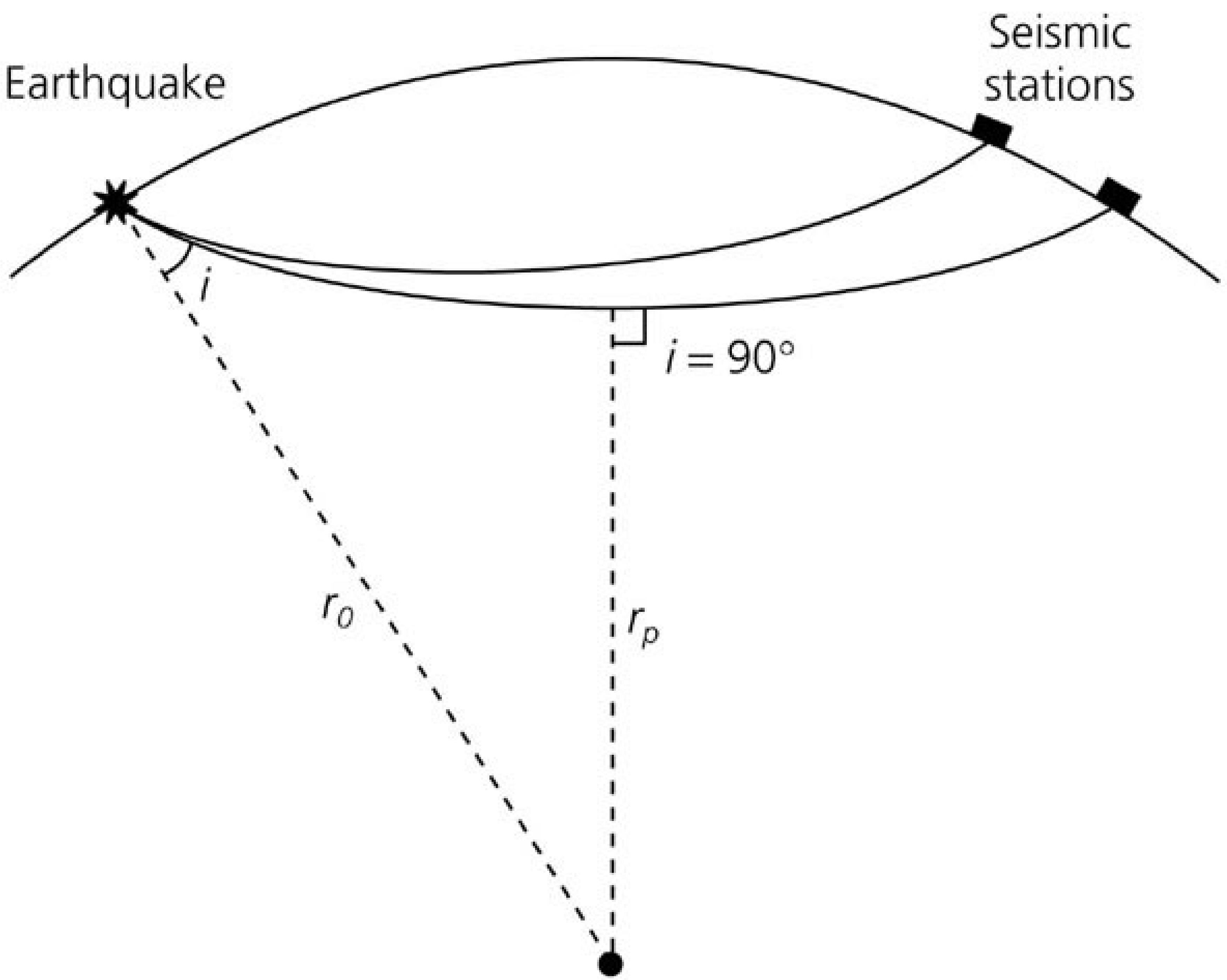
The meaning of p is the same as for a flat Earth: it is the slope of the travel time curve.

$$p = \frac{dT}{d\Delta}$$

The equations for the travel distance and travel time have very similar forms than for the flat Earth case!

Ray Parameter in a Spherical Earth

Figure 3.4-2: Geometry of a ray path in a spherical earth.

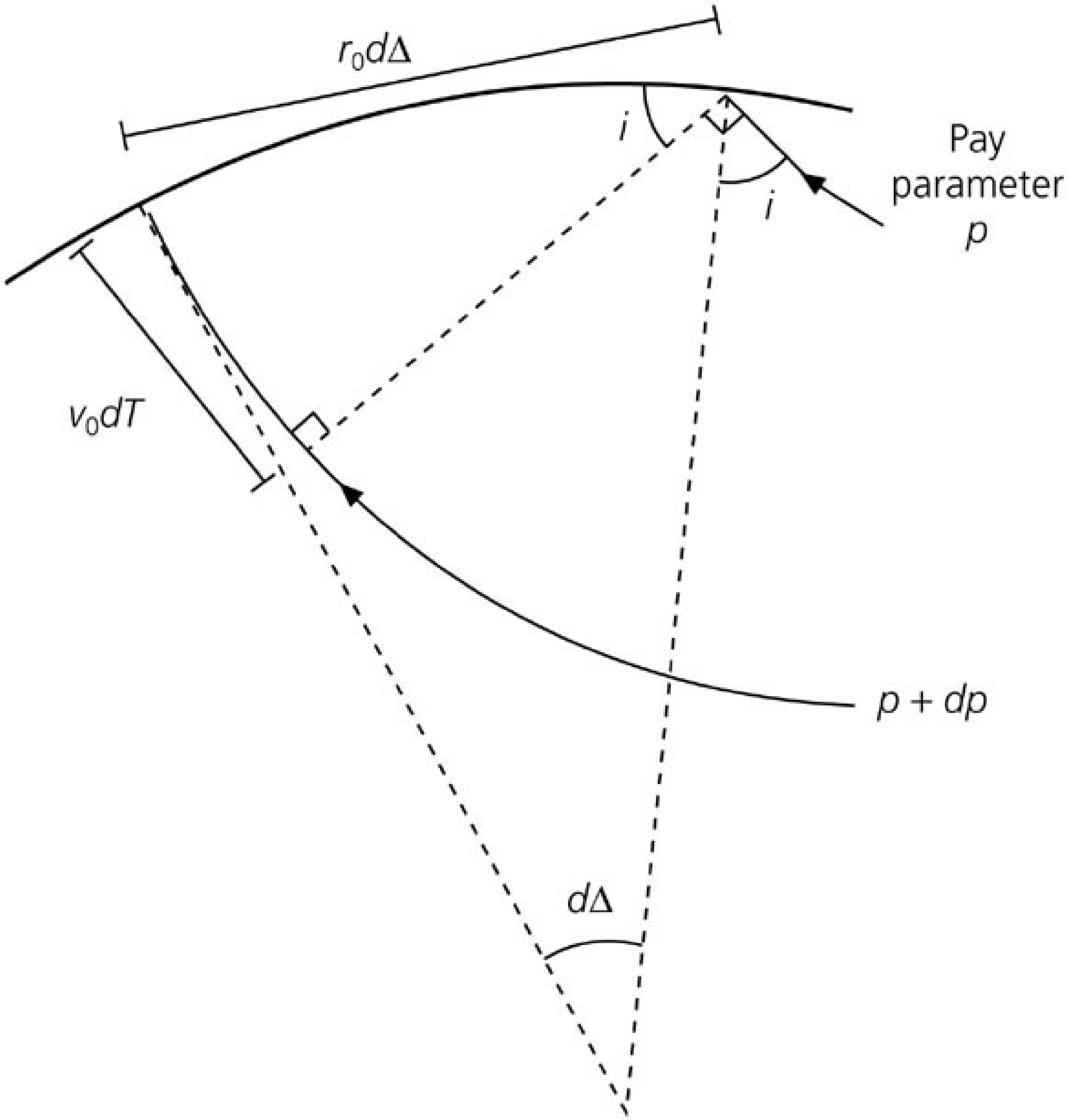


$$p = r_0 (\sin i)/v_0$$

$$p = r_P / v_P$$

Ray Parameter in a Spherical Earth

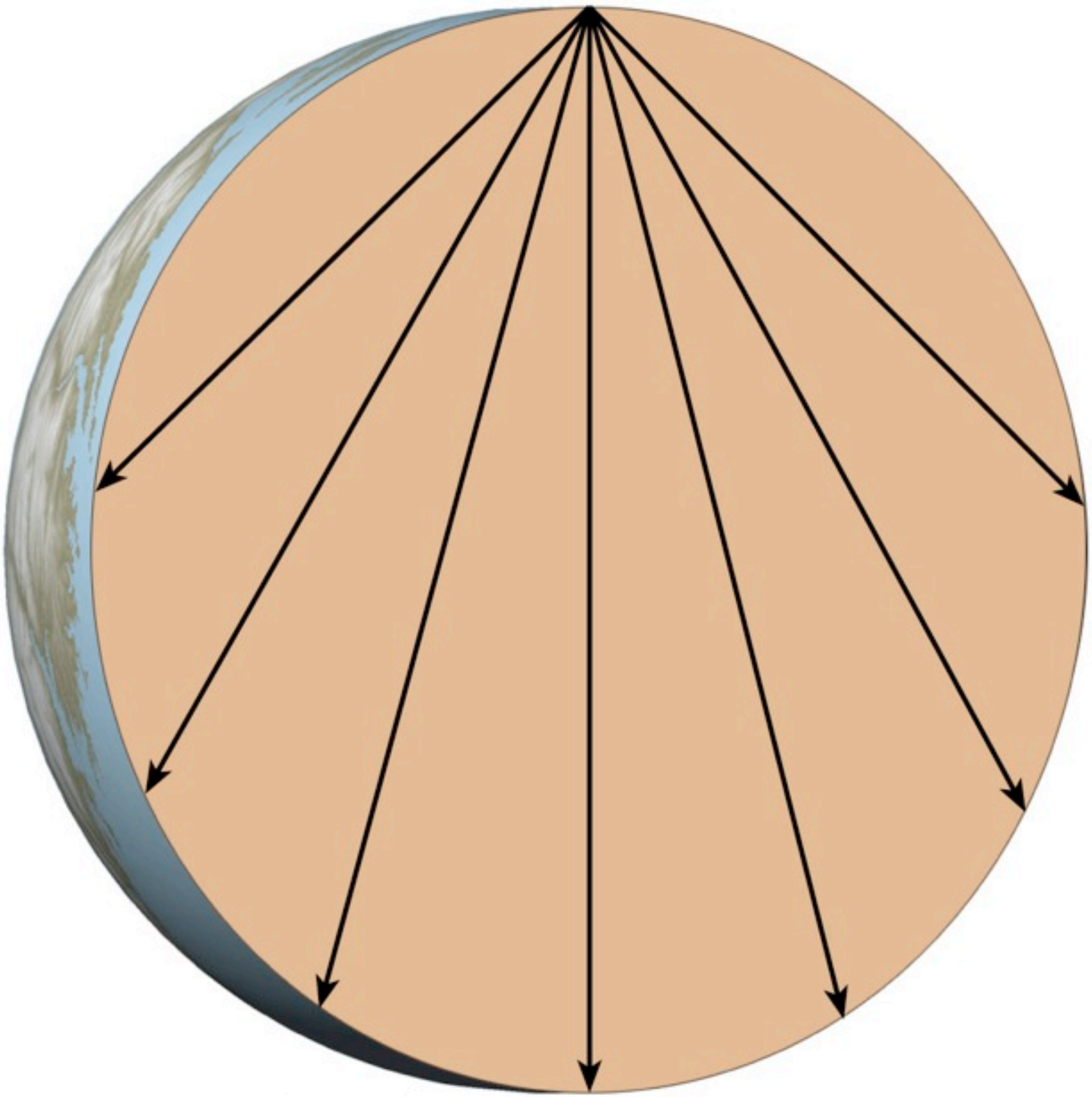
Figure 3.4-3: Derivation of the ray parameter in a spherical earth.



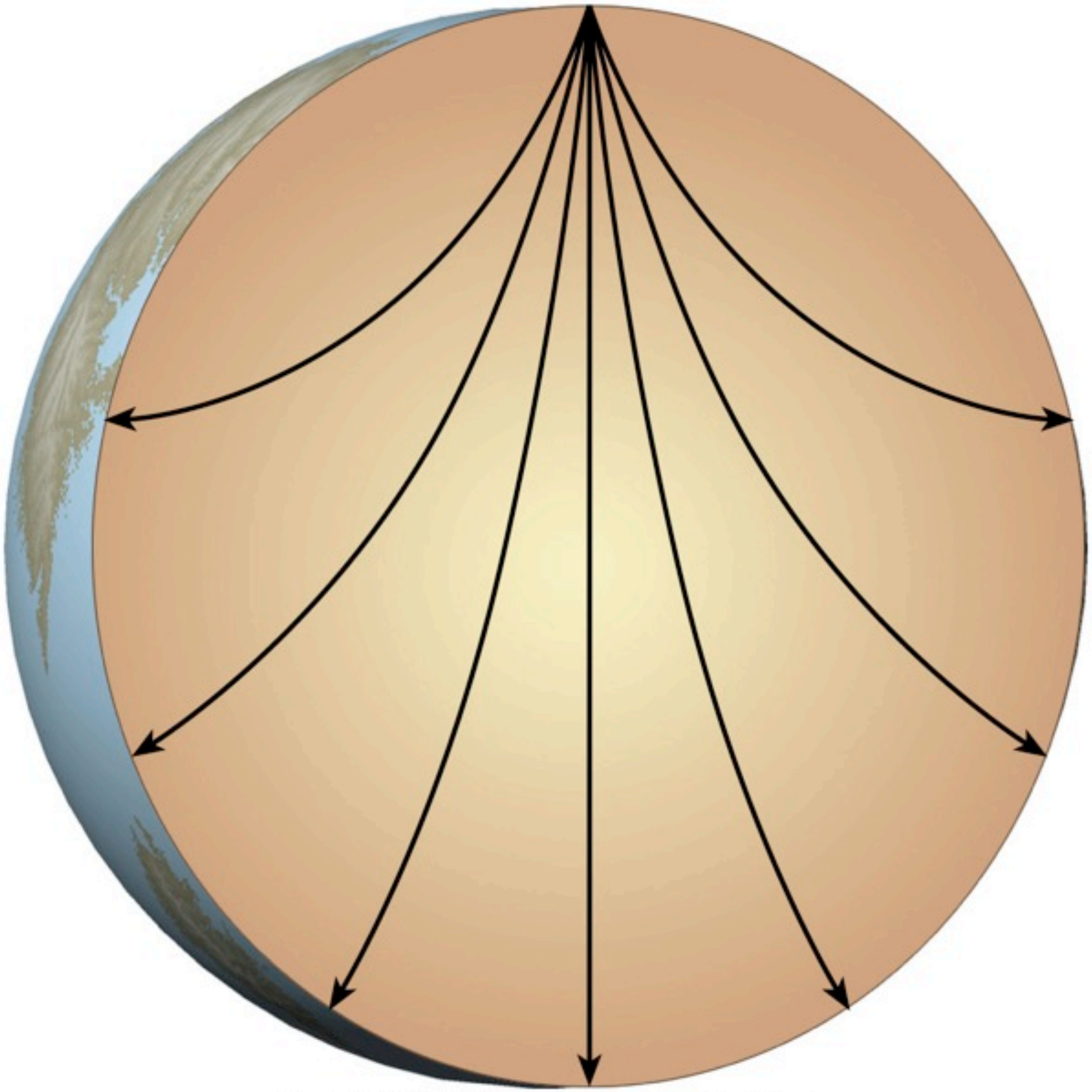
$v_0 dT/(r_0 d\Delta) = \sin i$

$p = dT/d\Delta$

Rays in homogeneous sphere



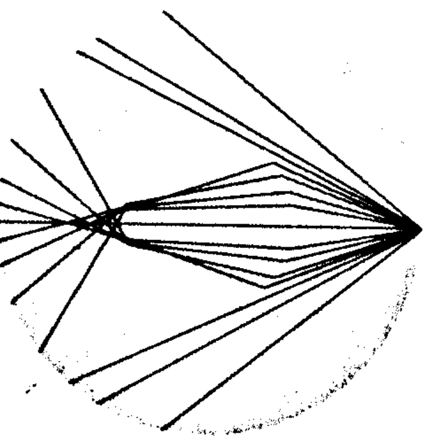
Sphere with increasing velocity...



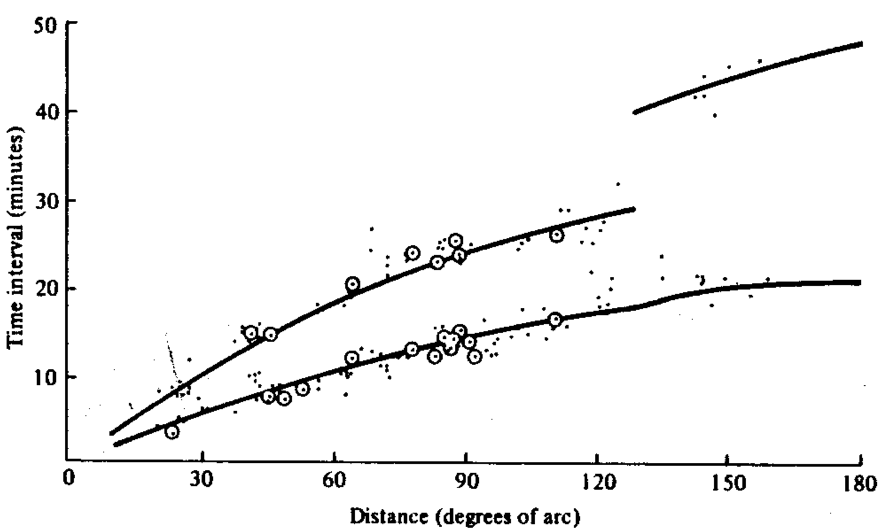
Earth's core

Richard Dixon Oldham

The Constitution of the Earth as revealed by earthquakes, Quart. J. Geological Soc. Lond., 62, 456-475, 1906



Paths of seismic waves through the Earth assuming a core of radius 0.4*R*, in which the speed is 3 km/sec, while the speed outside it is 6 km/sec. [From Oldham, 1906.]

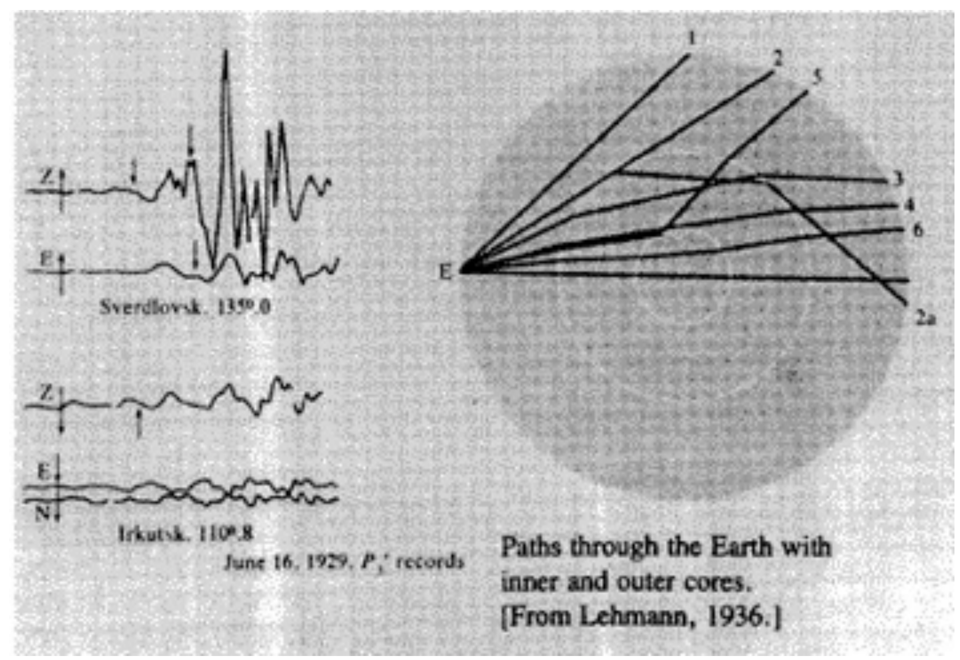


Time curves of first and second phases of preliminary tremors. The marks surrounded by circles are averages. [From Oldham, 1906.]

Beno Gutenberg

1914 Über Erdbebenwellen VIIA. Nachr. Ges. Wiss. Göttingen Math. Physik. Kl, 166.

who calculated depth of the core as 2900km or 0.545R



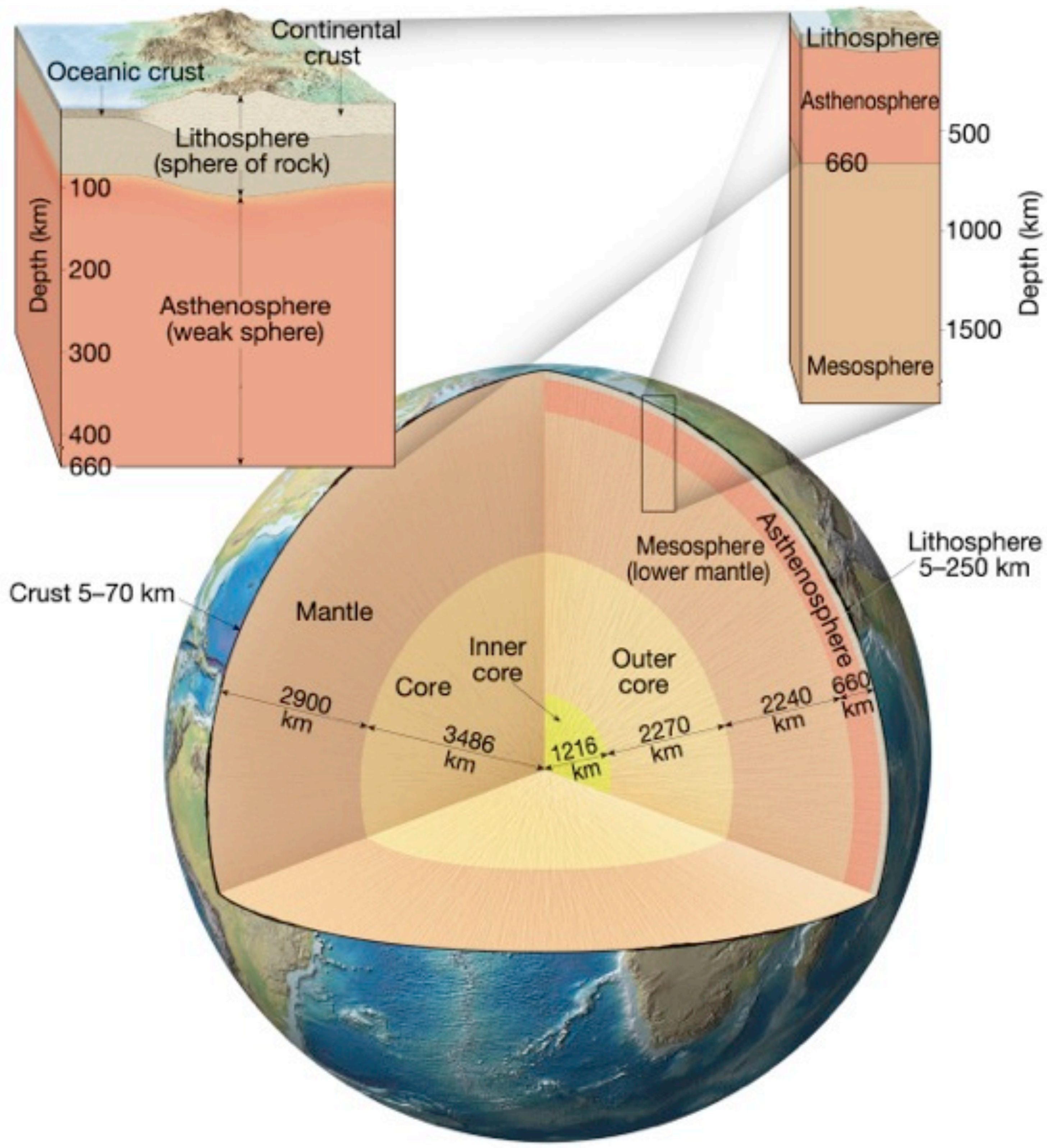
Paths through the Earth with inner and outer cores. [From Lehmann, 1936.]

Inge Lehmann

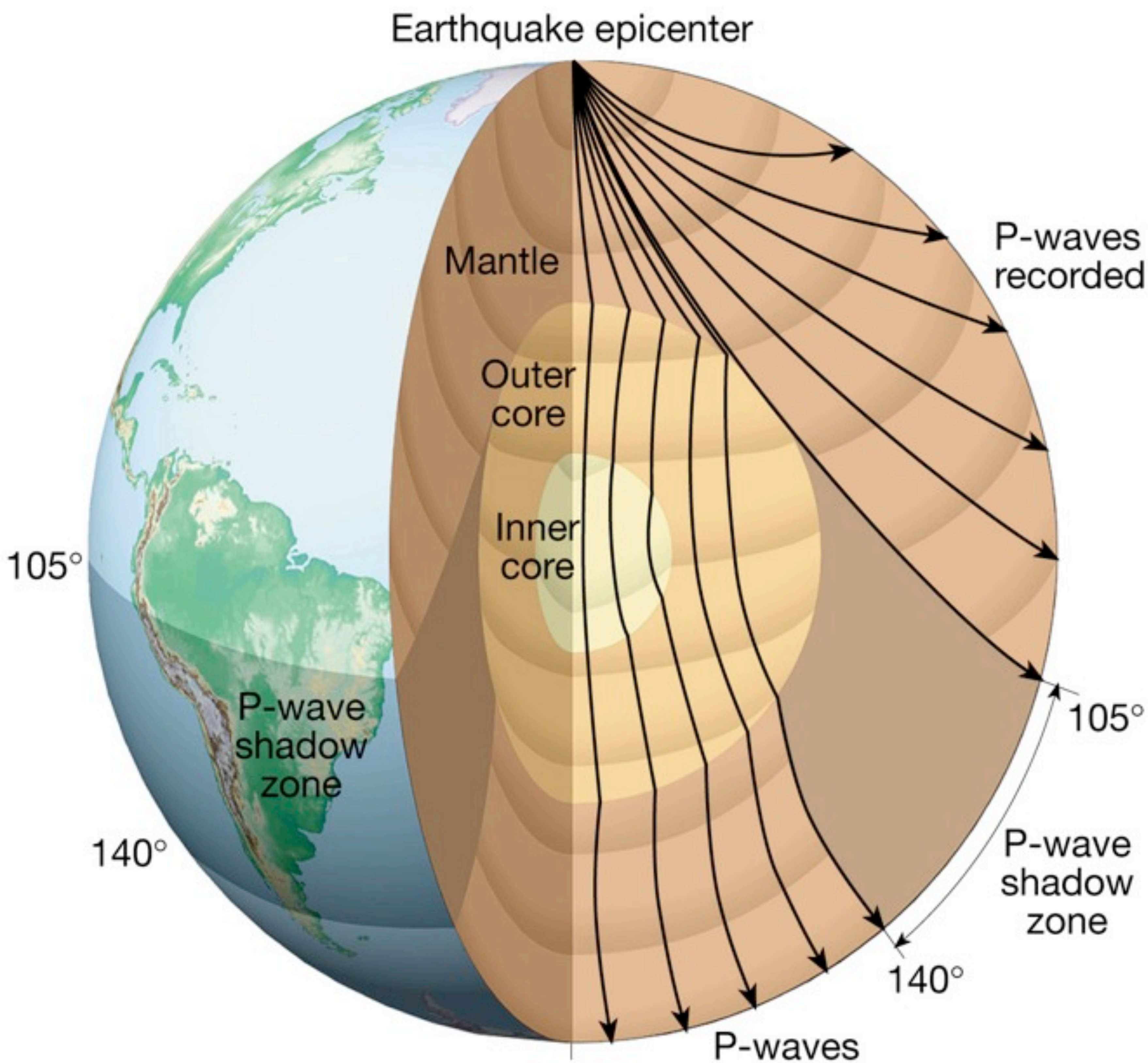
Bureau Central Seismologique International, Series A, Travaux Scientifiques, 14, 88, 1936.

who discovered of the earth's inner core.

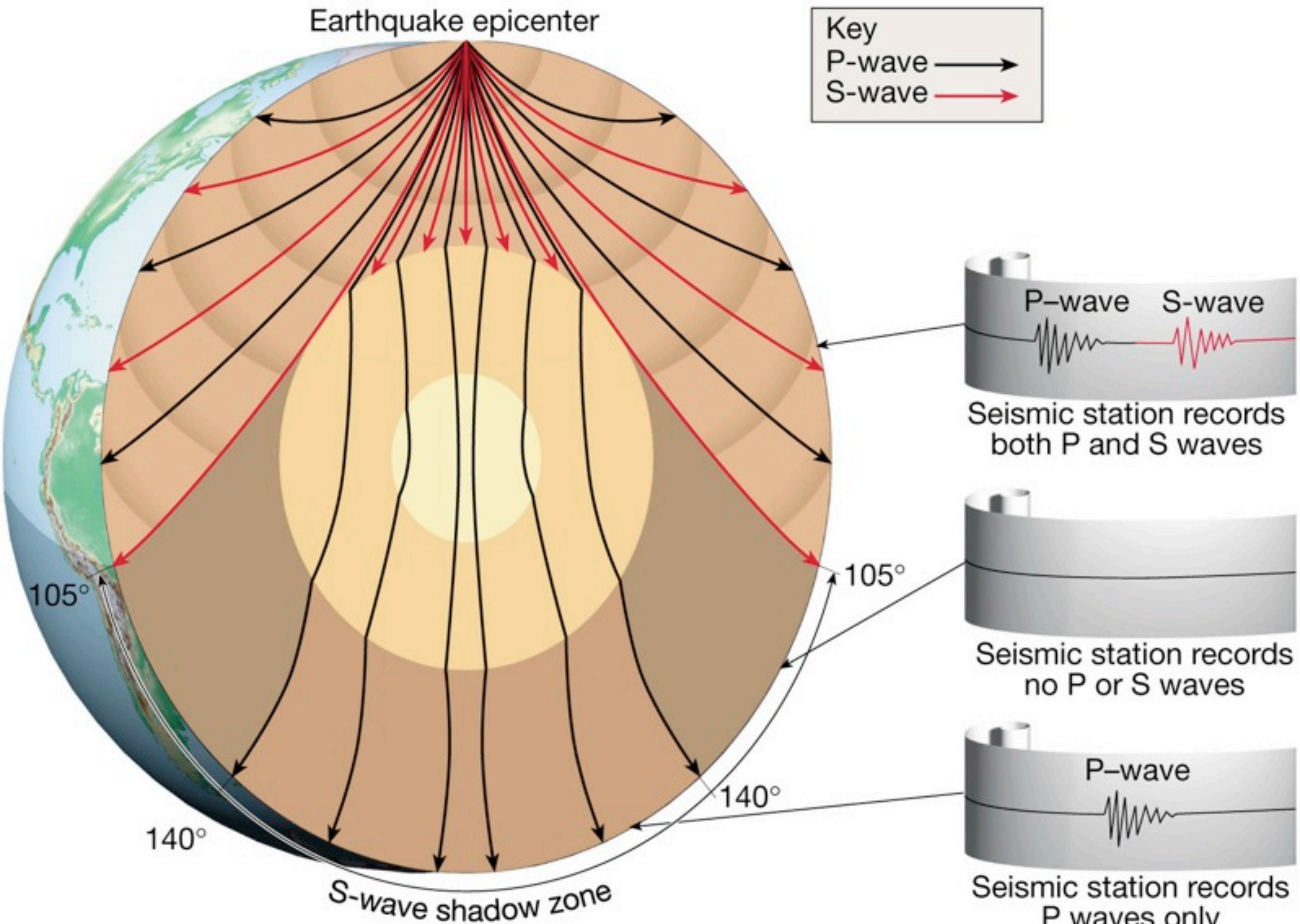
Earth layered structure

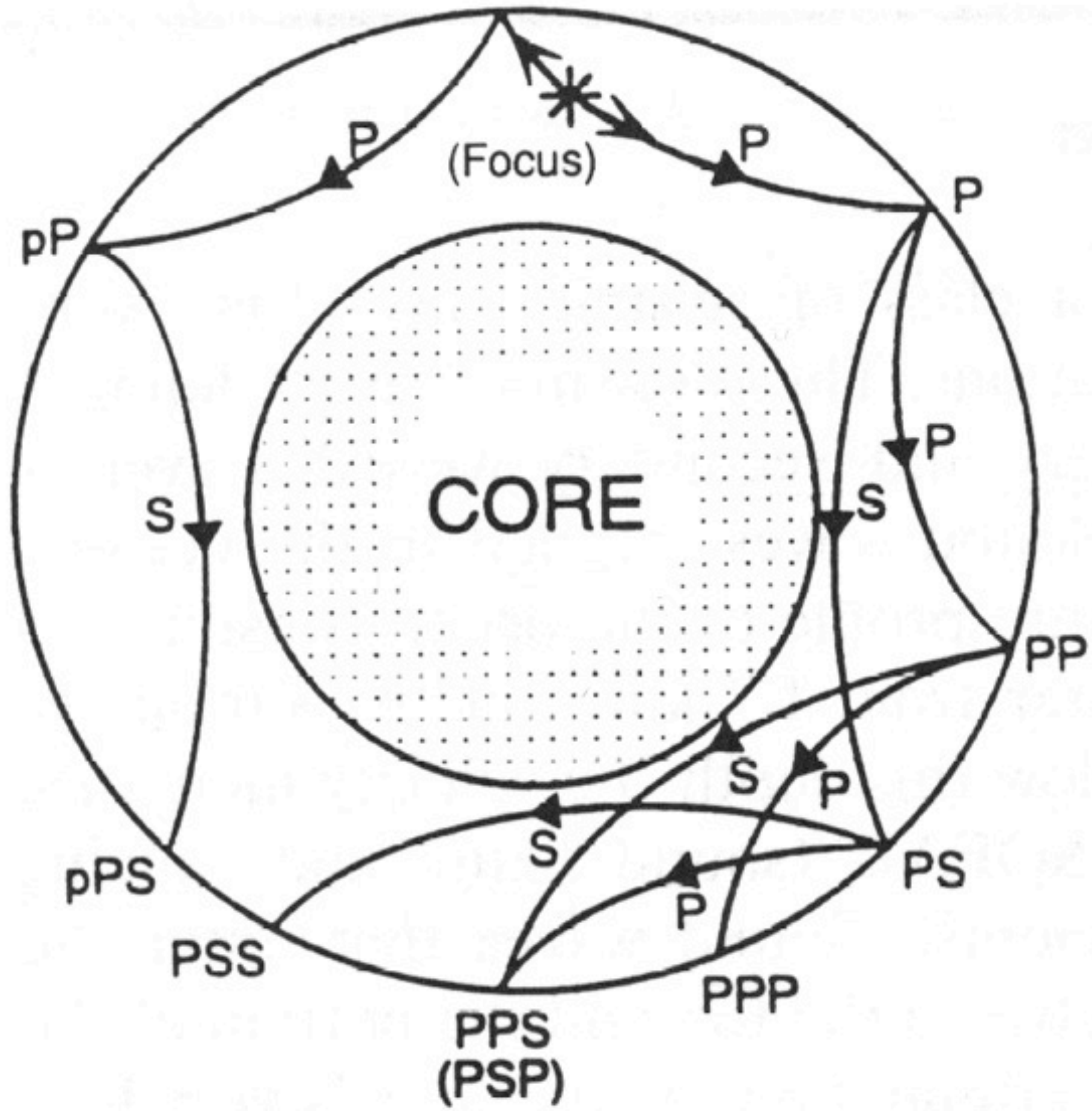


Ray Paths in the Earth (1)



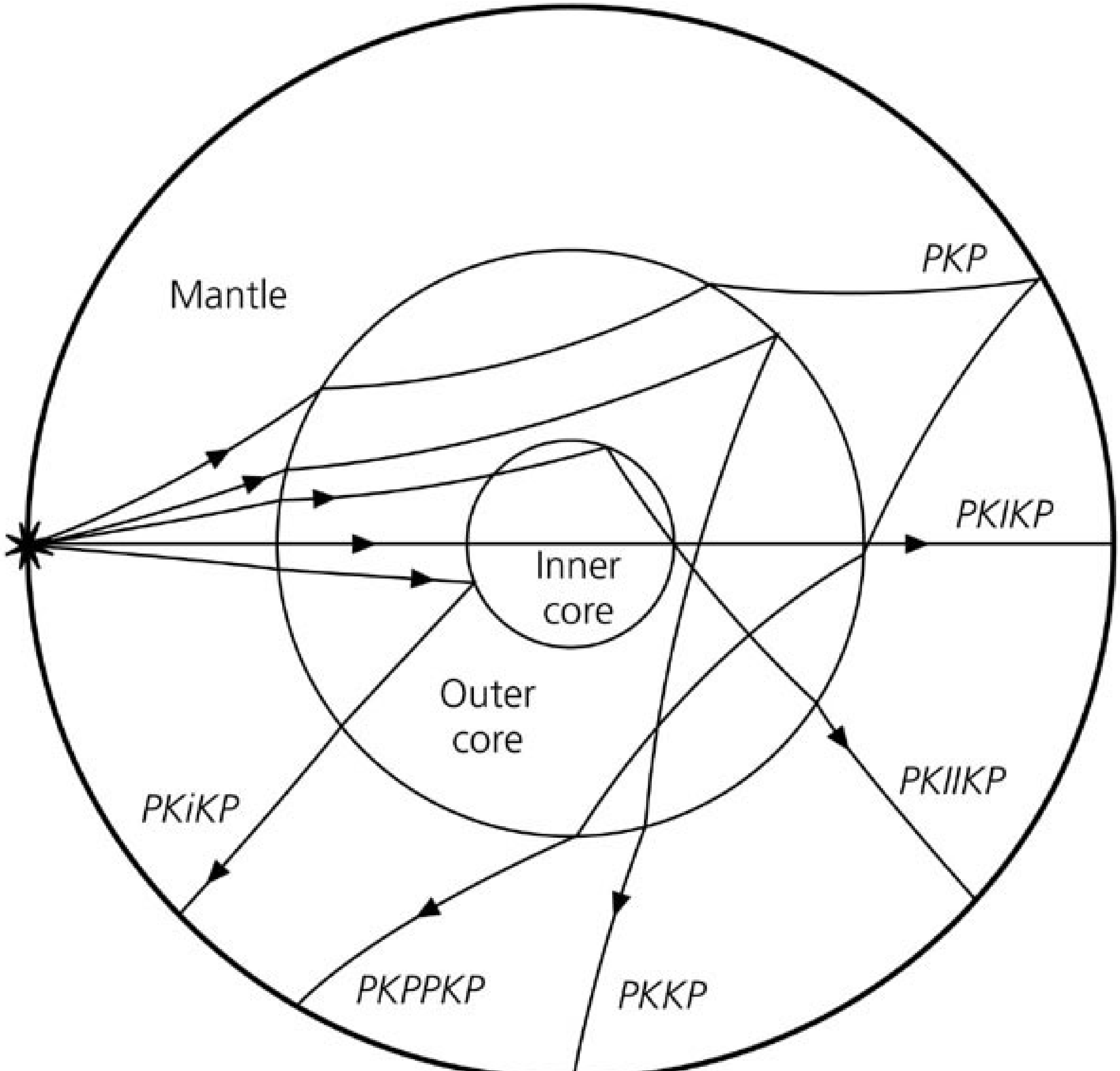
Ray Paths in the Earth (2)




(Focus)
P
P
P
pP
P
S
S
CORE
PP
S
P
S
PS
pPS
P
PSS
PPP
PPS
(PSP)

Ray Paths in the Earth (4)

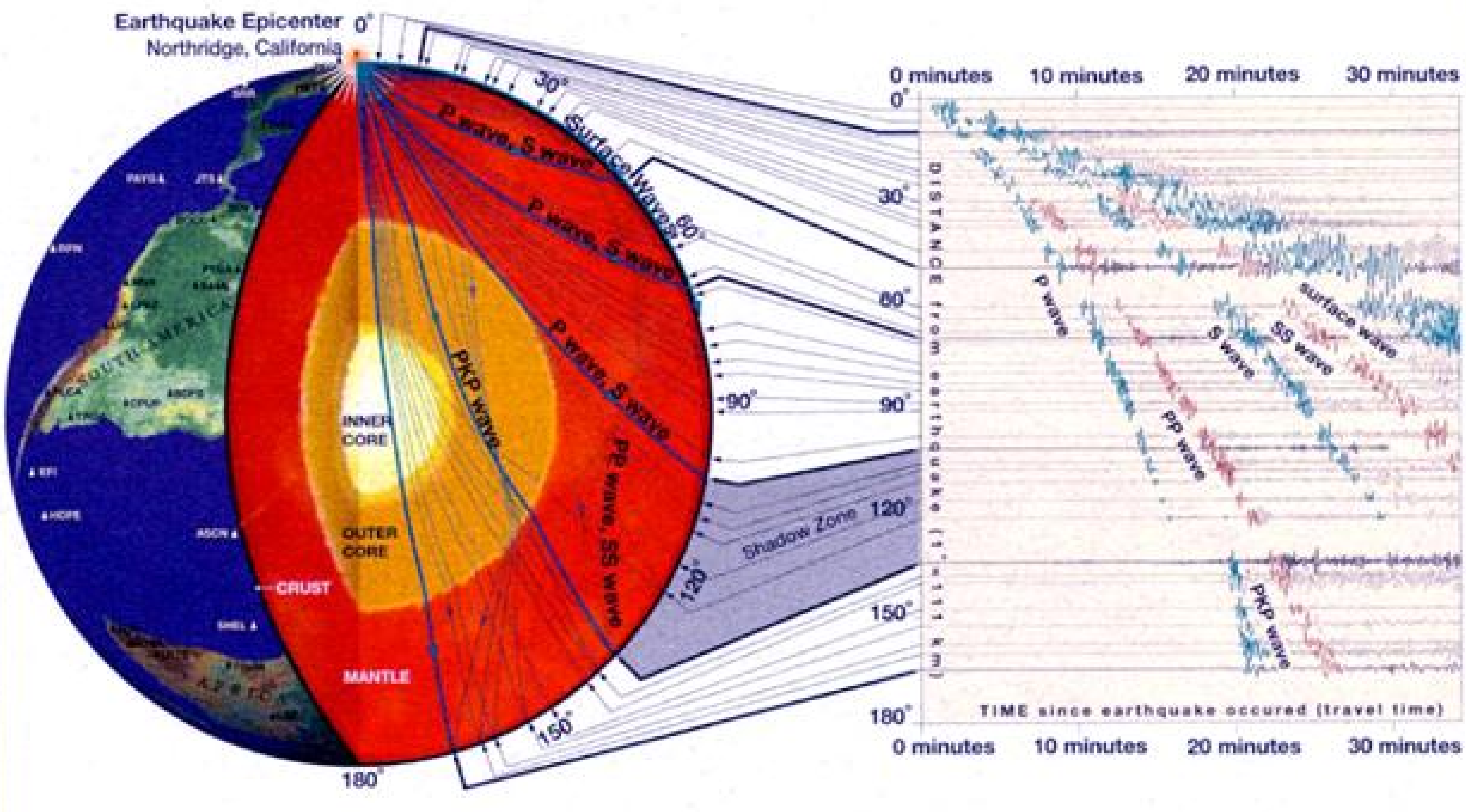
Figure 3.5-10: Ray paths for additional core phases.



Ray Paths in the Earth - Names

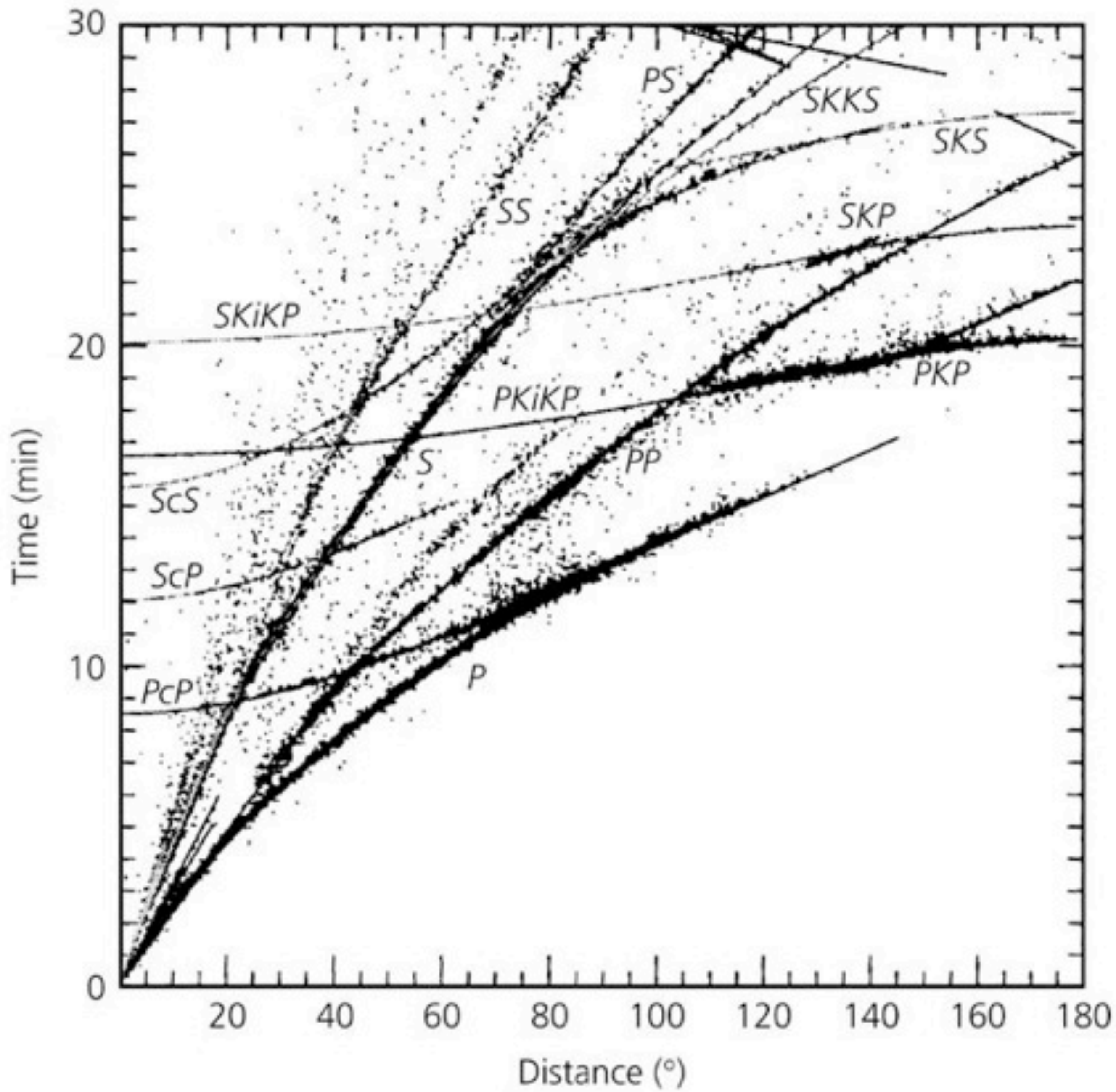
P	P waves
S	S waves
small p	depth phases (P)
small s	depth phases (S)
c	Reflection from CMB
K	wave inside core
i	Reflection from Inner core boundary
I	wave through inner core

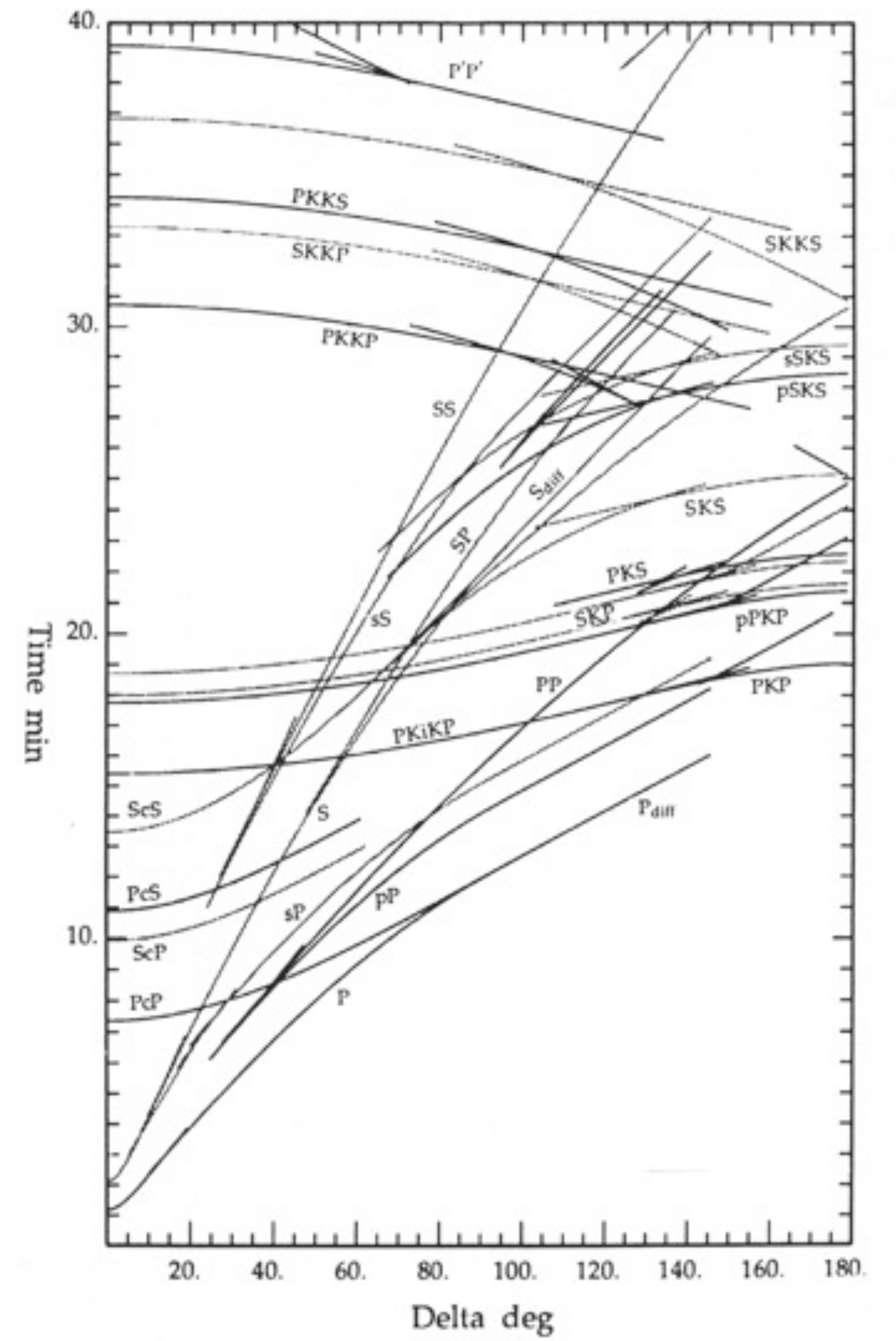
Travel times in the real Earth



Travel times in the Earth

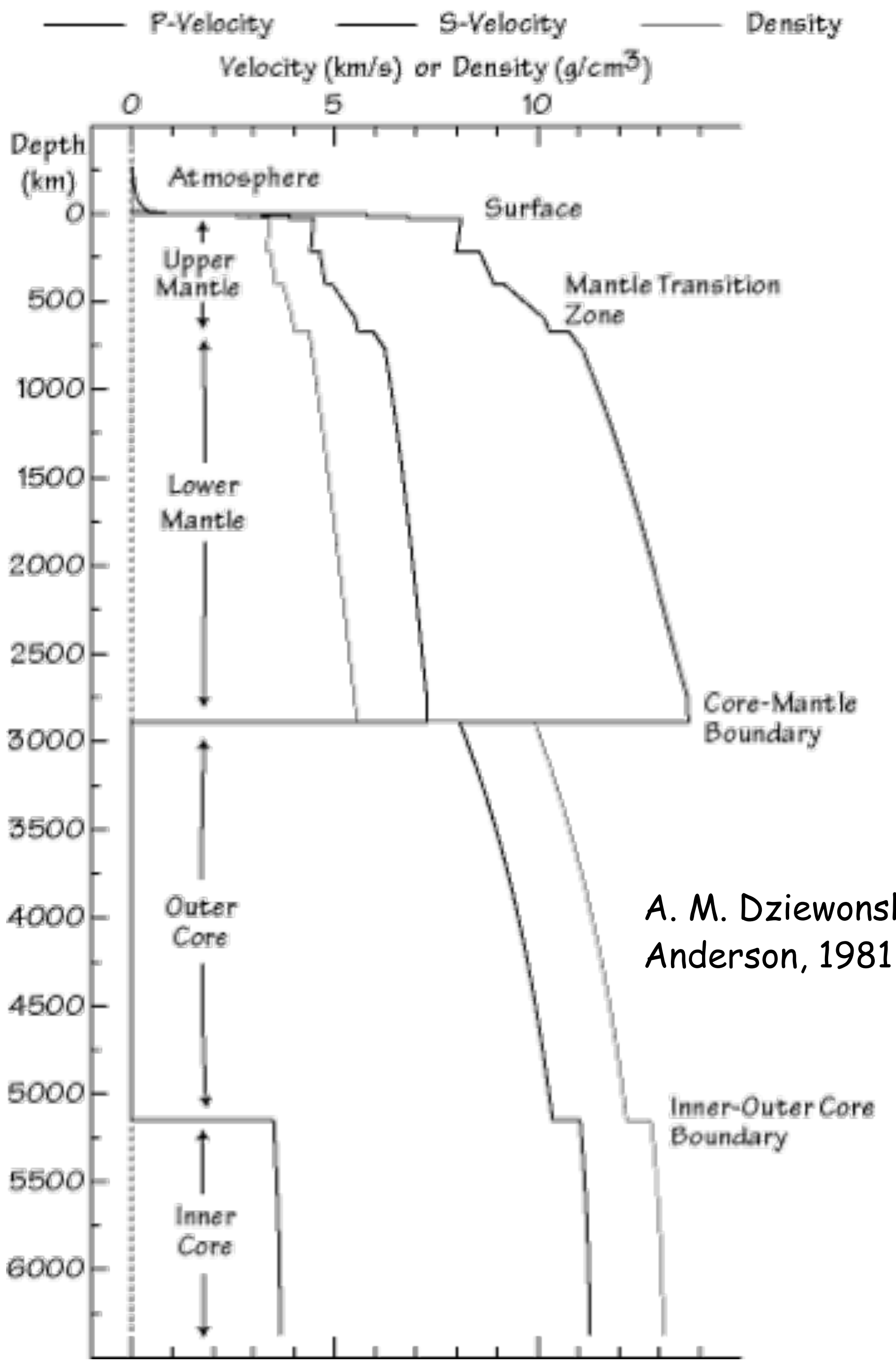
Figure 3.5-3: Travel time data and curves for the IASP91 model.





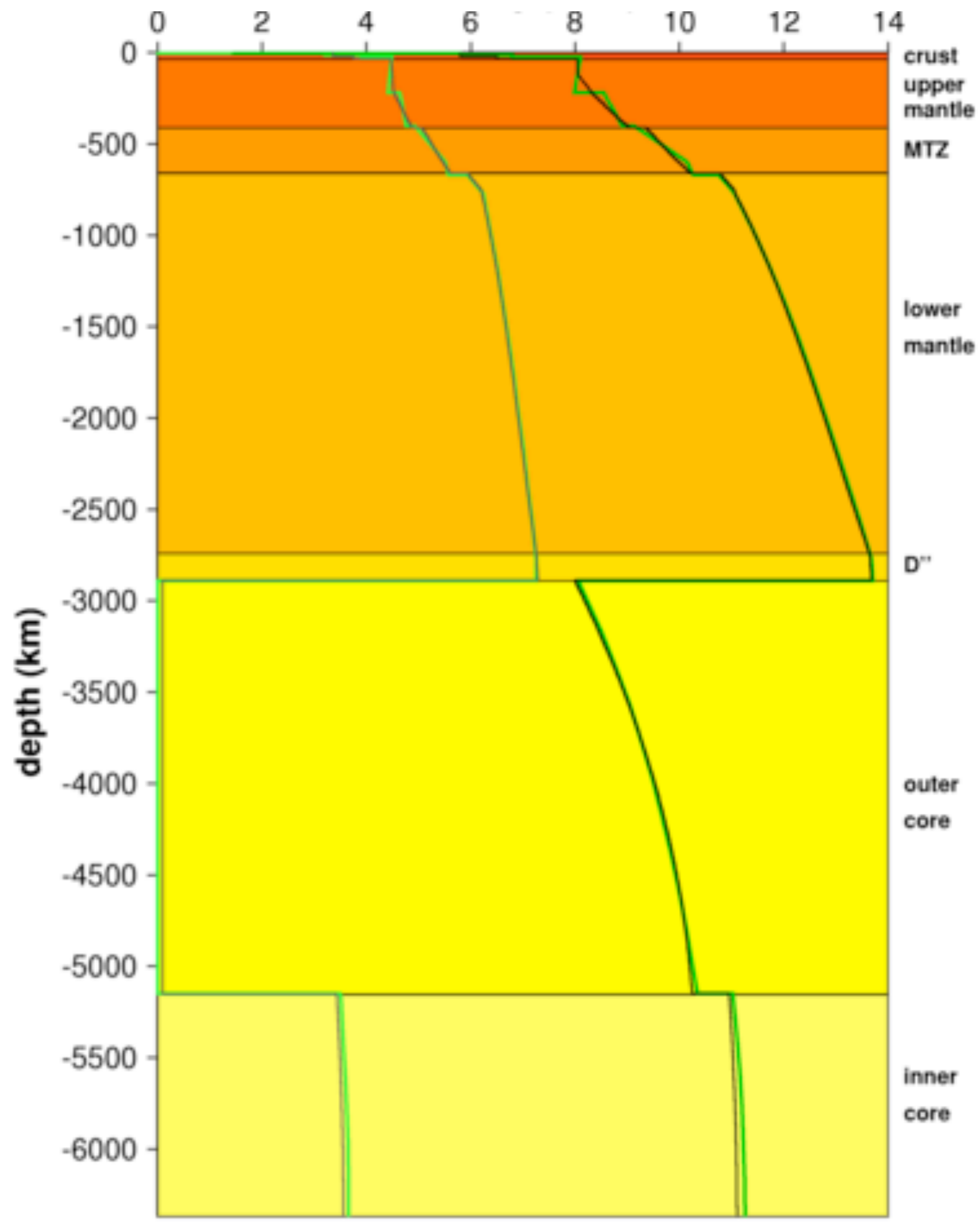
Kennett, B. L. N., and E. R. Engdahl (1991). Traveltimes for global earthquake location and phase identification. Geophysical Journal International 122, 429–465.

Spherically symmetric models



A. M. Dziewonski & D. L. Anderson, 1981

Velocity and density variations within Earth based on seismic observations. The main regions of Earth and important boundaries are labeled. This model was developed in the early 1980's and is called **PREM** for Preliminary Earth Reference Model.



Model **PREM** giving S and P wave velocities (light and dark green lines) in the earth's interior in comparison with the younger **IASP91** model (thin grey and black lines)

http://www.iris.edu/dms/products/emc/models/refModels.htm