SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Surface waves & dispersion

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Surface waves in an elastic half spaces: Rayleigh waves
- Potentials
- Free surface boundary conditions
- Solutions propagating along the surface, decaying with depth
- Lamb's problem

Surface waves in media with depth-dependent properties: Love waves
- Constructive interference in a low-velocity layer
- Dispersion curves
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Free Oscillations
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- Rotational Splitting

The Wave Equation: Potentials

On Waves Propagated along the Plane Surface of an Elastic Solid. *By* Lord RAYLEIGH, D.C.L., F.R.S.

[*Read November 12th*, 1885.]

It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity.*

Denoting the displacements by α, β, γ, and the dilatation by θ, we have the usual equations

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

$$\partial_t^2 \Phi = \alpha^2 \nabla^2 \Phi$$

$$\partial_t^2 \Psi_i = \beta^2 \nabla^2 \Psi_i$$

$\mathbf{u}$ displacement

Φ scalar potential

Ψ_i vector potential

α P-wave speed

β Shear wave speed

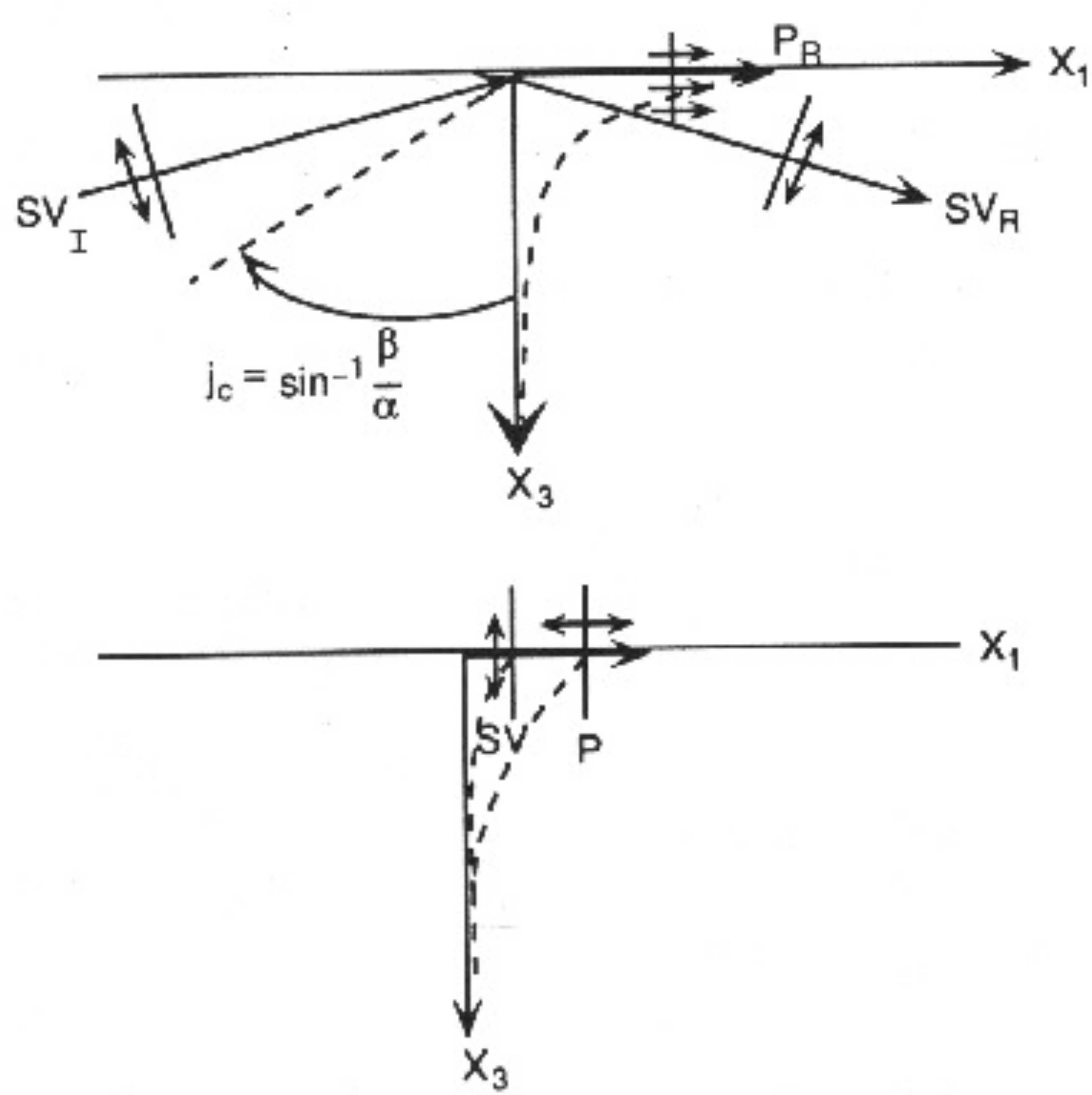
Rayleigh Waves

SV waves incident on a free surface: conversion and reflection

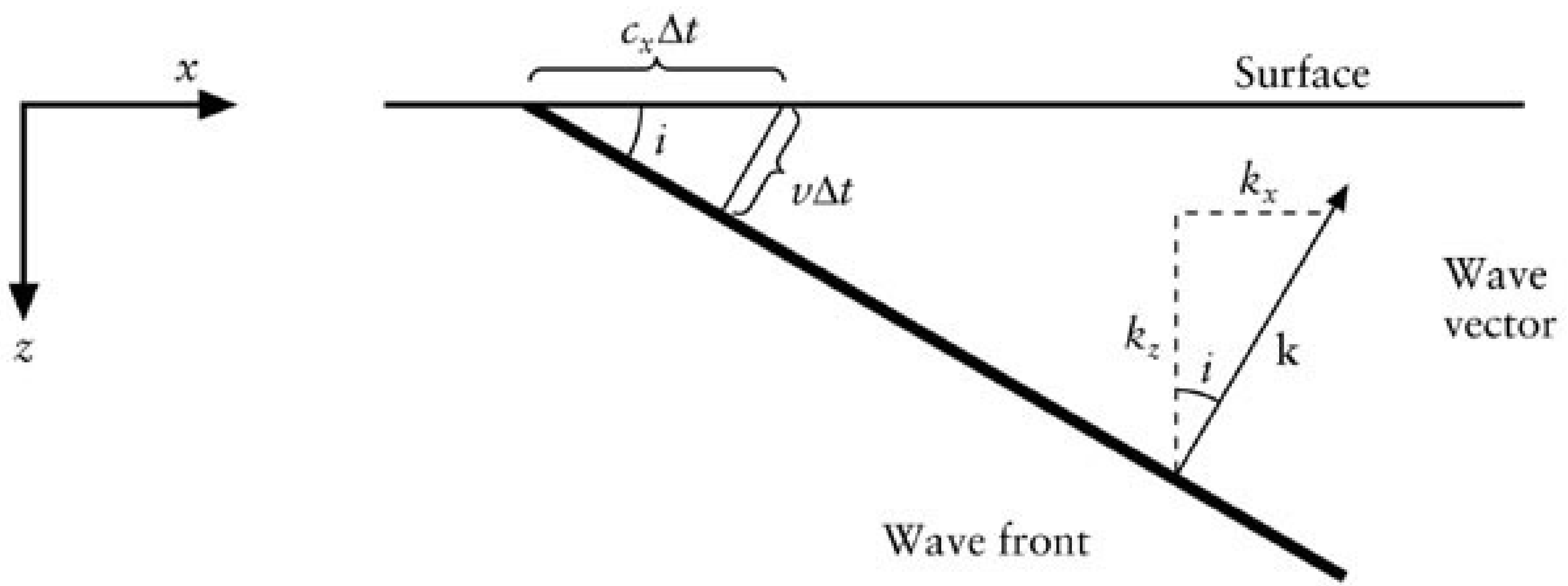
An evanescent P-wave propagates along the free surface decaying exponentially with depth.

The reflected post-crticially reflected SV wave is totally reflected and phase-shifted. These two wave types can only exist together, they both satisfy the free surface boundary condition:

-> Surface waves



Apparent horizontal velocity



$$k_x = k\sin(i) = \omega\frac{\sin(i)}{\alpha} = \frac{\omega}{c}$$

$$k_z = k\cos(i) = \sqrt{k^2 - k_x^2} = \omega\sqrt{\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c}\sqrt{\left(\frac{c}{\alpha}\right)^2 - 1} = k_x r_\alpha$$

In current terminology, k_x is k!

Surface waves: Geometry

We are looking for plane waves traveling along one horizontal coordinate axis, so we can - for example - set

$$\partial_y(.) = 0$$

And consider only wave motion in the x,z plane. Then

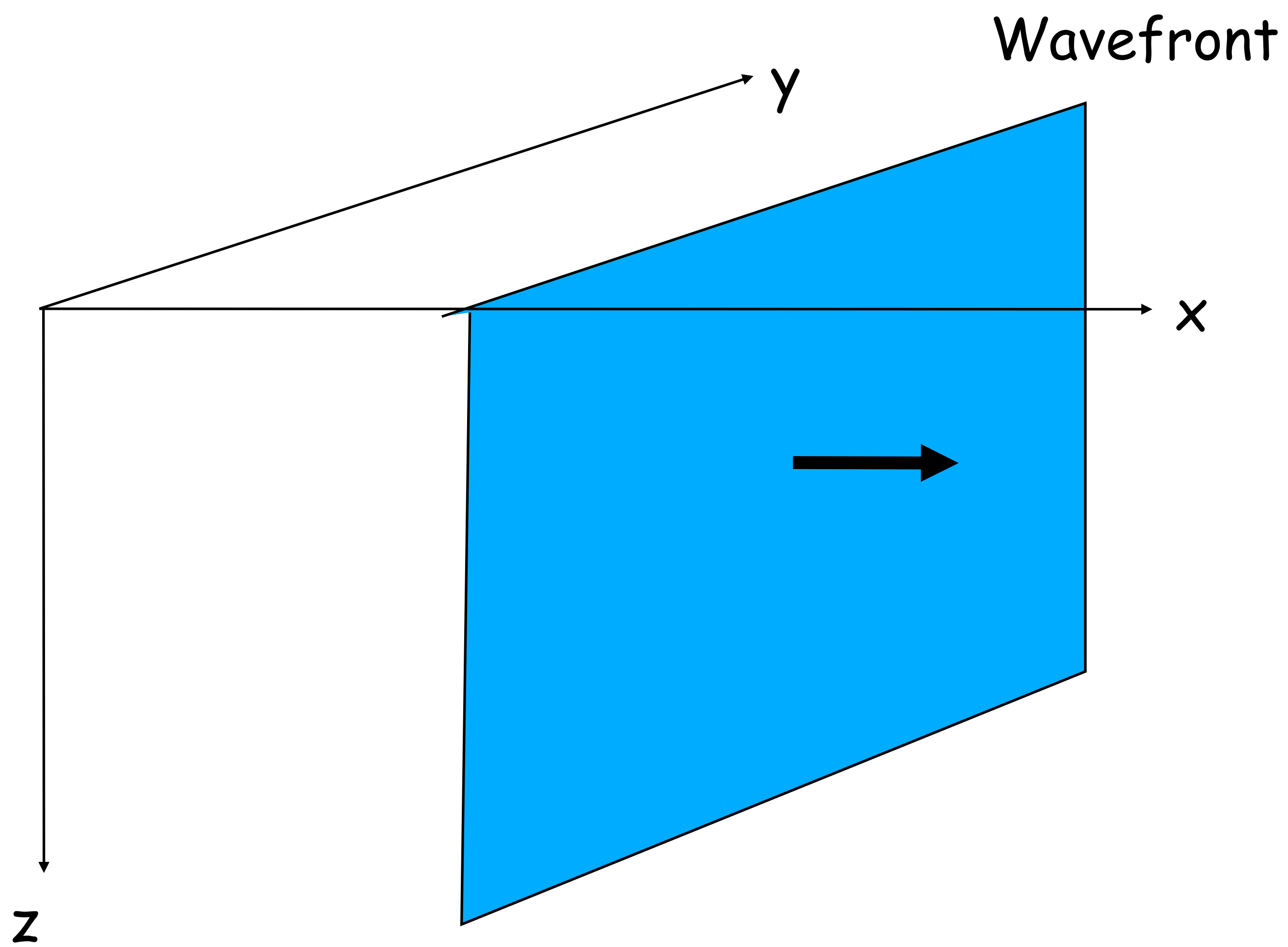
$$u_x = \partial_x \Phi - \partial_z \Psi_y$$

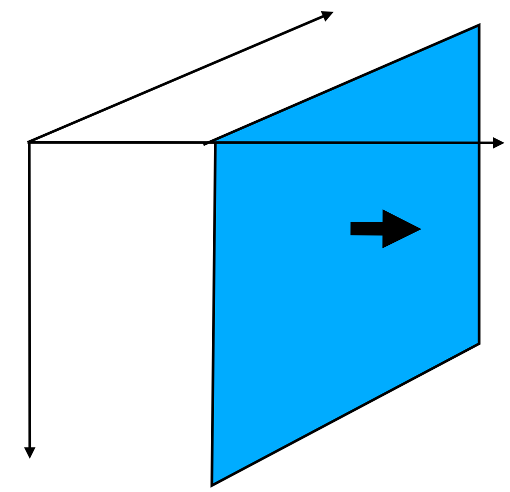
$$u_z = \partial_z \Phi + \partial_x \Psi_y$$

As we only require Ψ_y we set $\Psi_y = \Psi$ from now on. Our trial solution is thus

$$\Phi = A \exp[ik(ct \pm r_\alpha z - x)]$$

$$\Psi = B \exp[ik(ct \pm r_\beta z - x)]$$





With that ansatz one has that, in order to desired solution exists, the coefficients

$$r_\alpha = \pm\sqrt{\frac{c^2}{\alpha^2} - 1} \qquad r_\beta = \pm\sqrt{\frac{c^2}{\beta^2} - 1}$$

have to express a decay along z, i.e.

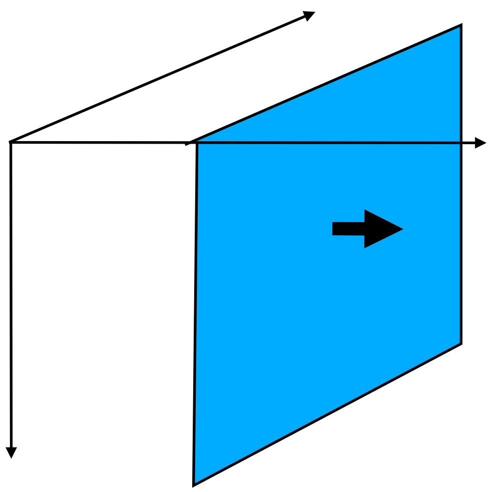
$$c < \beta < \alpha$$

to obtain

$$\Phi = A\exp\left[i(\omega t - kx) - kz\sqrt{1 - \frac{c^2}{\alpha^2}}\right] = A\exp\left(-kz\sqrt{1 - \frac{c^2}{\alpha^2}}\right)\exp\left[i(\omega t - kx)\right]$$

$$\Psi = B\exp\left[i(\omega t - kx) - kz\sqrt{1 - \frac{c^2}{\beta^2}}\right] = B\exp\left(-kz\sqrt{1 - \frac{c^2}{\beta^2}}\right)\exp\left[i(\omega t - kx)\right]$$

Surface waves: Boundary Conditions



Analogous to the problem of finding the reflection-transmission coefficients we now have to satisfy the boundary conditions at the free surface (stress free)

$$\sigma_{xz} = \sigma_{zz} = 0$$

In isotropic media we have

$$\sigma_{zz} = \lambda(\partial_x u_x + \partial_z u_z) + 2\mu\partial_z u_z$$
$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu(\partial_x u_z + \partial_z u_x)$$

where

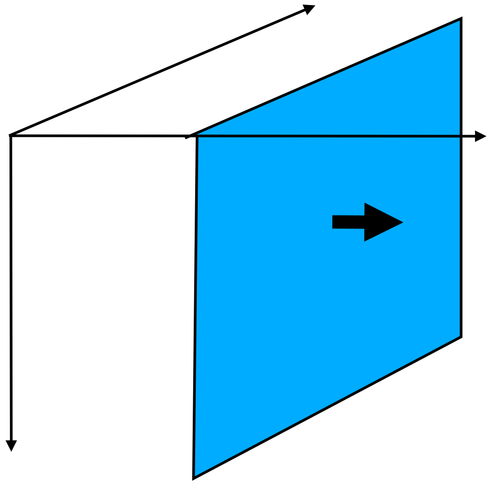
$$u_x = \partial_x\Phi - \partial_z\Psi$$
$$u_z = \partial_z\Phi + \partial_x\Psi$$

and

$$\Phi = A\exp[i(\omega t \pm k r_\alpha z - kx)]$$

$$\Psi = B\exp[i(\omega t \pm k r_\beta z - kx)]$$

Rayleigh waves: solutions



This leads to the following relationship for c, the phase velocity:

$$\left(2-\frac{c^2}{\beta^2}\right)^2 = 4\left(1-\frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{\beta^2}\right)^{\frac{1}{2}}$$

For simplicity we take a fixed relationship between P and shear-wave velocity (Poisson's medium):

$$\alpha = \sqrt{3}\beta$$

… to obtain

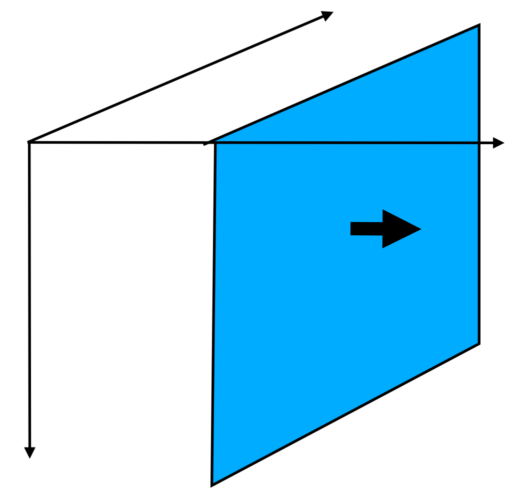
$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + \frac{56}{3}\frac{c^2}{\beta^2} - \frac{32}{3} = 0$$

… and the only root which fulfills the condition $c < \beta$ is

$$c = 0.9194\beta$$

Displacement

Putting this value back into our solutions we finally obtain the displacement in the x-z plane for a plane harmonic surface wave propagating along direction x



$$u_x = C(e^{-0.8475kz} - 0.5773e^{-0.3933kz})\sin k(ct - x)$$

$$u_z = C(-0.8475e^{-0.8475kz} + 1.4679e^{-0.3933kz})\cos k(ct - x)$$

This development was first made by Lord Rayleigh in 1885.

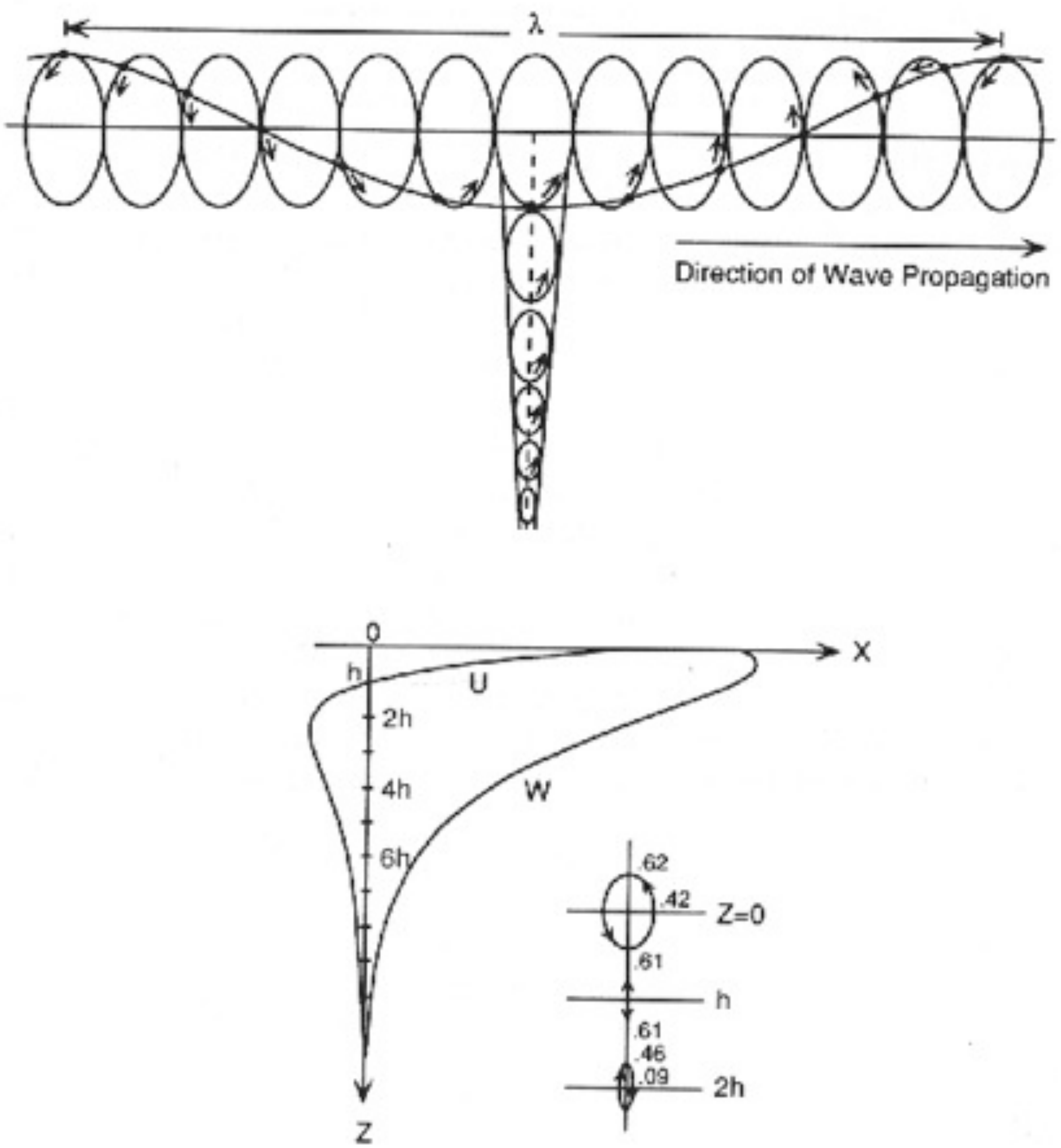
It demonstrates that YES there are solutions to the wave equation propagating along a **free surface!**

Some remarkable facts can be drawn from this particular form:

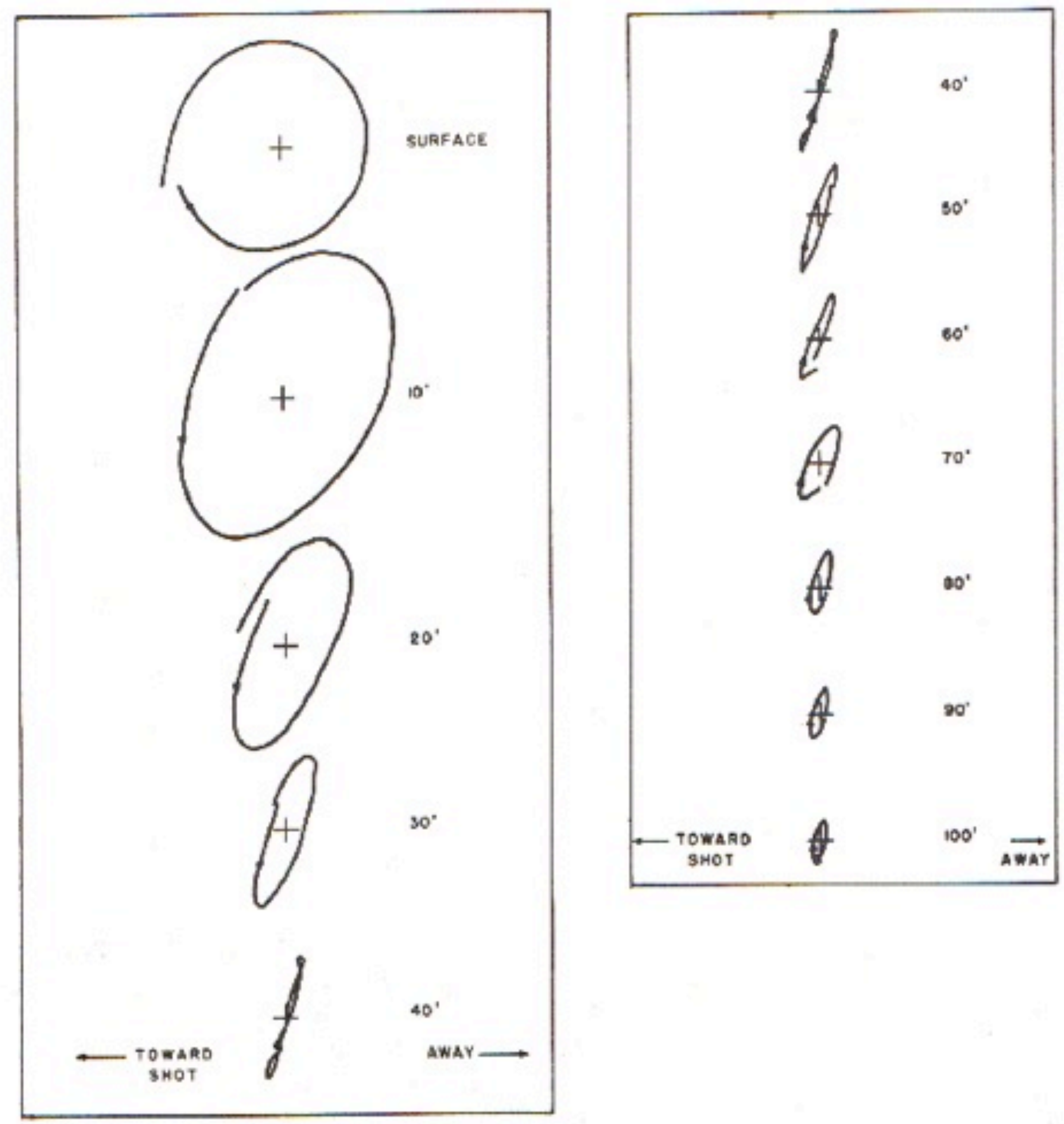
Particle Motion (1)

How does the particle motion look like?

theoretical



experimental



Lamb's Problem and Rayleigh waves

Transient solution to an impulsive vertical point force at the surface of a half space is called **Lamb's problem** (after Horace Lamb, 1904).

-the two components are out of phase by $\pi/2$

– for small values of z a particle describes an ellipse and the motion is retrograde

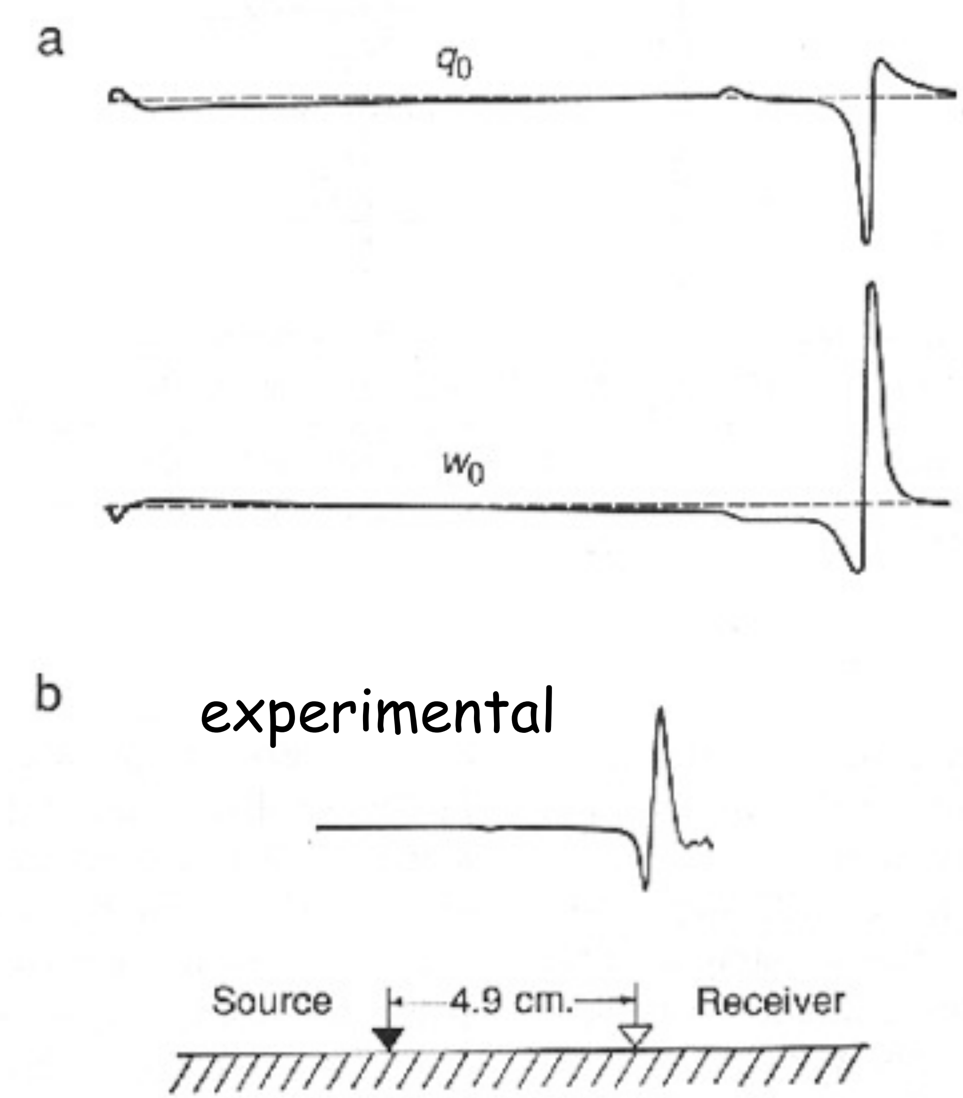
- at some depth z the motion is linear in z

- below that depth the motion is again elliptical but prograde

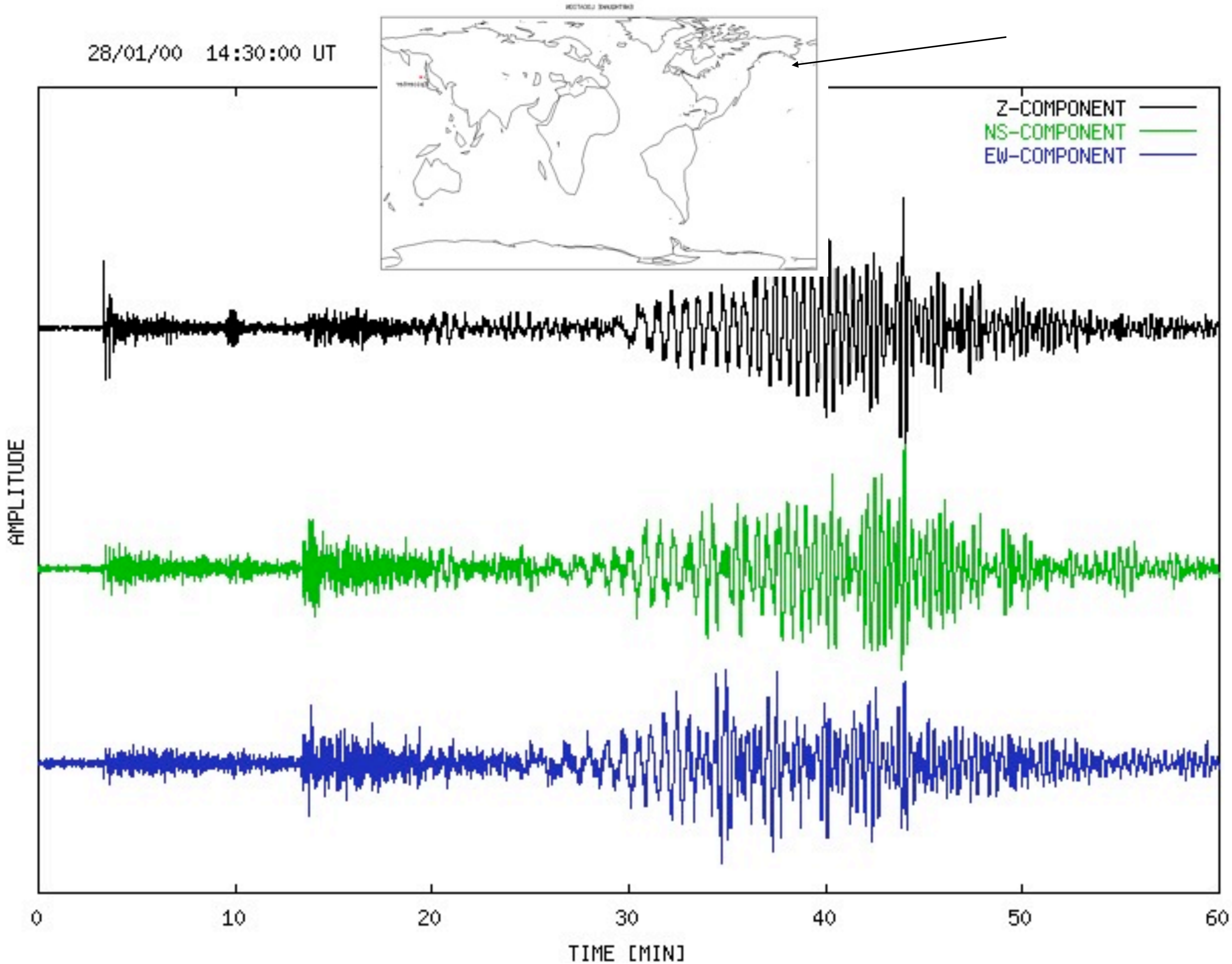
- the phase velocity is independent of k: **there is no dispersion** for a homogeneous half space

- Right Figure: radial and vertical motion for a source at the surface

theoretical



Data Example



Dispersion relation

- In physics, the dispersion relation is the relation between the energy of a system and its corresponding momentum. For example, for massive particles in free space, the dispersion relation can easily be calculated from the definition of kinetic energy:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

- For electromagnetic waves, the energy is proportional to the frequency of the wave and the momentum to the wavenumber. In this case, Maxwell's equations tell us that the dispersion relation for vacuum is linear: $\omega = ck$.

- The name "**dispersion relation**" originally comes from optics. It is possible to make the effective speed of light dependent on wavelength by making light pass through a material which has a non-constant index of refraction, or by using light in a non-uniform medium such as a waveguide. In this case, the waveform will spread over time, such that a narrow pulse will become an extended pulse, i.e. be dispersed.

Dispersion...

- In optics, dispersion is a phenomenon that causes the separation of a wave into spectral components with different wavelengths, due to a dependence of the wave's speed on its wavelength. It is most often described in light waves, but it may happen to any kind of wave that interacts with a medium or can be confined to a waveguide, such as sound waves. There are generally two sources of dispersion: **material dispersion**, which comes from a frequency-dependent response of a material to waves; and **waveguide dispersion**, which occurs when the speed of a wave in a waveguide depends on its frequency.

- In optics, the phase velocity of a wave v in a given uniform medium is given by: **v=c/n**, where c is the speed of light in a vacuum and n is the refractive index of the medium. In general, the refractive index is some function of the frequency v of the light, thus n = n(f), or alternately, with respect to the wave's wavelength n = n(λ). For visible light, most transparent materials (e.g. glasses) have a refractive index n decreases with increasing wavelength λ (**dn/dλ<0**, i.e. **dv/dλ>0**). In this case, the medium is said to have **normal dispersion** and if the index increases with increasing wavelength the medium has **anomalous dispersion**.

Demonstration: sum two harmonic waves with slightly different angular frequencies and wavenumbers:

$$u(x, t) = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$

$$\omega_1 = \omega + \delta\omega \qquad \omega_2 = \omega - \delta\omega \qquad \omega \gg \delta\omega$$

$$k_1 = k + \delta k \qquad k_2 = k - \delta k \qquad k \gg \delta k$$

Add the two cosines:

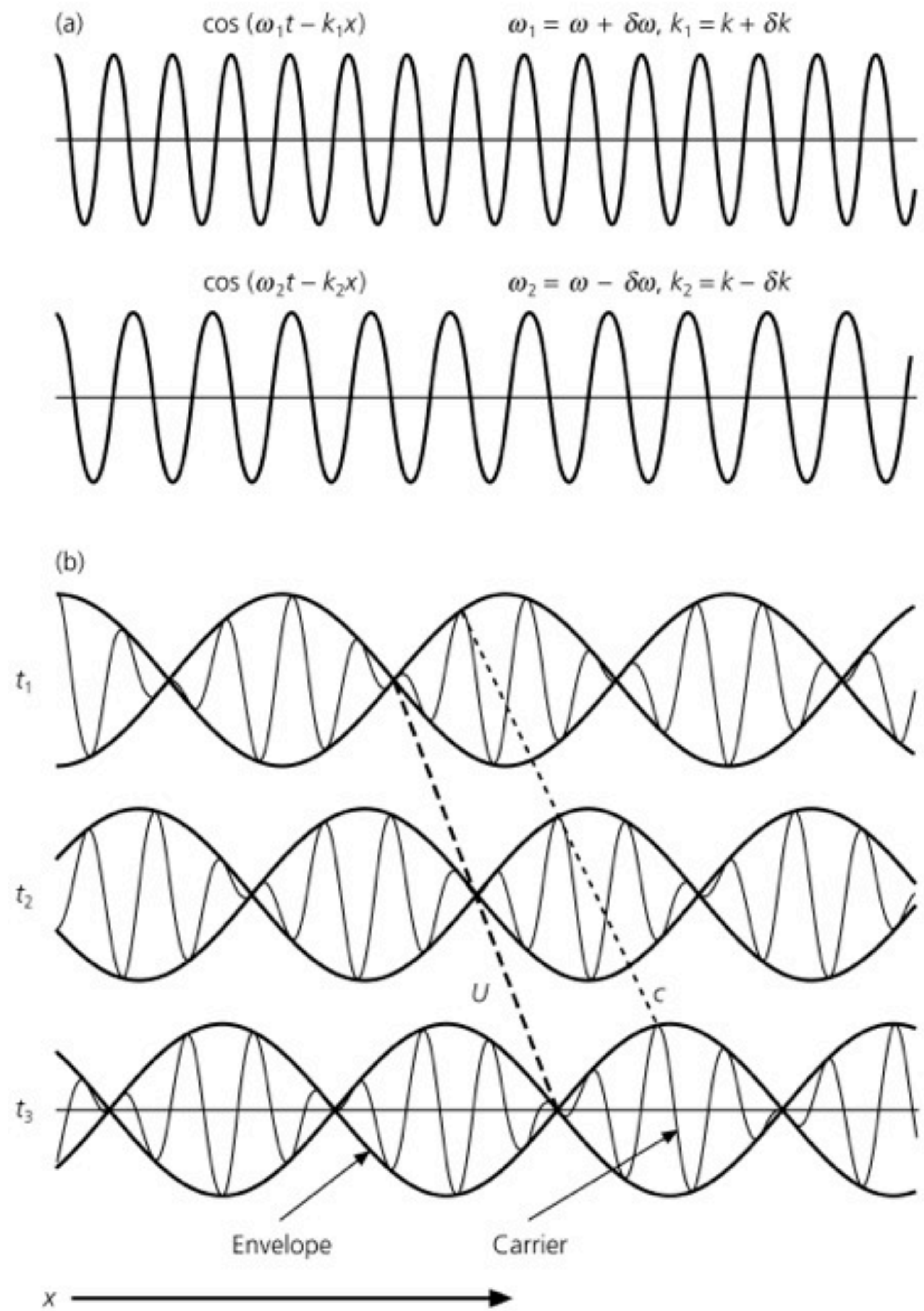
$$\begin{aligned} u(x, t) &= \cos(\omega t + \delta\omega t - kx - \delta kx) \\ &\quad + \cos(\omega t - \delta\omega t - kx + \delta kx) \\ &= 2\cos(\omega t - kx)\cos(\delta\omega t - \delta kx) \end{aligned}$$

The envelope (beat) has a *group velocity*:

$$U = \delta\omega / \delta k$$

The individual peaks move with a *phase velocity*:

$$c = \omega / k$$



Fourier domain

Fourier transform: $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$

Inverse Fourier transform: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$

$F(\omega) = A(\omega)e^{i\phi(\omega)}$

with a magnitude, $A(\omega) = |F(\omega)|$, and phase, $\phi(\omega)$.

So the Fourier transform represents a time series by two real functions of angular frequency: the *amplitude spectrum,* $A(\omega)$, and the *phase spectrum,* $\phi(\omega)$.

The displacements are: $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)\exp i[\omega t - k(\omega)x + \phi_i(\omega)]\,d\omega$

The phase has two parts (propagation and initial phase): $\Phi(\omega) = \omega t - k(\omega)x + \phi_i(\omega)$

The phase velocity $c(\omega) = \omega/k(\omega)$ describes wave surfaces of constant phase (individual peaks).

To find the group velocity of energy propagation in the angular frequency band between $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$, first approximate the wavenumber $k(\omega)$ by the first term of a Taylor series about ω_0:

$$k(\omega) \approx k(\omega_0) + \frac{dk}{d\omega}\bigg|_{\omega_0} (\omega - \omega_0)$$

This gives: $$u(x, t) \approx \frac{1}{2\pi} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} A(\omega) \exp\left[i\left(\omega t - k(\omega_0)x - \frac{dk}{d\omega}\bigg|_{\omega_0} (\omega - \omega_0)x + \phi_i(\omega) \right)\right] d\omega$$

$$u(x, t) \approx \frac{1}{2\pi} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} A(\omega) \exp\left[i\left((\omega - \omega_0)(t - \frac{dk}{d\omega}\bigg|_{\omega_0} x) + (\omega_0 t - k(\omega_0)x) + \phi_i(\omega) \right)\right] d\omega$$

Compare to the simple situation of two cosine waves:

$$u(x, t) = 2\cos(\omega t - kx)\cos(\delta\omega t - \delta kx)$$

Similar to the cosine waves, the group velocity is defined as $U(\omega) = \dfrac{d\omega}{dk}$

Group velocity

- Another consequence of dispersion manifests itself as a temporal effect. The phase velocity is the velocity at which the phase of any one frequency component of the wave will propagate. This is not the same as the **group velocity of the wave, which is the rate that changes in amplitude** (known as the envelope of the wave) will propagate. The group velocity v_g is related to the phase velocity v by, for a homogeneous medium (here λ is the wavelength in vacuum, not in the medium):

-

$$v_g = \frac{d\omega}{dk} = \frac{d(vk)}{dk} = v + k\frac{dv}{dk} = v - \lambda\frac{dv}{d\lambda}$$

and thus in the normal dispersion case

v_g is always < v !

Dispersion relation

In classical mechanics, the Hamilton's principle the perturbation scheme applied to an averaged Lagrangian for an harmonic wave field gives a characteristic equation: $\Delta(\omega, k_i)=0$

Transverse wave in a string

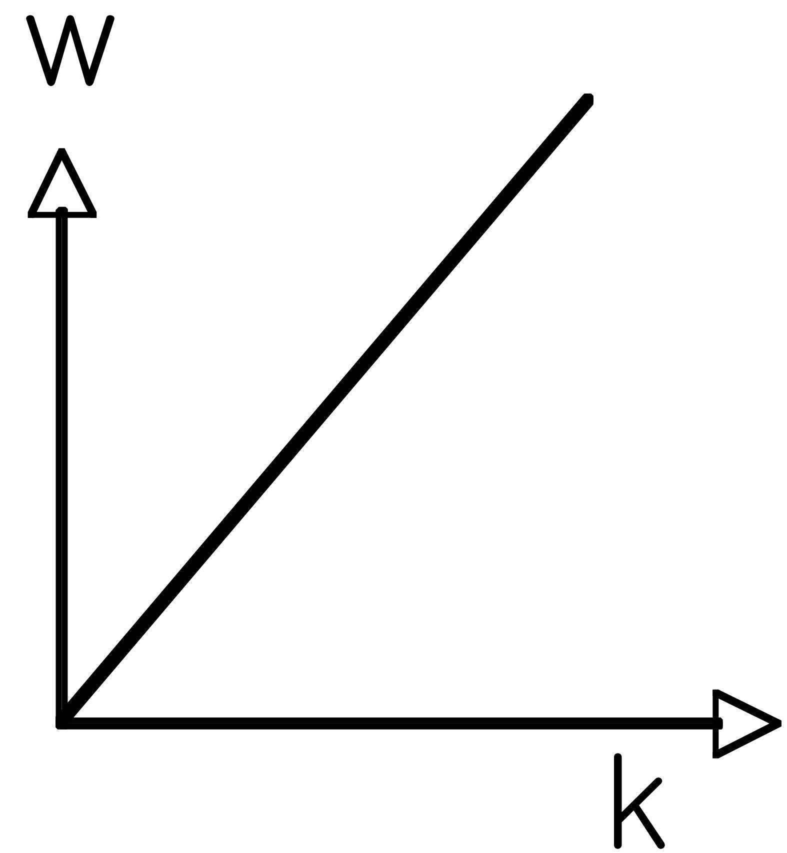
$$\left(\frac{\partial^2}{\partial x^2} - \frac{\mu}{F}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$

Acoustic wave

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{B}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$

Longitudinal wave in a rod

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$



Dispersion examples

- Discrete systems: lattices

- Stiff systems: rods and thin plates

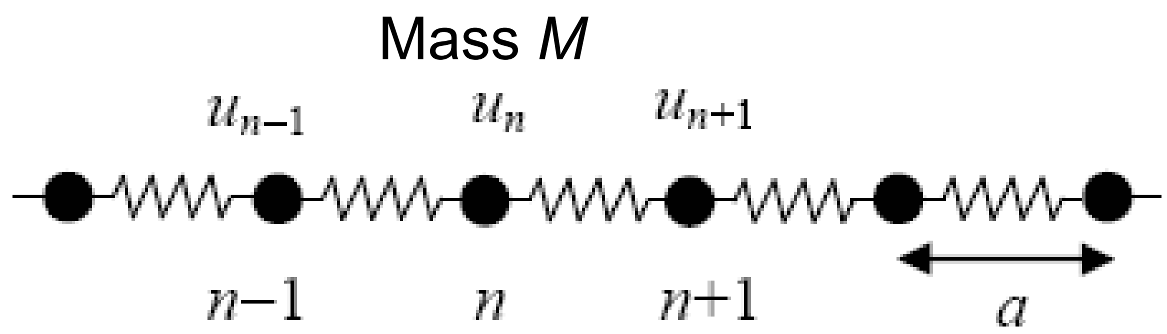
- Boundary waves: plates and rods

  Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

Monatomic 1D lattice

Let us examine the simplest periodic system within the context of harmonic approximation ($F = dU/du = Cu$) - a one-dimensional crystal lattice, which is a sequence of masses m connected with springs of force constant C and separation a.

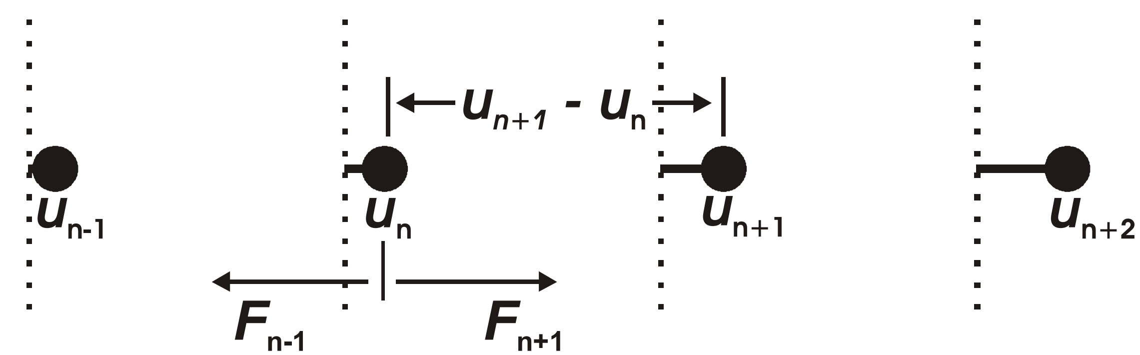
The collective motion of these springs will correspond to solutions of a wave equation. Note: by construction we can see that 3 types of wave motion are possible, 2 transverse, 1 longitudinal (or compressional)



How does the system appear with a longitudinal wave?:

The force exerted on the n-th atom in the lattice is given by

$F_n = F_{n+1,n} - F_{n-1,n} = C[(u_{n+1} - u_n) - (u_n - u_{n-1})]$.



Applying Newton's second law to the motion of the n-th atom we obtain

$$M\frac{d^2u_n}{dt^2} = F_n = -C(2u_n - u_{n+1} - u_{n-1})$$

Note that we neglected hereby the interaction of the n-th atom with all but its nearest neighbors. A similar equation should be written for each atom in the lattice, resulting in N coupled differential equations, which should be solved simultaneously (N - total number of atoms in the lattice). In addition the boundary conditions applied to end atoms in the lattice should be taken into account.

Monatomic 1D lattice - continued

Now let us attempt a solution of the form: $u_n = Ae^{i(kx_n - \omega t)}$,

where x_n is the equilibrium position of the n-th atom so that $x_n = na$. This equation represents a traveling wave, in which all atoms oscillate with the same frequency ω and the same amplitude A and have a wavevector k. Now substituting the guess solution into the equation and canceling the common quantities (the amplitude and the time-dependent factor) we obtain

$$M(-\omega^2)e^{ikna} = -C[2e^{ikna} - e^{ik(n+1)a} - e^{ik(n-1)a}].$$

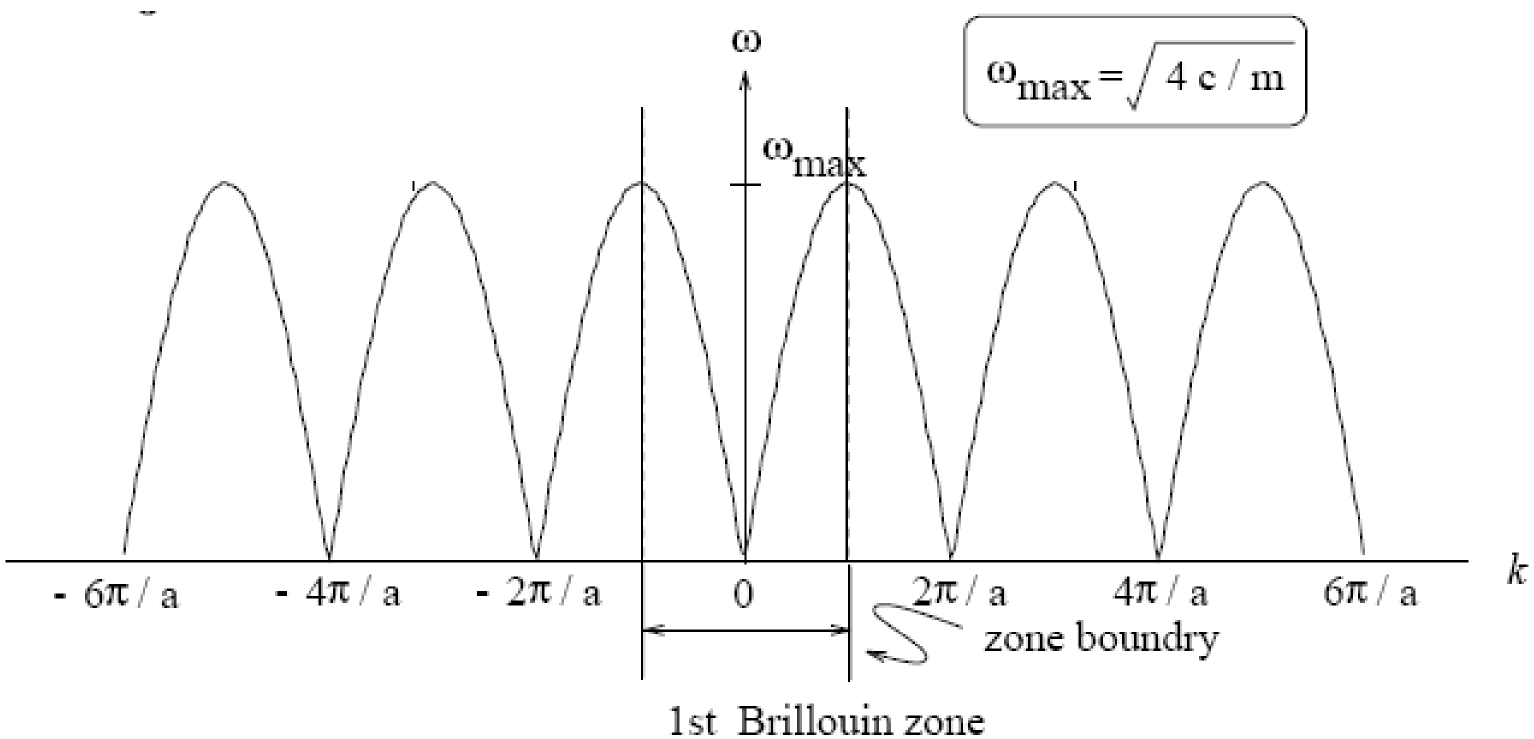
This equation can be further simplified by canceling the common factor e^{ikna} , which leads to

$$M\omega^2 = C\left(2 - e^{ika} - e^{-ika}\right) = 2C(1 - \cos ka) = 4C\sin^2\frac{ka}{2}.$$

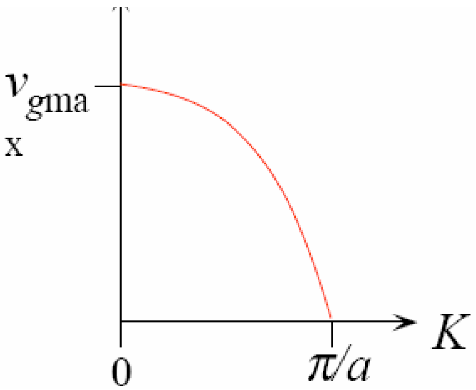
We find thus the dispersion relation for the frequency:

$$\omega = \sqrt{\frac{4C}{M}}\left|\sin\frac{ka}{2}\right|$$

which is the relationship between the frequency of vibrations and the wavevector k. The dispersion relation has a number of important properties.



Monatomic 1D lattice – continued



Phase and group velocity. The phase velocity is defined by

$v_p = \dfrac{\omega}{k}$ and the group velocity by $v_g = \dfrac{d\omega}{dk}$

The physical distinction between the two velocities is that v_p is the velocity of propagation of the plane wave, whereas the v_g is the velocity of the propagation of the wave packet. The latter is the velocity for the propagation of energy in the medium. For the particular dispersion relation $\omega = \sqrt{\dfrac{4C}{M}}\left|\sin\dfrac{ka}{2}\right|$ the group velocity is given by $v_g = \sqrt{\dfrac{Ca^2}{M}}\cos\dfrac{ka}{2}$.

Apparently, the group velocity is zero at the edge of the zone where $k = \pm \pi/a$. Here the wave is standing and therefore the transmission velocity for the energy is zero.

Long wavelength limit. The long wavelength limit implies that $\lambda >> a$. In this limit $ka << 1$.

We can then expand the sine in 'ω' and obtain for the positive frequencies: $\omega = \sqrt{\dfrac{C}{M}}ka$.

We see that the frequency of vibration is proportional to the wavevector. This is equivalent to the statement that velocity is independent of frequency. In this case:

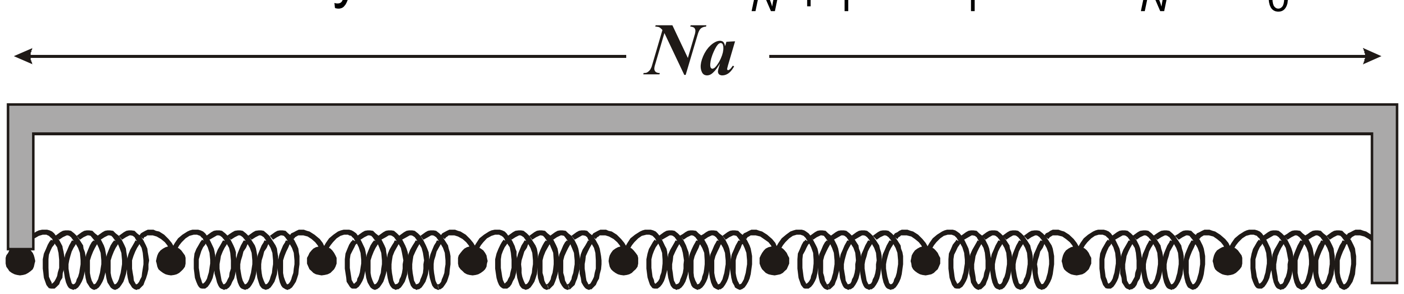
$v_p = \dfrac{\omega}{k} = \sqrt{\dfrac{C}{M}}a$. This is the velocity of sound for the one dimensional lattice which is consistent with the expression we obtained earlier for elastic waves.

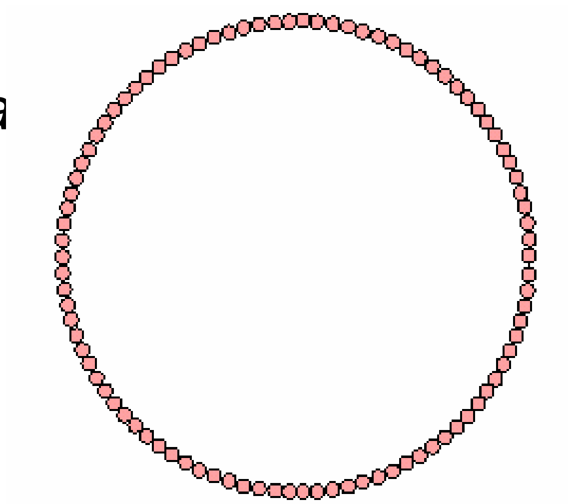
Monatomic 1D lattice – continued

<u>Finite chain – Born – von Karman periodic boundary condition</u>.

Unlike a continuum, there is only a finite number of distinguishable vibrational modes. But how many?

Let us impose on the chain ends the Born – von Karman periodic boundary conditions specified as following: we simply join the two remote ends by one more spring in a ring or device in the figure below forcing atom N to interact with ion 1 via a spring with a spring constant C. If the atoms occupy sites a, $2a$, …, Na The boundary condition is $u_{N+1} = u_1$ or $u_N = u_0$.

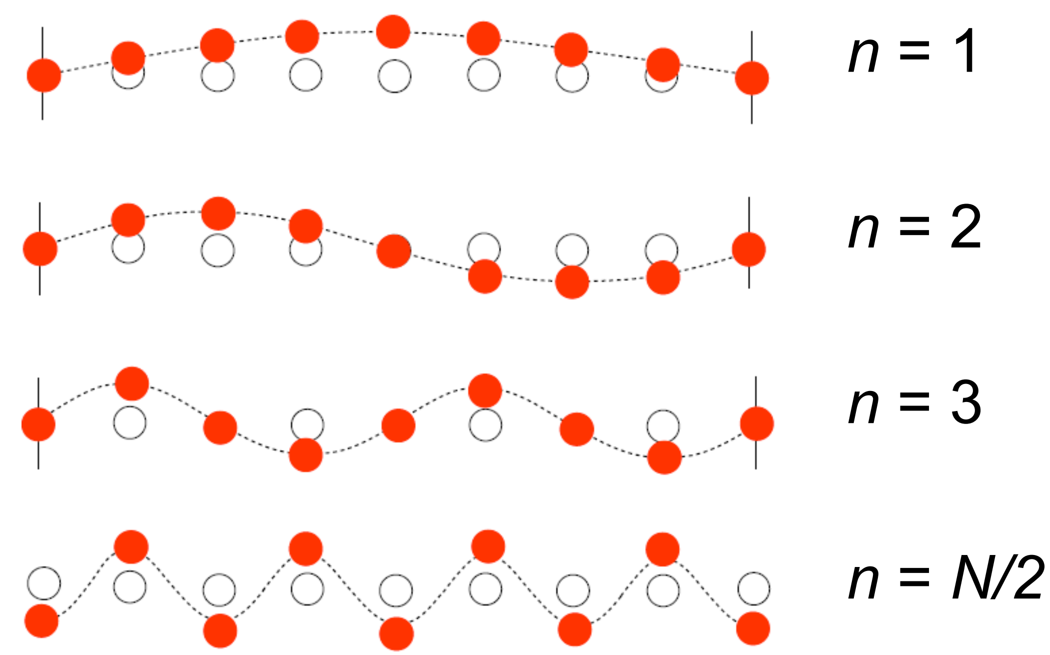


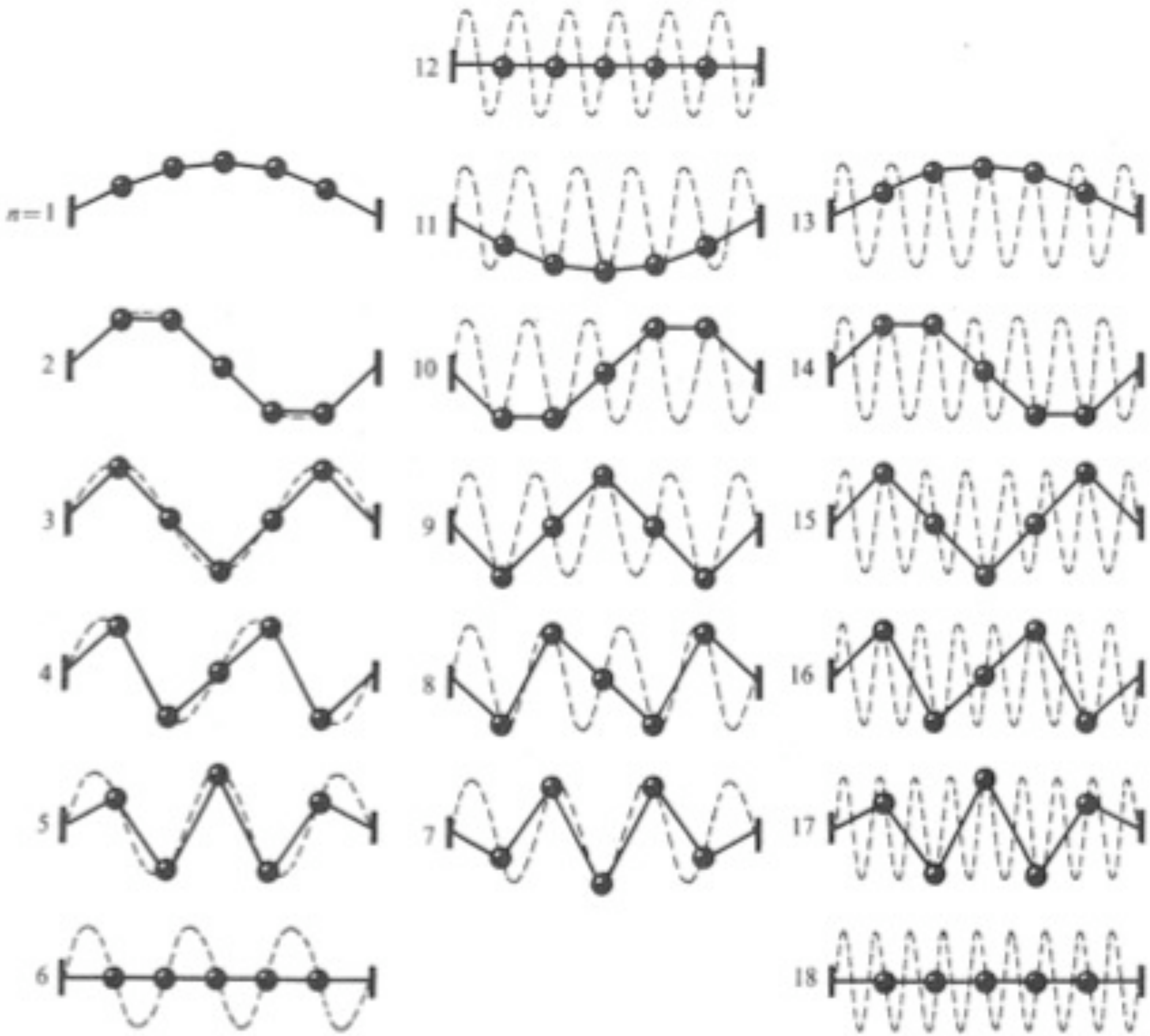


With the displacement solution of the form $u_n = A\exp[i(kna\text{-}wt)]$, the periodic boundary condition requires that $\exp(\pm ikNa) = 1$, which in turn requires 'k' to have the form:

$$k = \frac{2\pi}{a}\frac{n}{N} \quad \text{(n – an integer), and} \quad -\frac{N}{2} \le n \le \frac{N}{2}, \quad \text{or}$$

$$k = \pm\frac{2\pi}{Na}, \pm\frac{4\pi}{Na}, \pm\frac{6\pi}{Na}, \ldots, \pm\frac{\pi}{a} \qquad (N \text{ values of } k).$$




12
n=1
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18

Diatomic 1D lattice

We can treat the motion of this lattice in a similar fashion as for the monatomic lattice. However, in this case, because we have two different kinds of atoms, we should write two equations of motion:

$$M_1 \frac{d^2 u_n}{dt^2} = -C(2u_n - u_{n+1} - u_{n-1})$$

$$M_2 \frac{d^2 u_{n+1}}{dt^2} = -C(2u_{n+1} - u_{n+2} - u_n)$$

In analogy with the monatomic lattice we are looking for the solution in the form of traveling mode for the two atoms:

$$\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} A_1 e^{ikna} \\ A_2 e^{ik(n+1)a} \end{bmatrix} e^{-i\omega t} \qquad \text{in matrix form.}$$

Substituting this solution into the equations of the previous slide we obtain:

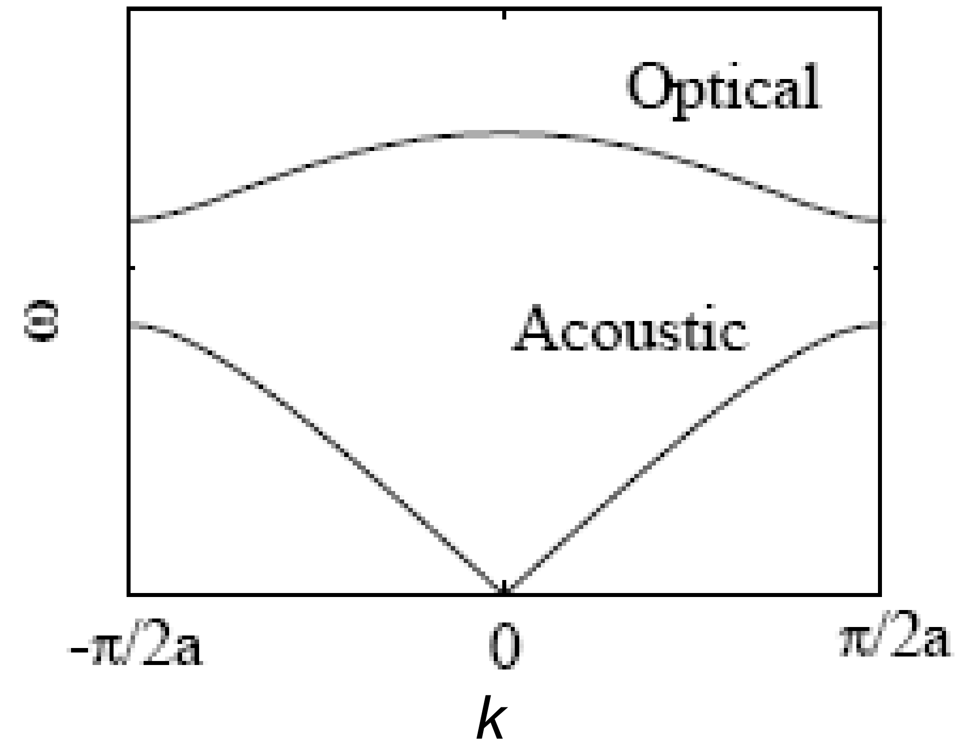
$$\begin{bmatrix} 2C - M_1\omega^2 & -2C\cos ka \\ -2C\cos ka & 2C - M_2\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0.$$

This is a system of linear homogeneous equations for the unknowns A_1 and A_2. A nontrivial solution exists only if the determinant of the matrix is zero. This leads to the secular equation

$$\left(2C - M_1\omega^2\right)\left(2C - M_2\omega^2\right) - 4C^2\cos^2 ka = 0.$$

This is a quadratic equation, which can be readily solved:

$$\omega^2 = C\left(\frac{1}{M_1}+\frac{1}{M_2}\right) \pm C\sqrt{\left(\frac{1}{M_1}+\frac{1}{M_2}\right)^2 - \frac{4\sin^2 ka}{M_1 M_2}}$$



Depending on sign in this formula there are two different solutions corresponding to two different dispersion curves, as is shown in the figure:

The lower curve is called the ***acoustic*** *branch*, while the upper curve is called the ***optical*** *branch*.

The acoustic branch begins at $k = 0$ and $\omega = 0$, and as k $\Rightarrow$ 0:

$$\omega_a(0) = \sqrt{\frac{C}{2(M_1+M_2)}}\cdot ka$$

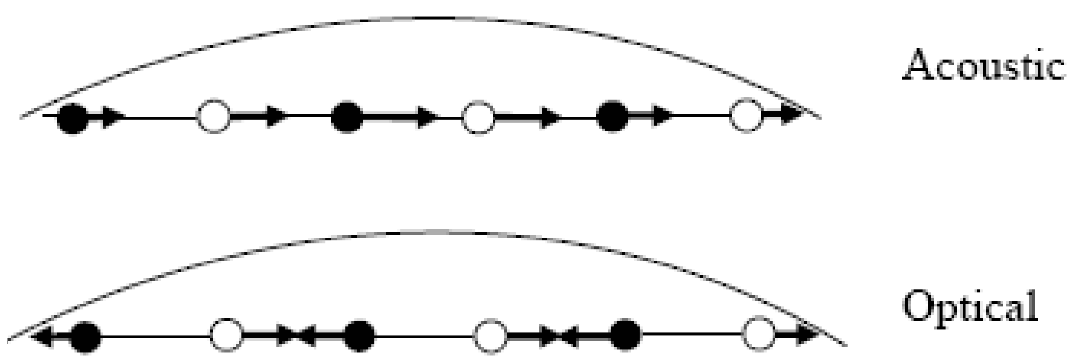
With increasing *k* the frequency increases in a linear fashion. This is why this branch is called *acoustic*: it corresponds to elastic waves, or sound. Eventually, this curve saturates at the edge of the Brillouin zone. On the other hand, the optical branch Has a nonzero frequency at zero *k*,

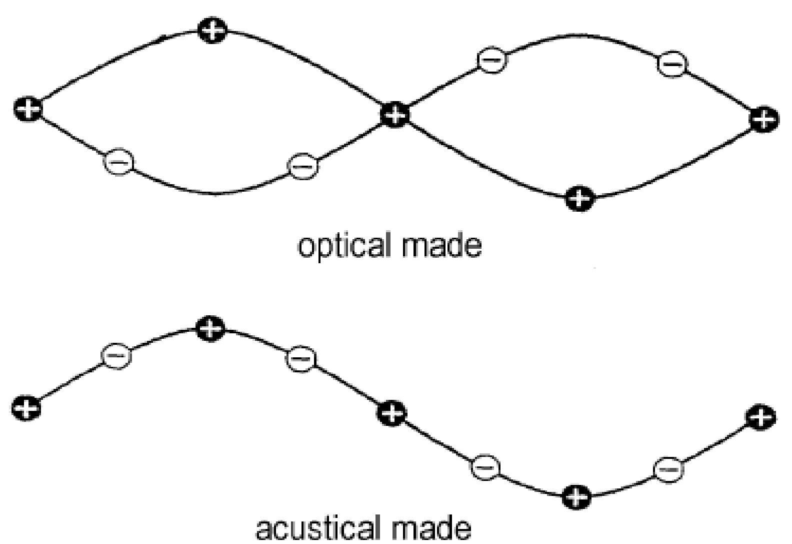
$$\omega_o = \sqrt{2C\left(\frac{1}{M_1}+\frac{1}{M_2}\right)}$$

and it does not change much with *k.*

Another feature of the dispersion curves is the existence of a forbidden gap between $\omega_a = (2C/M_1)^{1/2}$ and $\omega_o = (2C/M_2)^{1/2}$ at the zone boundaries ($k = \pm \pi/2a$).
The forbidden region corresponds to frequencies in which lattice waves cannot propagate through the linear chain without attenuation. It is interesting to note that a similar situation also exists in the energy band scheme of a solid to be discussed later.

The distinction between the acoustic and optical branches of lattice vibrations can be seen most clearly by comparing them at $k = 0$ (infinite wavelength). As follows from the equations of motion, for the acoustic branch $\omega = 0$ and $A_1 = A_2$. So, in this limit the two atoms in the cell have the same amplitude and phase. Therefore, the molecule oscillates as a rigid body, as shown in the left figure for the acoustic mode.



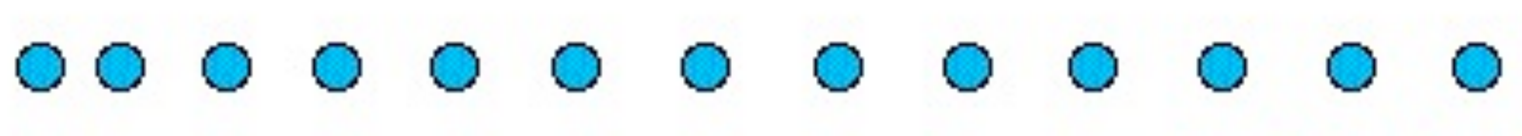


On the other hand, for the optical vibrations, by substituting ω_o we obtain for $k = 0$:

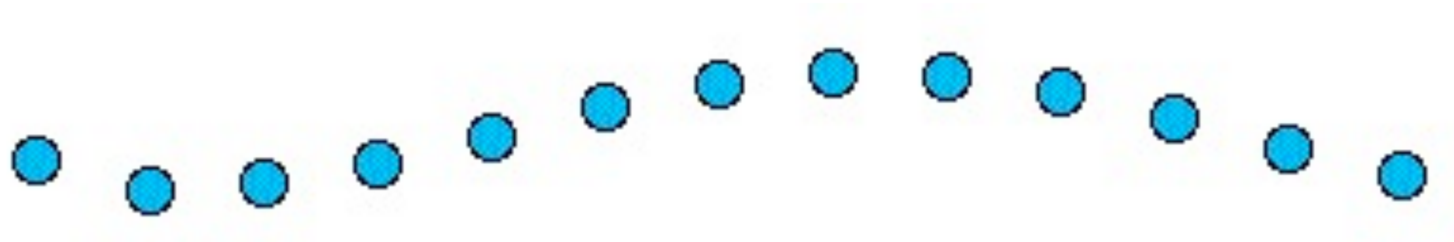
$$M_1A_1 + M_2A_2 = 0 \quad (M_1/M_2 = -A_2/A_1).$$

This implies that the optical oscillation takes place in such a way that the center of mass of a molecule remains fixed. The two atoms move in out of phase as shown. The frequency of these vibrations lies in the infrared region (10^{12} to 10^{14} Hz) which is the reason for referring to this branch as *optical*. If the two atoms carry opposite charges, we may excite a standing wave motion with the electric field of a light wave.

Acoustic and optical modes

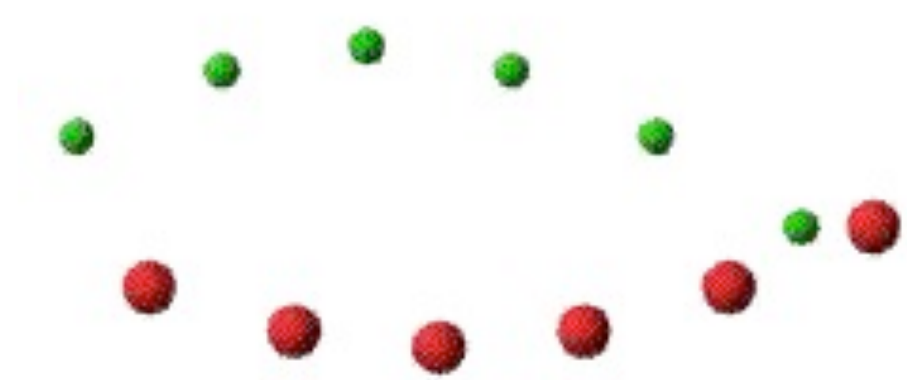


Monoatomic chain
acoustic longitudinal mode



Monoatomic chain
acoustic transverse mode

Diatomic chain
acoustic transverse mode



Diatomic chain
optical transverse mode

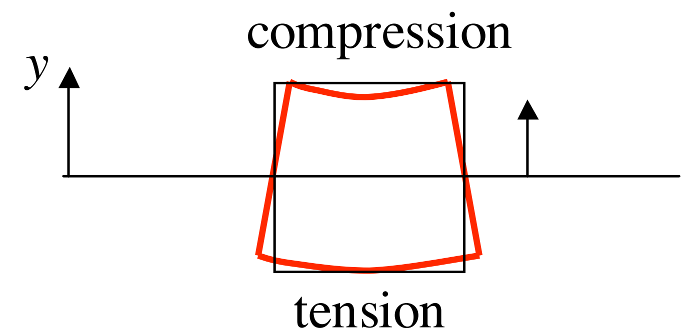
Dispersion examples

- ☑ Discrete systems: lattices
- ☑ Stiff systems: rods and thin plates
- ☑ Boundary waves: plates and rods

  Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

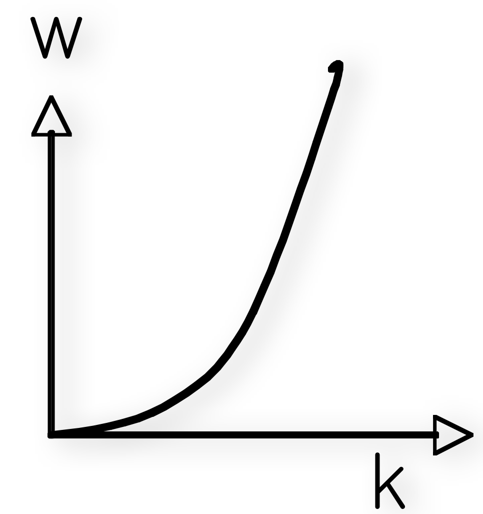
Stiffness...

- How "**stiff**" or "flexible" is a material? It depends on whether we pull on it, twist it, bend it, or simply compress it. In the simplest case the material is characterized by two independent "stiffness constants" and that different combinations of these constants determine the response to a pull, twist, bend, or pressure.



Euler-Bernoulli equation

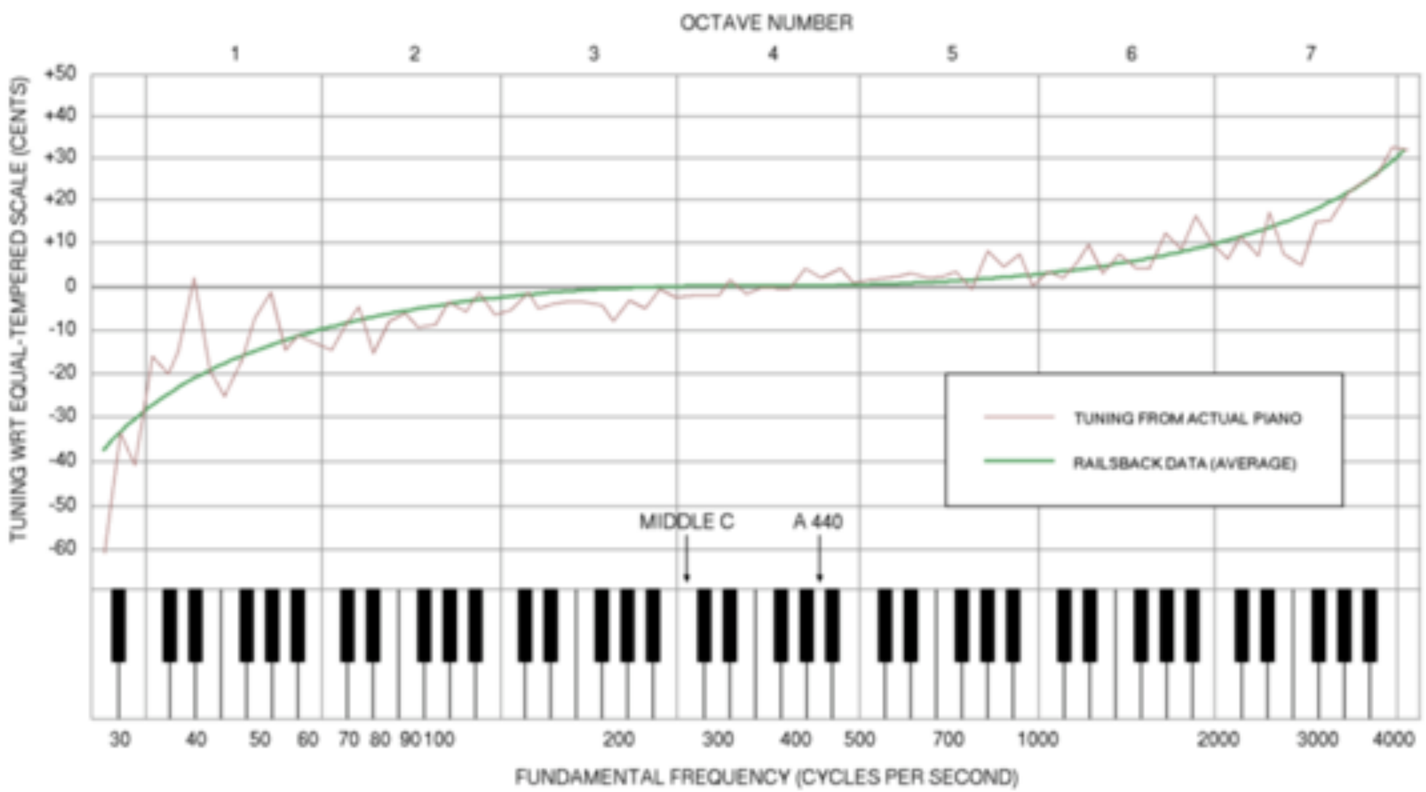
$$\left(\frac{\partial^4}{\partial x^4} - \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2}\right)w = 0 \Rightarrow \quad \omega = \pm k^2\sqrt{\frac{EI}{\rho A}}$$



Stiffness...

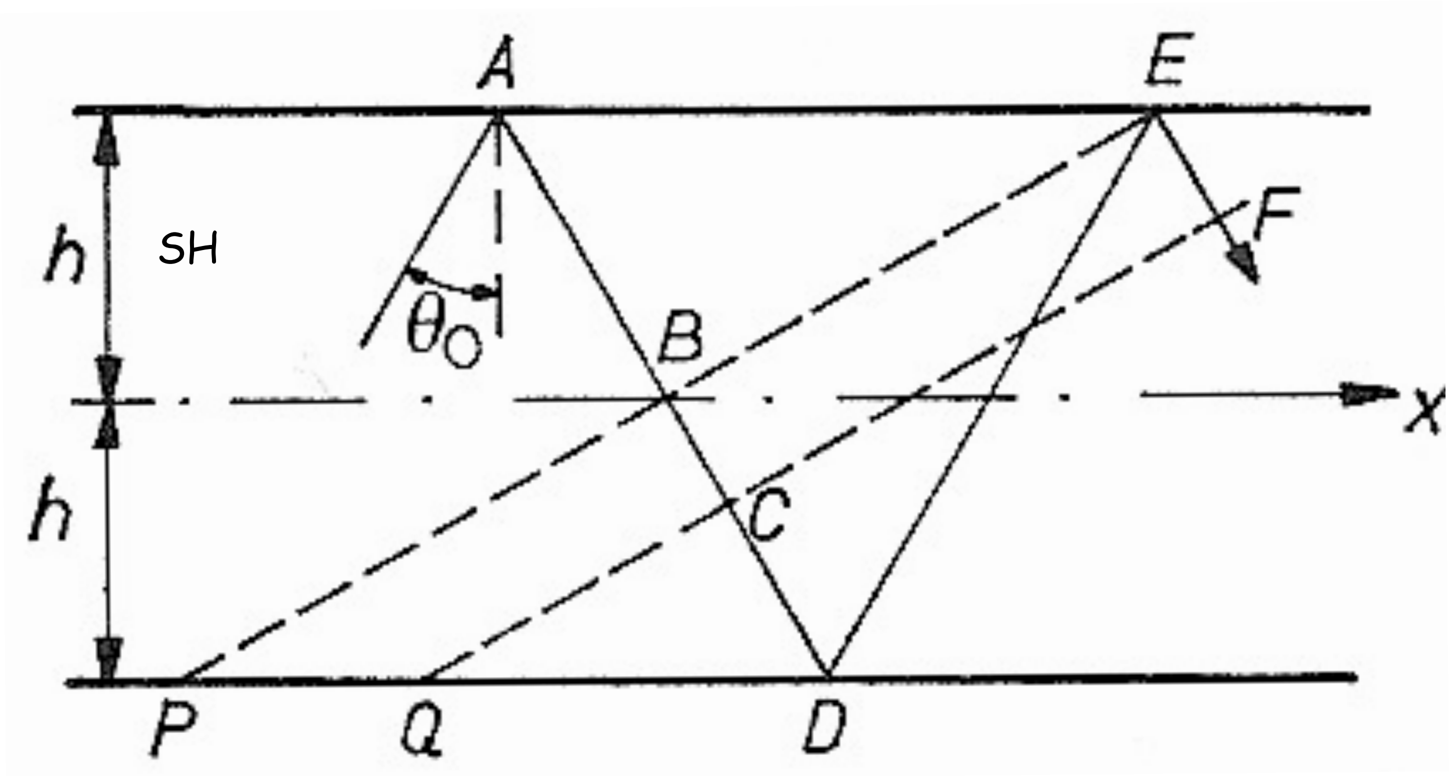
- Stiffness in a vibrating string introduces a restoring force proportional to the bending angle of the string and the usual stiffness term added to the wave equation for the ideal string. Stiff-string models are commonly used in piano synthesis and they have to be included in tuning of piano strings due to inharmonic effects.

$$\left(\frac{\partial^4}{\partial x^4} + \frac{E}{\rho}\frac{\partial^2}{\partial x^2} - \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2}\right)w = 0 \Rightarrow \quad \omega = \pm k\sqrt{\frac{E}{\rho}}\left(1 + k^2\sqrt{\frac{I}{A}}\right)^{1/2}$$



$$\Rightarrow \quad \omega \approx \pm k\sqrt{\frac{E}{\rho}}\left(1 + \frac{1}{2}k^2\sqrt{\frac{I}{A}}\right)$$

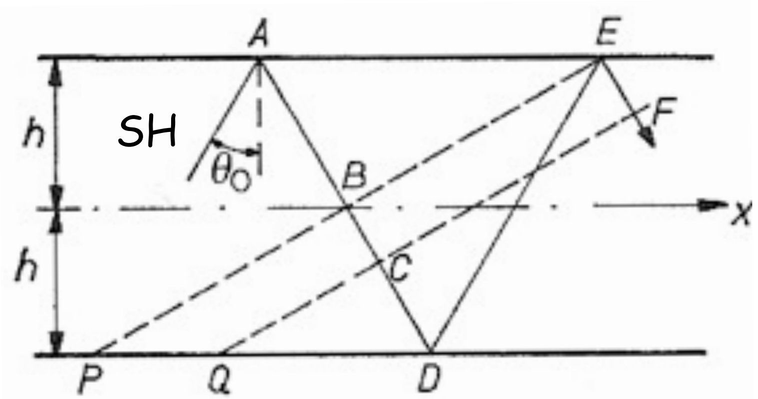
In an elastic half-space no SH type surface waves exist. Why?
Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer delimited by two free boundaries, i.e. a homogeneous plate?



Repeated reflection in the layer allow interference between incident and reflected SH waves: SH reverberations can be totally trapped.

SH waves: trapping

$$u_y = A\exp[i(\omega t + \omega\eta_\beta z - kx)] + B\exp[i(\omega t - \omega\eta_\beta z - kx)]$$



$$k = k_x = \frac{\omega}{c}; \quad \omega\eta_\beta = k_z = \frac{\omega}{c}\sqrt{\frac{c^2}{\beta^2} - 1} = kr_\beta$$

$$u_y = A\exp[i(\omega t + kr_\beta z - kx)] + B\exp[i(\omega t - kr_\beta z - kx)]$$

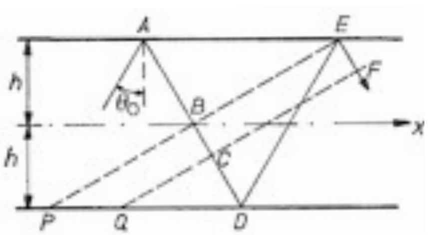
The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are: free surface conditions

$$\sigma_{zy}(0) = \mu \left.\frac{\partial u_y}{\partial z}\right|_0 = ikr_\beta\mu\left\{A\exp[i(\omega t - kx)] - B\exp[i(\omega t - kx)]\right\} = 0$$

$$\sigma_{zy}(2h) = \mu \left.\frac{\partial u_y}{\partial z}\right|_{2h} = ikr_\beta\mu\left\{A\exp[i(\omega t + kr_\beta 2h - kx)] - B\exp[i(\omega t - kr_\beta 2h - kx)]\right\} = 0$$

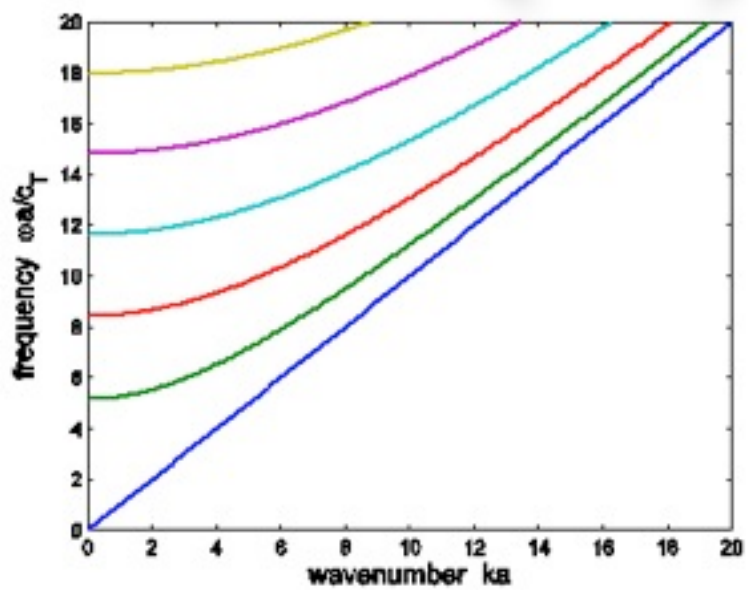
SH waves: eigenvalues...

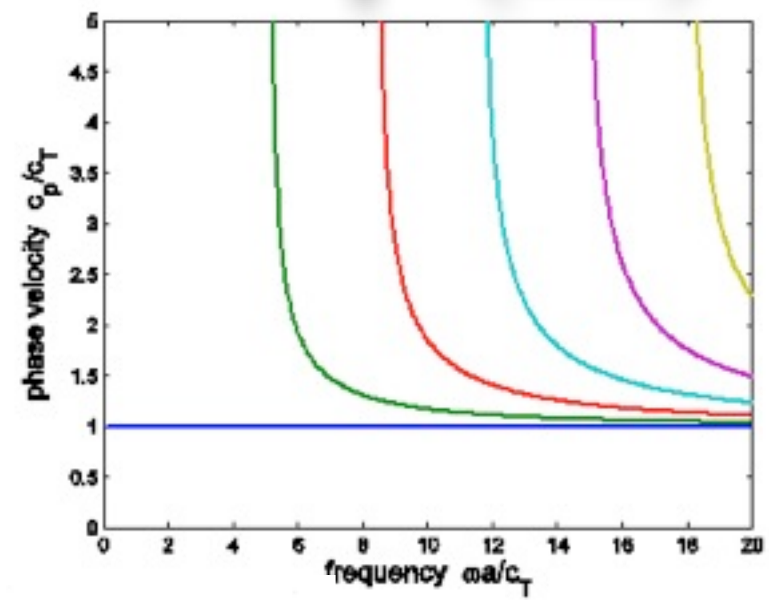
that leads to: $kr_\beta 2h = n\pi$ with n=0,1,2,... NB: REMEMBER THE "STRING PROBLEM": $kL=n\pi$

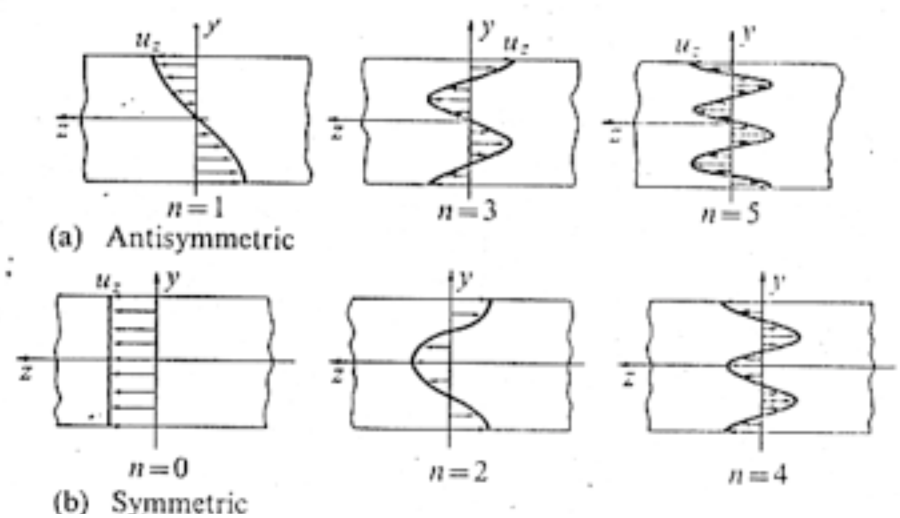


$$\omega^2 = k^2\beta^2 + \left(\frac{n\pi\beta}{2h}\right)^2$$

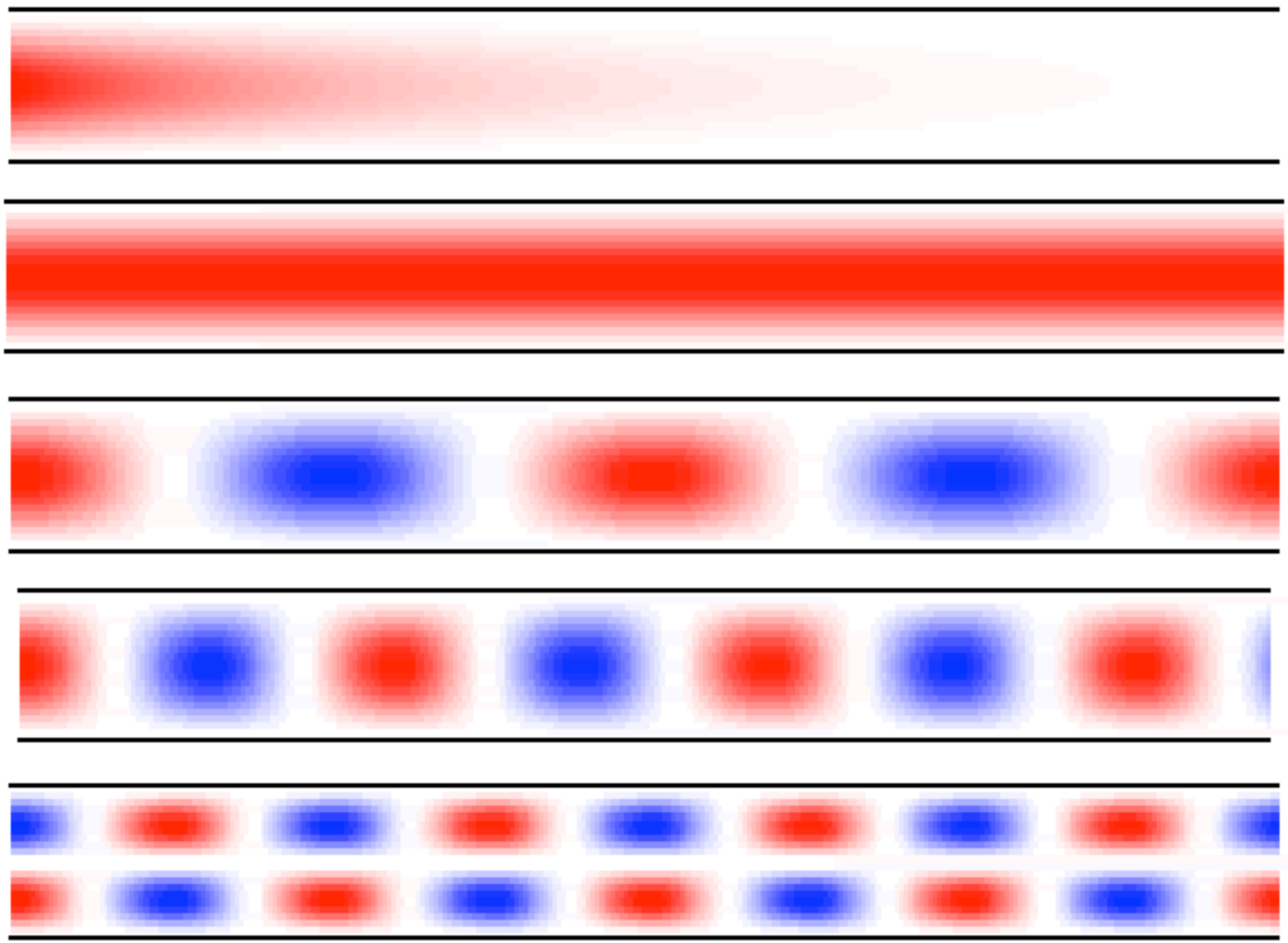
$$c = \frac{\beta}{\sqrt{1-\left(\frac{n\pi\beta}{2h\omega}\right)^2}}$$





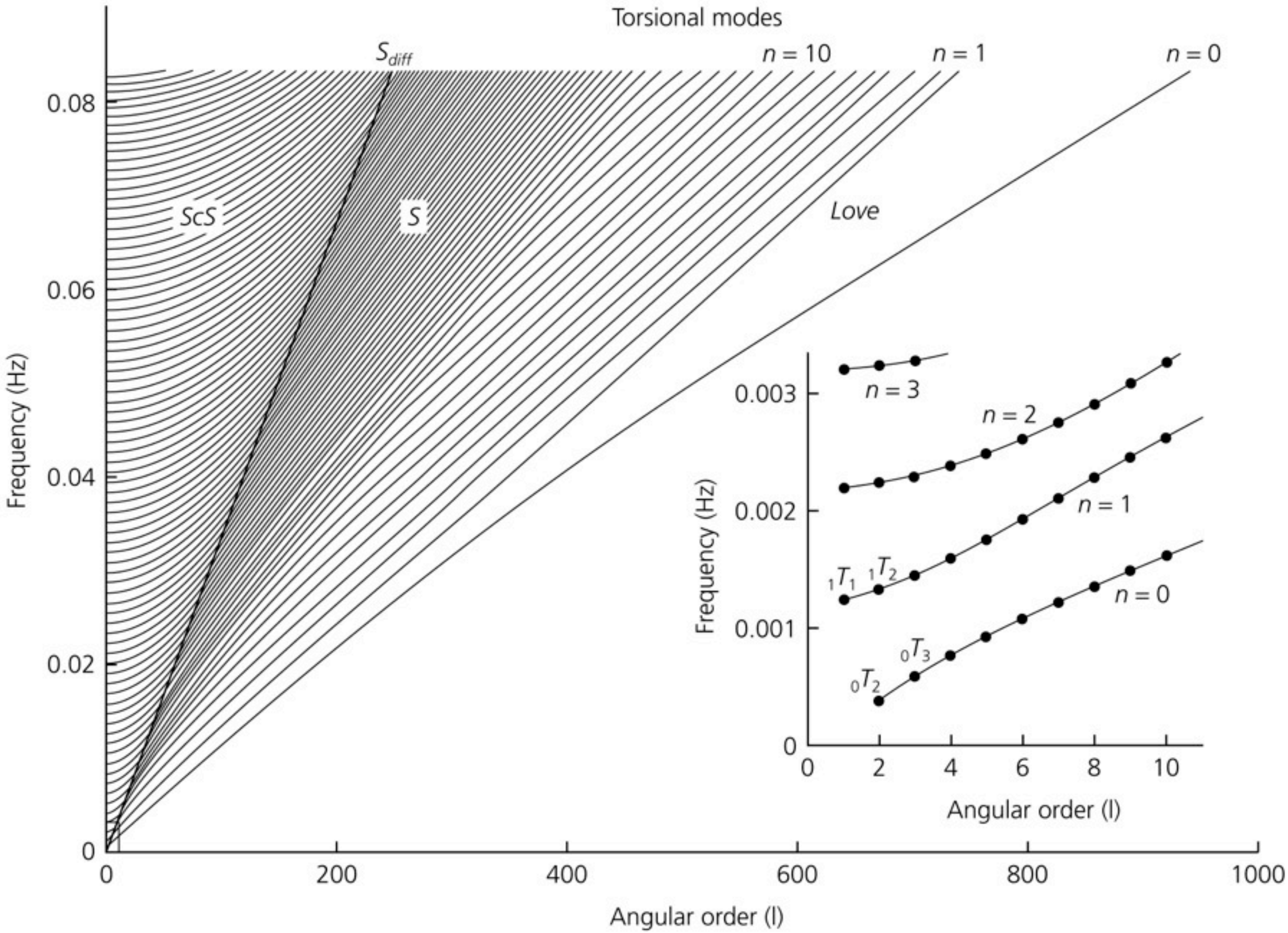


EM waveguide animations



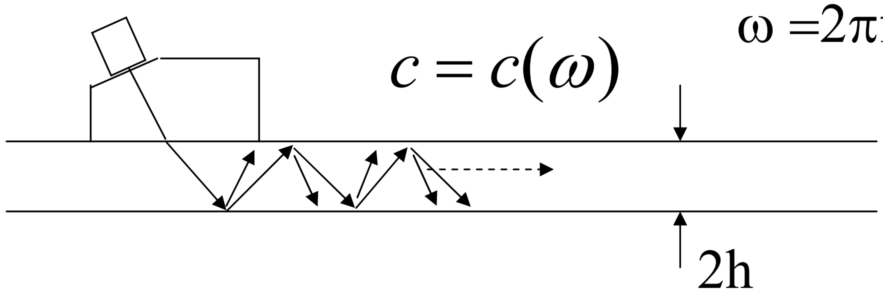
http://www.ee.iastate.edu/~hsiu/descriptions/paral.html

Torsional modes dispersion



Waves in plates

In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.

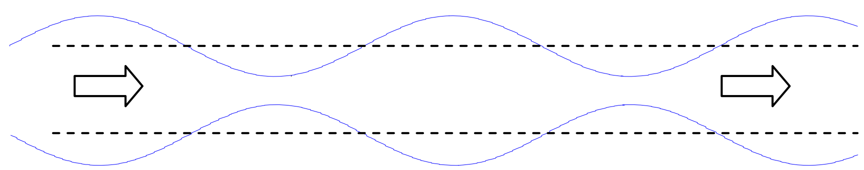


$$c = c(\omega)$$

ω =2πf … frequency (rad/sec)

$$\phi = f(y)\exp[ik(x-ct)]$$

$$\psi = g(y)\exp[ik(x-ct)]$$

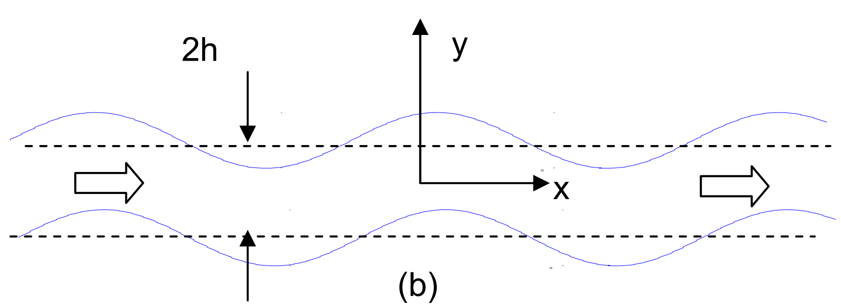


(a)

extensional waves

$$f = A\cosh(\alpha y)$$

$$g = B\sinh(\beta y)$$



(b)

flexural waves

$$f = A'\sinh(\alpha y)$$

$$g = B'\cosh(\beta y)$$

satisfying the boundary conditions $\tau_{yy} = \tau_{xy} = 0$ on $y = \pm h$ gives the Rayleigh-Lamb equations:

$$\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \left[\frac{4\omega^2 \alpha\beta}{c^2 \left(\omega^2 / c^2 + \beta^2\right)^2} \right]^{\pm 1}$$

\+ … extensional waves

\- … flexural waves

$$\alpha = \left|\frac{\omega}{c}\right| \sqrt{1 - \frac{c^2}{c_p^2}} \quad , \beta = \left|\frac{\omega}{c}\right| \sqrt{1 - \frac{c^2}{c_s^2}}$$

consider the extensional waves

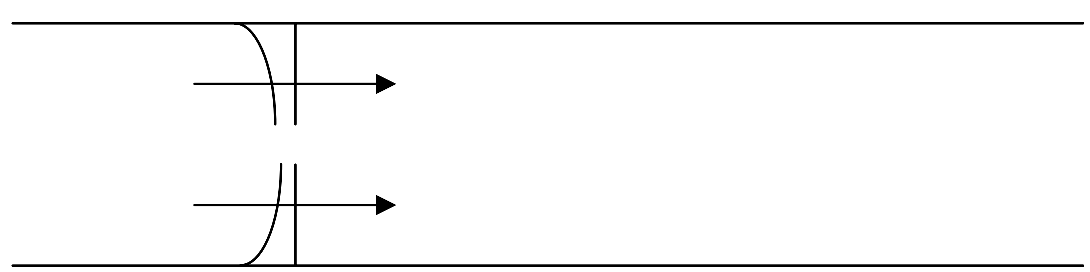
$$\frac{\tanh\left[2\pi fh\sqrt{1/c^2-1/c_s^2}\right]}{\tanh\left[2\pi fh\sqrt{1/c^2-1/c_p^2}\right]}=\frac{4\sqrt{1-c^2/c_s^2}\sqrt{1-c^2/c_p^2}}{\left(2-c^2/c_s^2\right)^2}$$

If we let $kh=\dfrac{2\pi fh}{c}>>1$ (high frequency)

then both tanh functions are $\simeq 1$

and we find $\left(2-c^2/c_s^2\right)^2=4\sqrt{1-c^2/c_s^2}\sqrt{1-c^2/c_p^2}$

so we just have Rayleigh waves on both stress-free surfaces:



In contrast for $kh \ll 1$ (low frequency)

we find
$$\tanh(\alpha h) \cong \alpha h$$
$$\tanh(\beta h) \cong \beta h$$

and the Rayleigh-Lamb equation reduces to

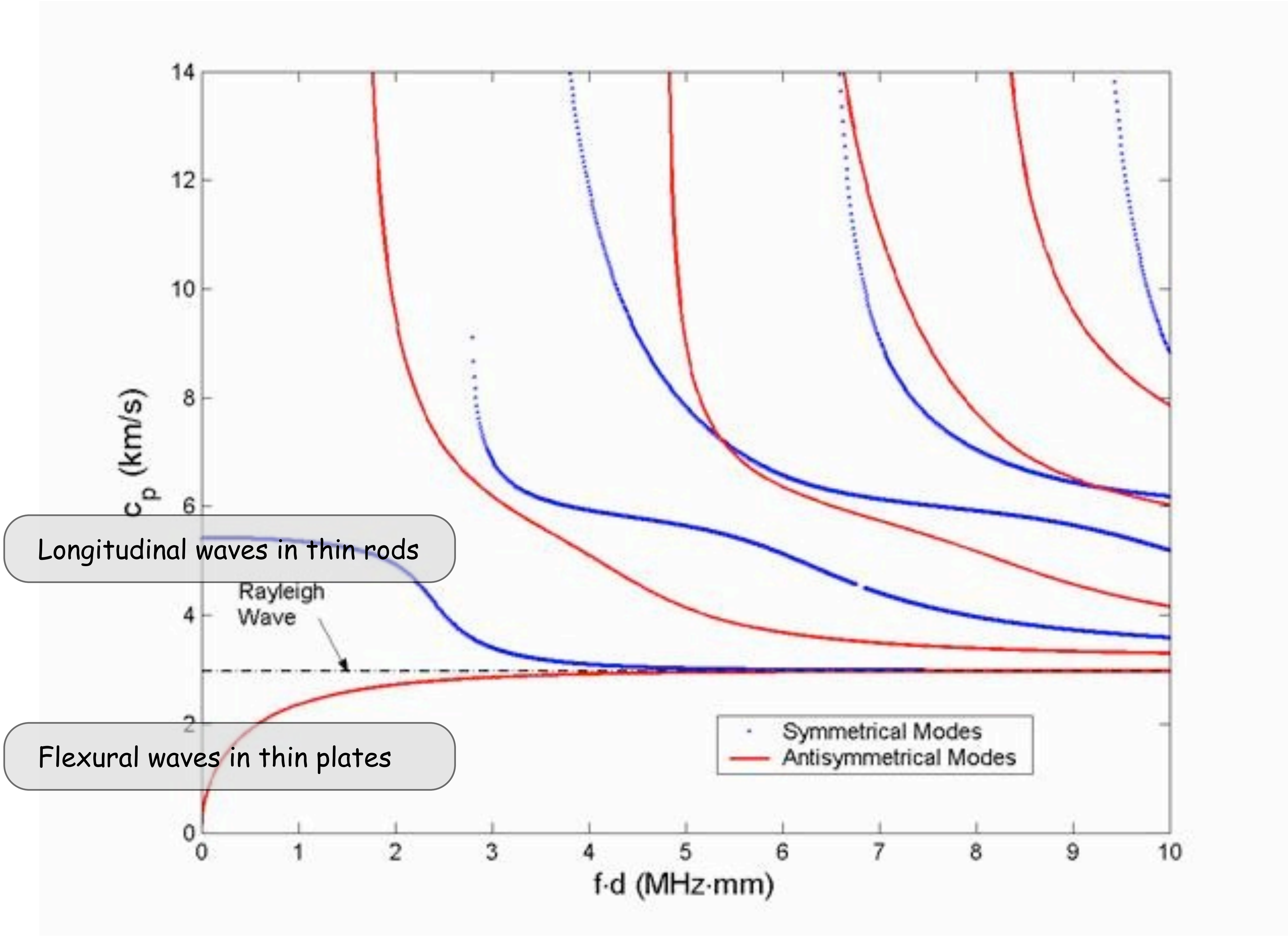
$$\left(2 - c^2 / c_s^2\right)^2 = 4\left(1 - c^2 / c_p^2\right)$$

which can be solved for c to give

$$c = c_{plate} = \sqrt{\frac{E}{\rho\left(1 - \nu^2\right)}}$$

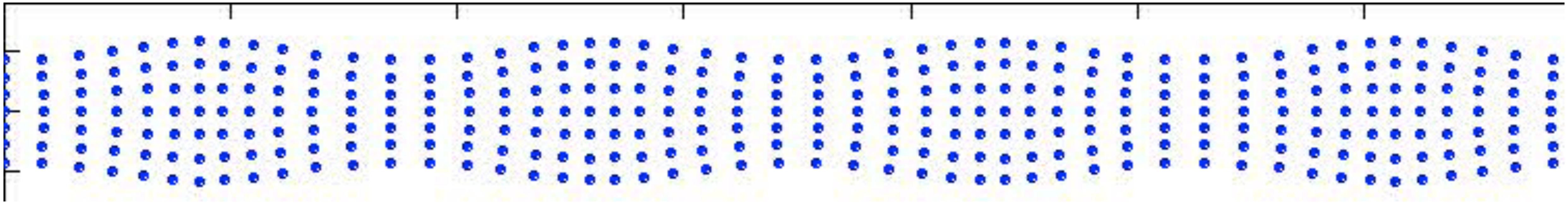
Waves in plates

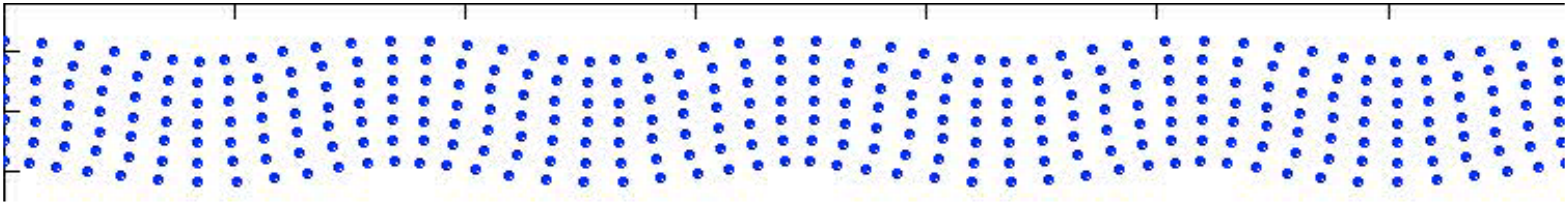
In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.



Lamb waves

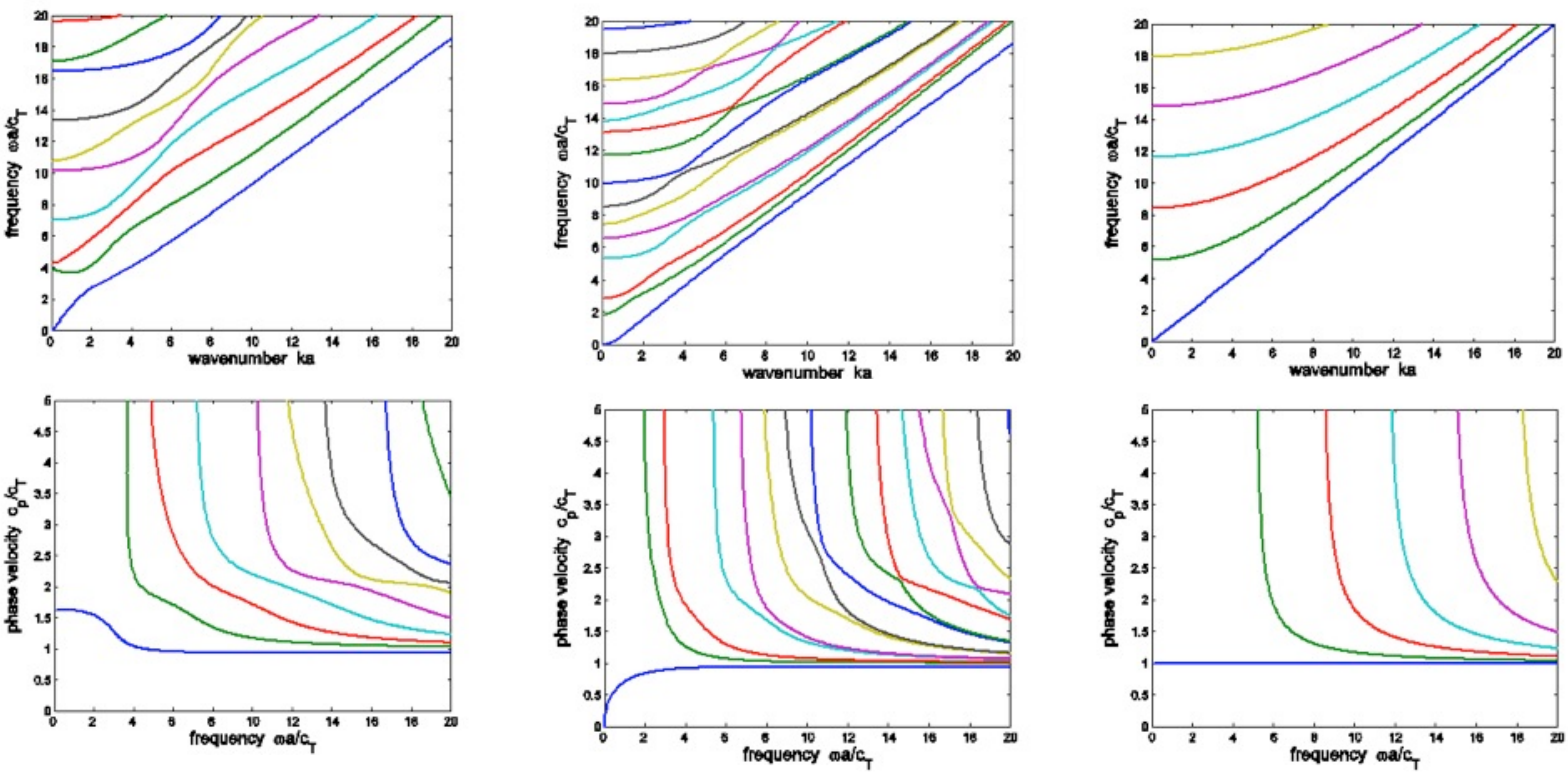
Lamb waves are waves of plane strain that occur in a free plate, and the traction force must vanish on the upper and lower surface of the plate. In a free plate, a line source along y axis and all wave vectors must lie in the x-z plane. This requirement implies that response of the plate will be independent of the in-plane coordinate normal to the propagation direction.





Elastic waves in rods

Three types of elastic waves can propagate in rods: (1) **longitudinal waves**, (2) **flexural waves**, and (3) **torsional waves**. Longitudinal waves are similar to the symmetric Lamb waves, flexural waves are similar to antisymmetric Lamb waves, and torsional waves are similar to horizontal shear (SH) waves in plates.



Elastic waves in rods

QuickTime™ and a
decompressor
are needed to see this picture.