SEISMOLOGY I

Laurea Magistralis in Physics of the Earth and of the Environment

Linear systems

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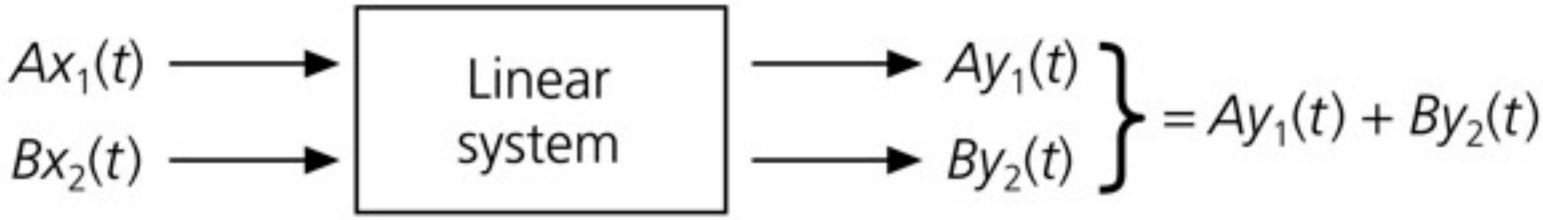
Figure 6.3-1: Definition of a linear system.



$$x(t) = \int x(\tau)\delta(\tau - t)d\tau$$

$$\downarrow$$

$$\int x(\tau)h(\tau - t)d\tau$$

$$x(t) * h(t) = y(t)$$

(remember GF definition)

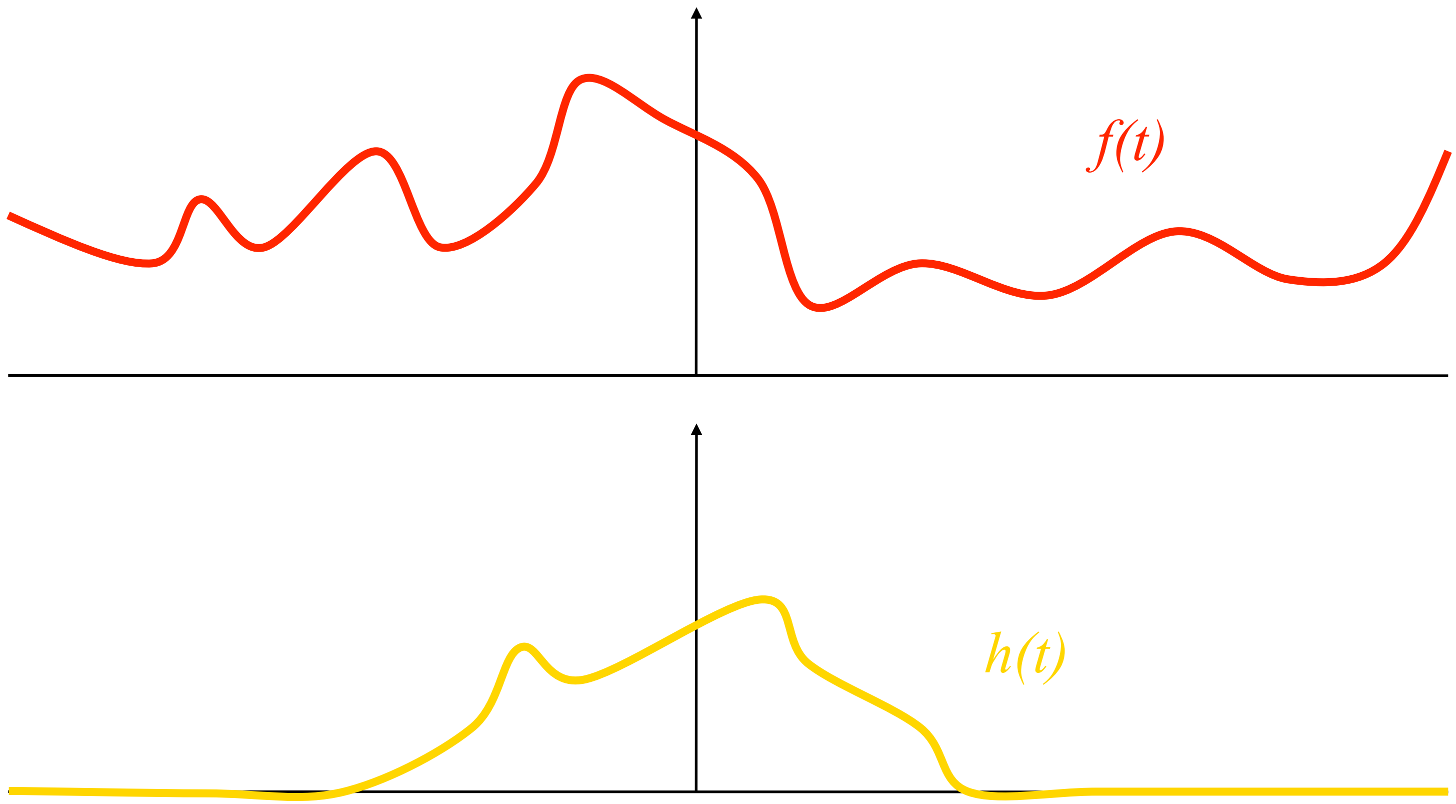
Convolution

- Definition:

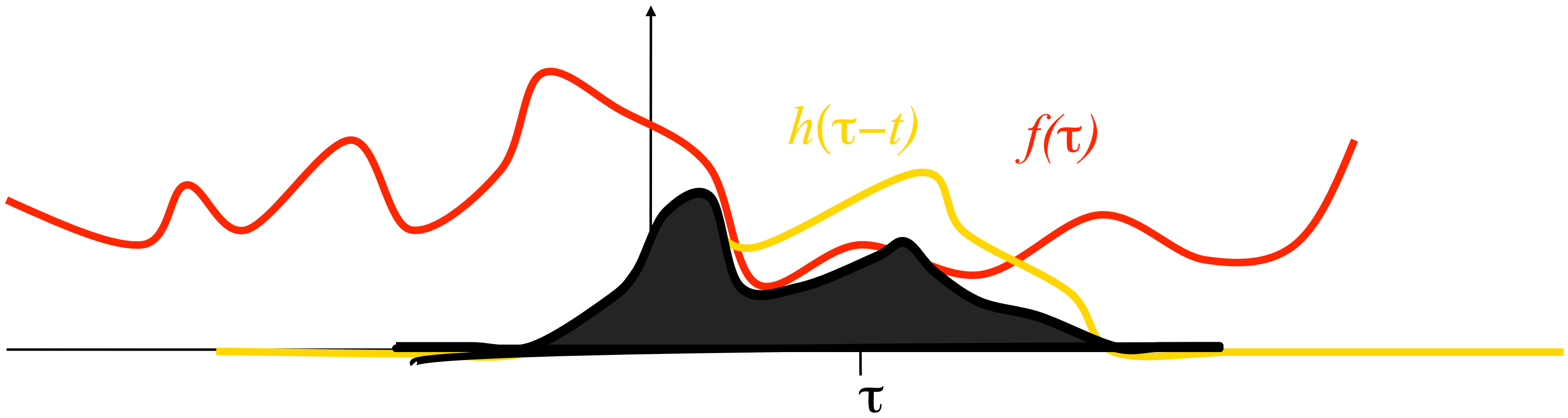
$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

Convolution

- Pictorially



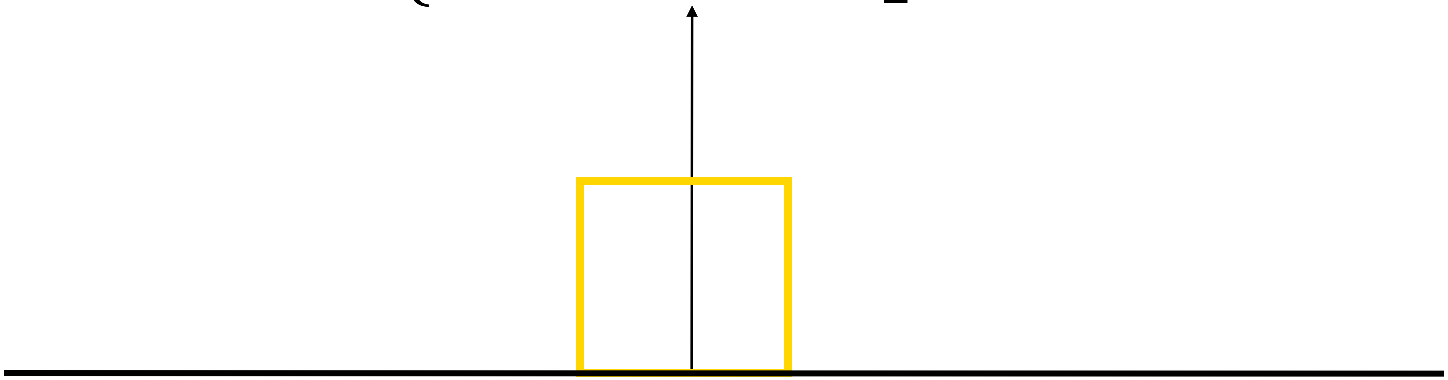
Convolution



Convolution

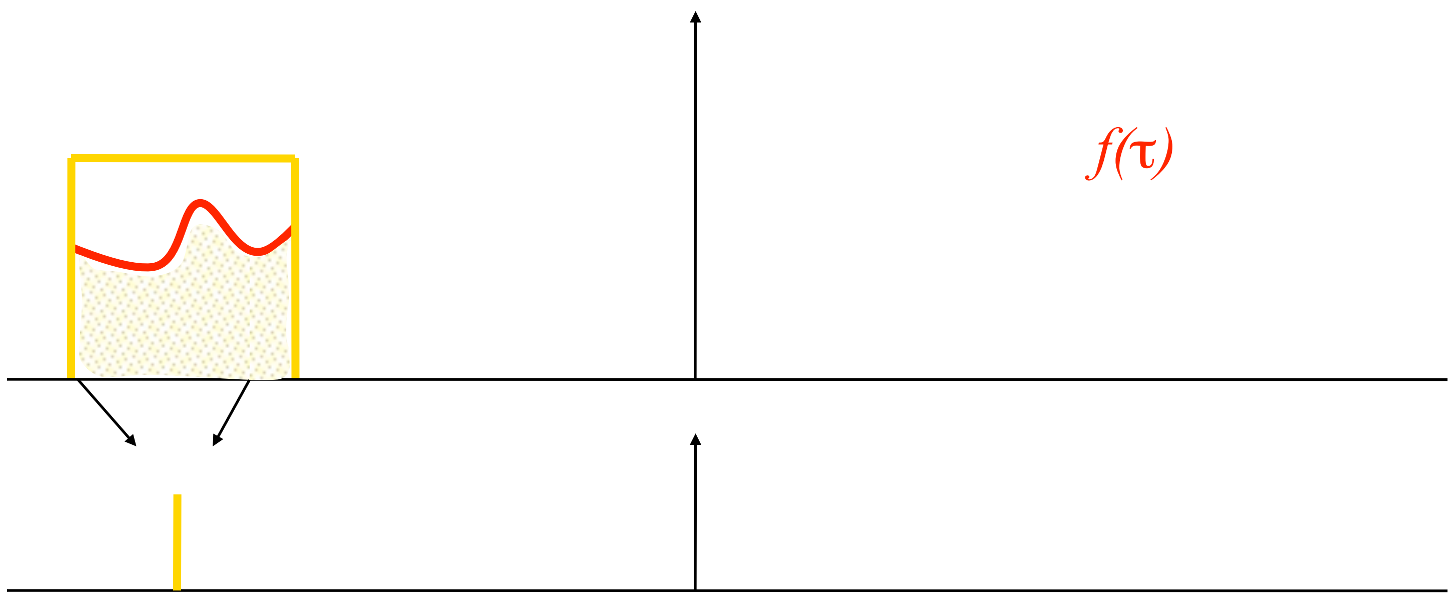
- Consider the function (box filter):

$$h(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$



Convolution

- This function windows our function f(t)



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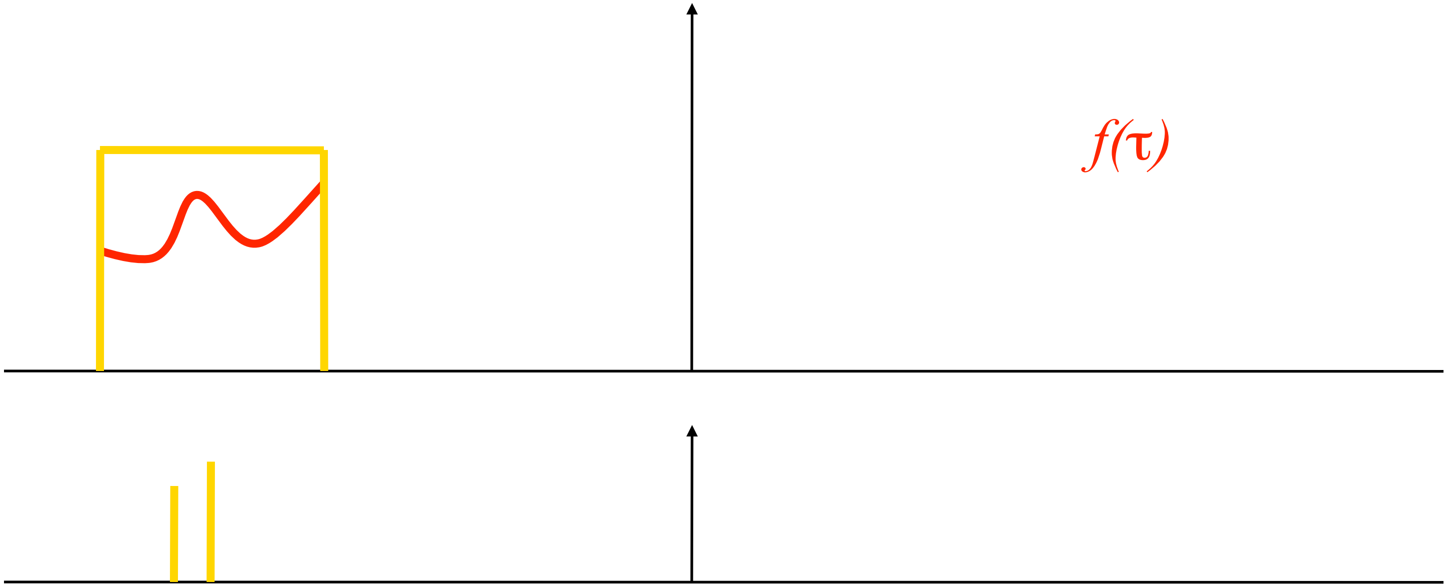

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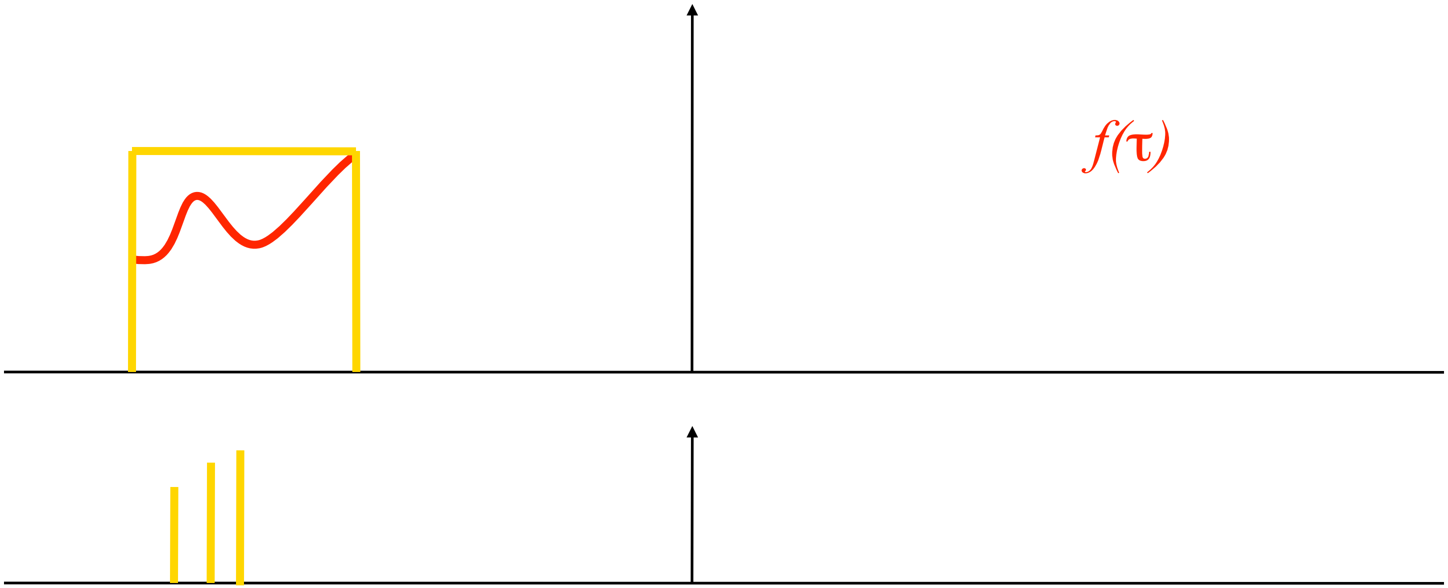

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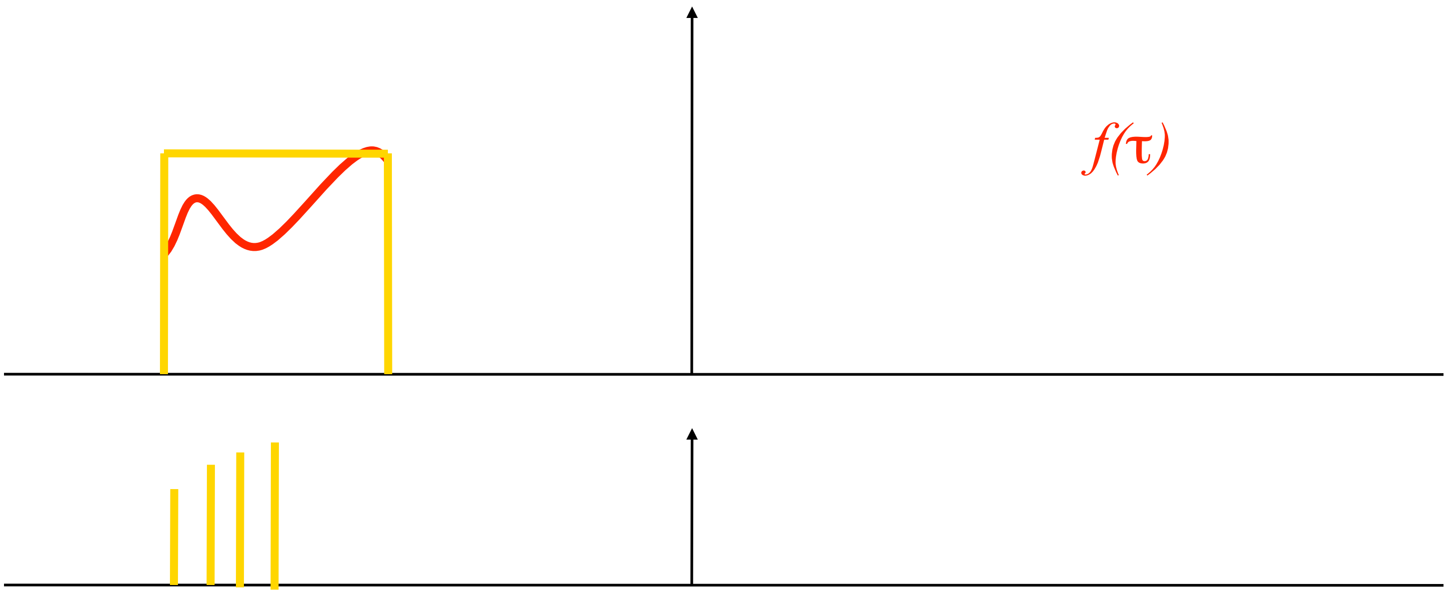

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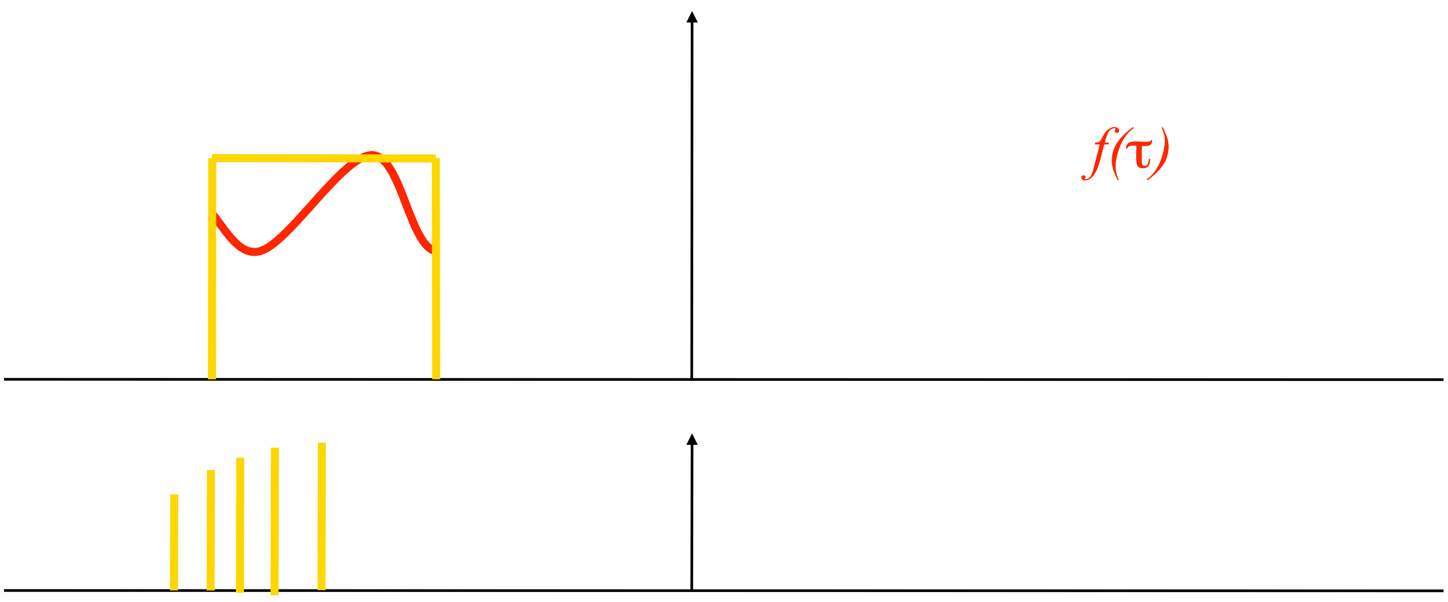

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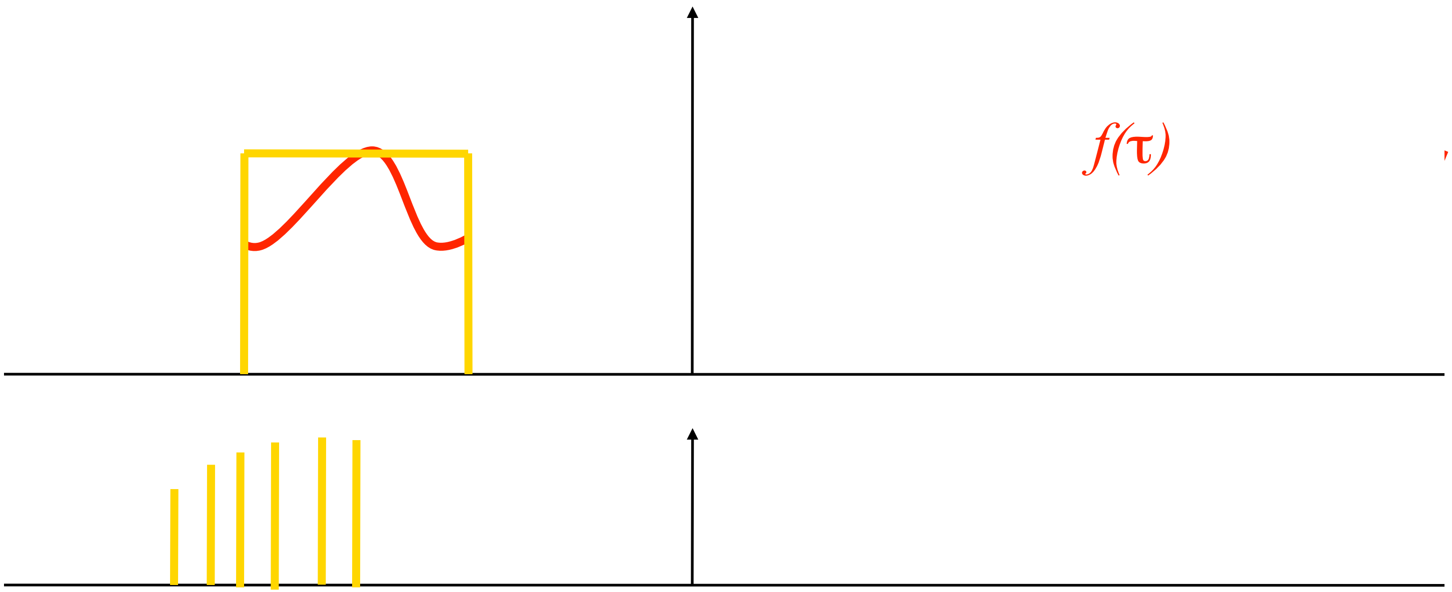

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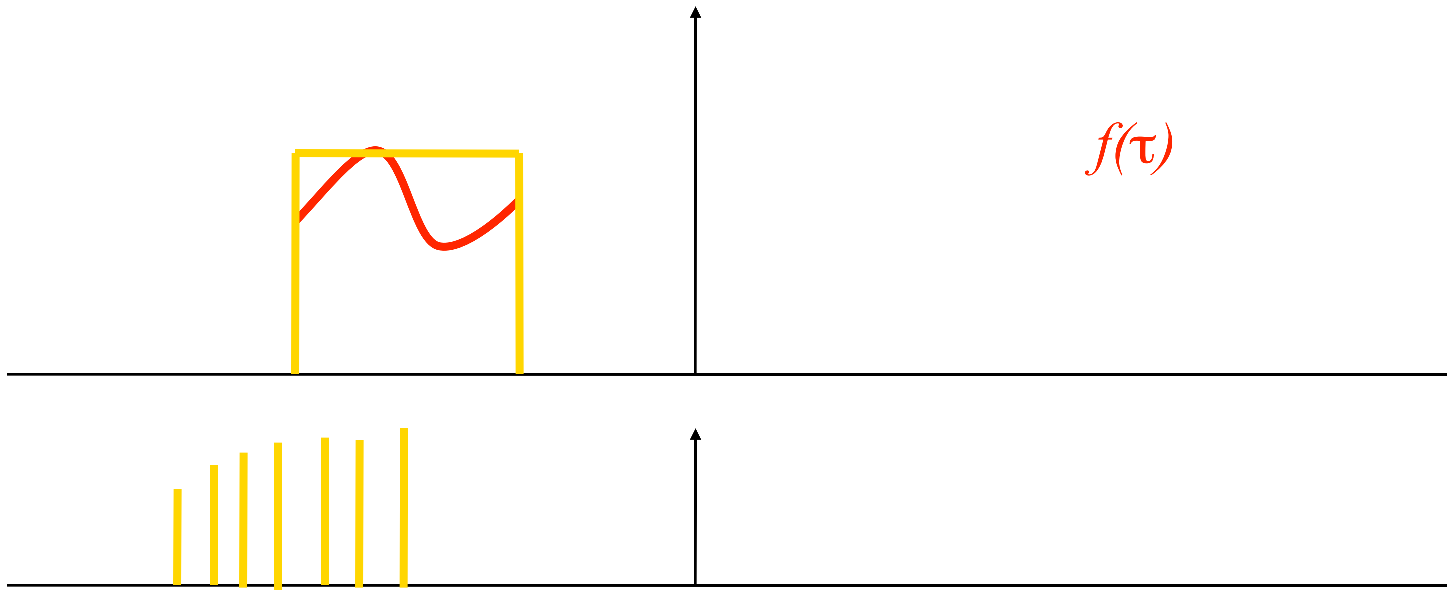

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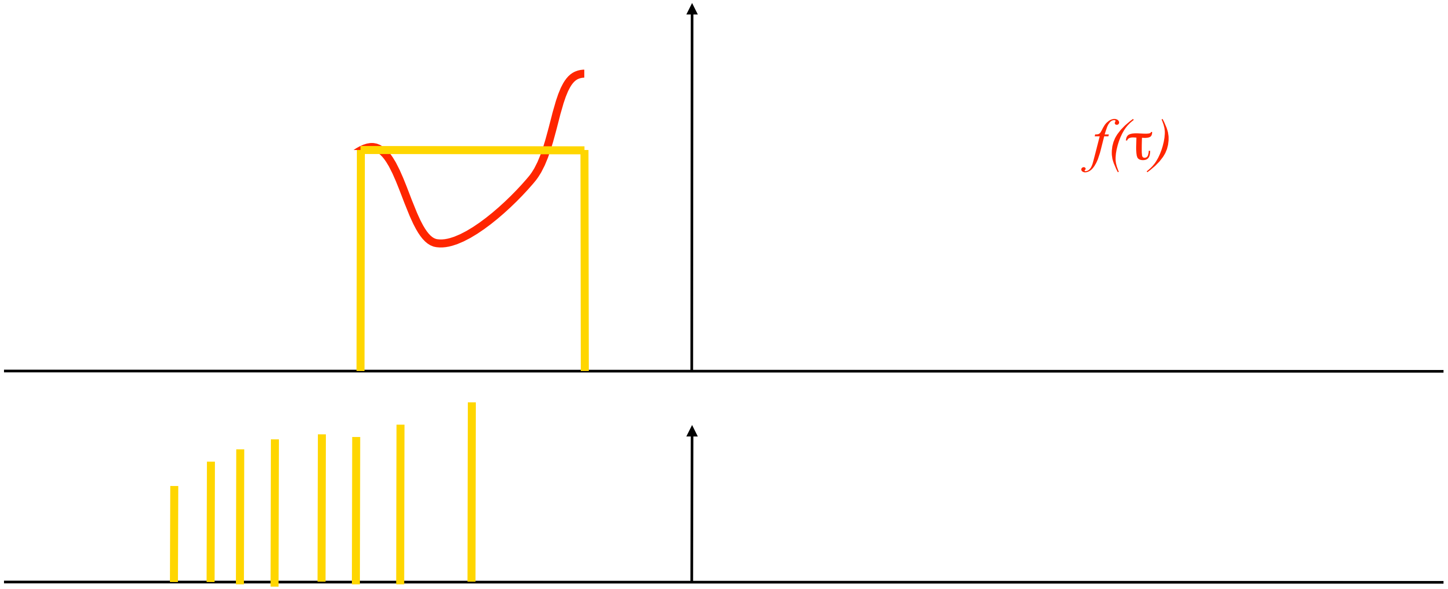

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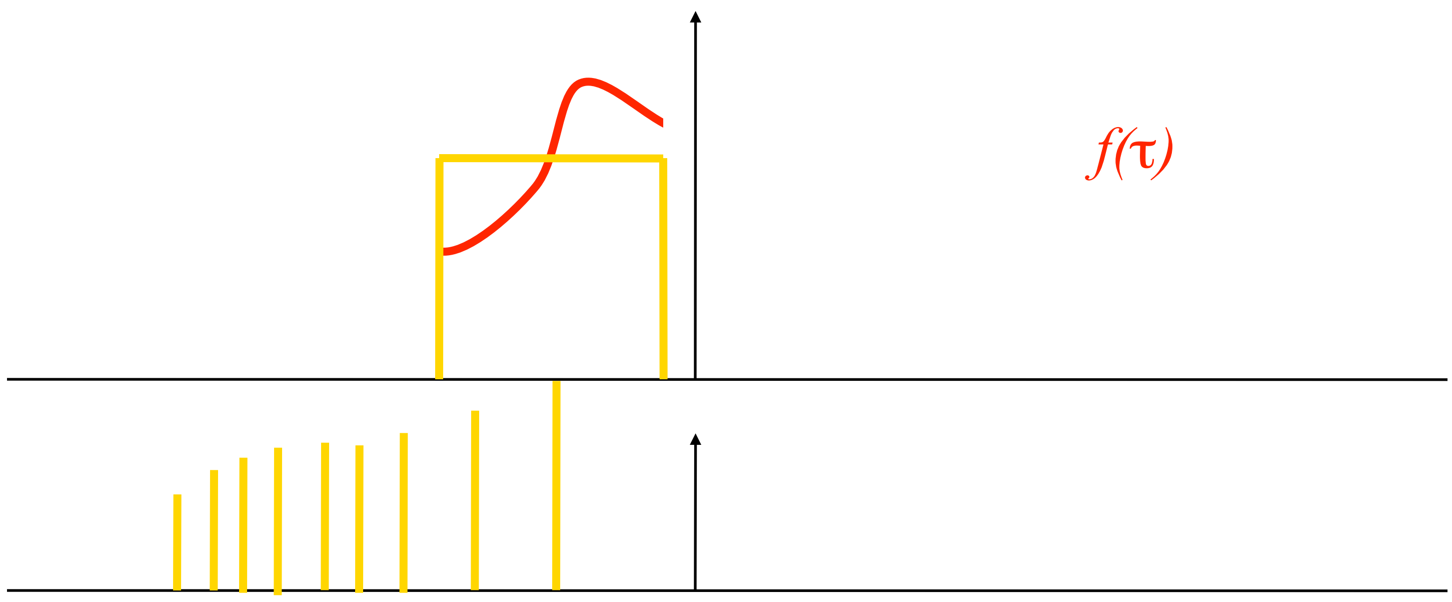

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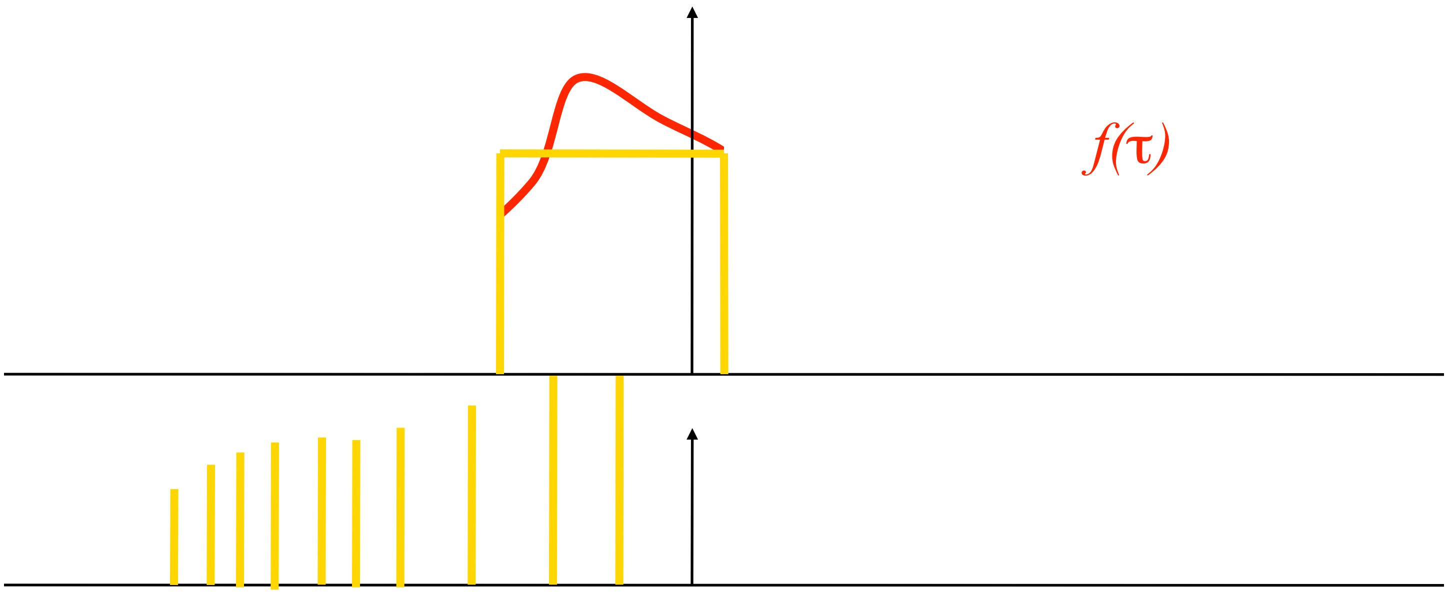

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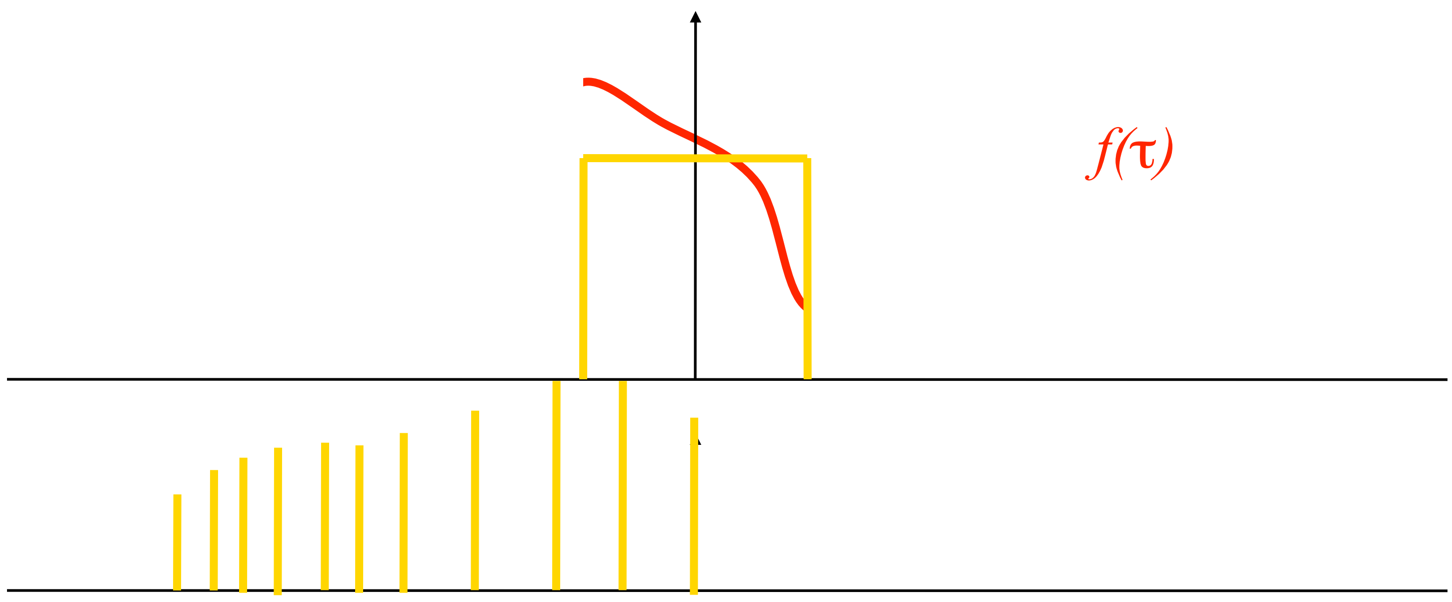

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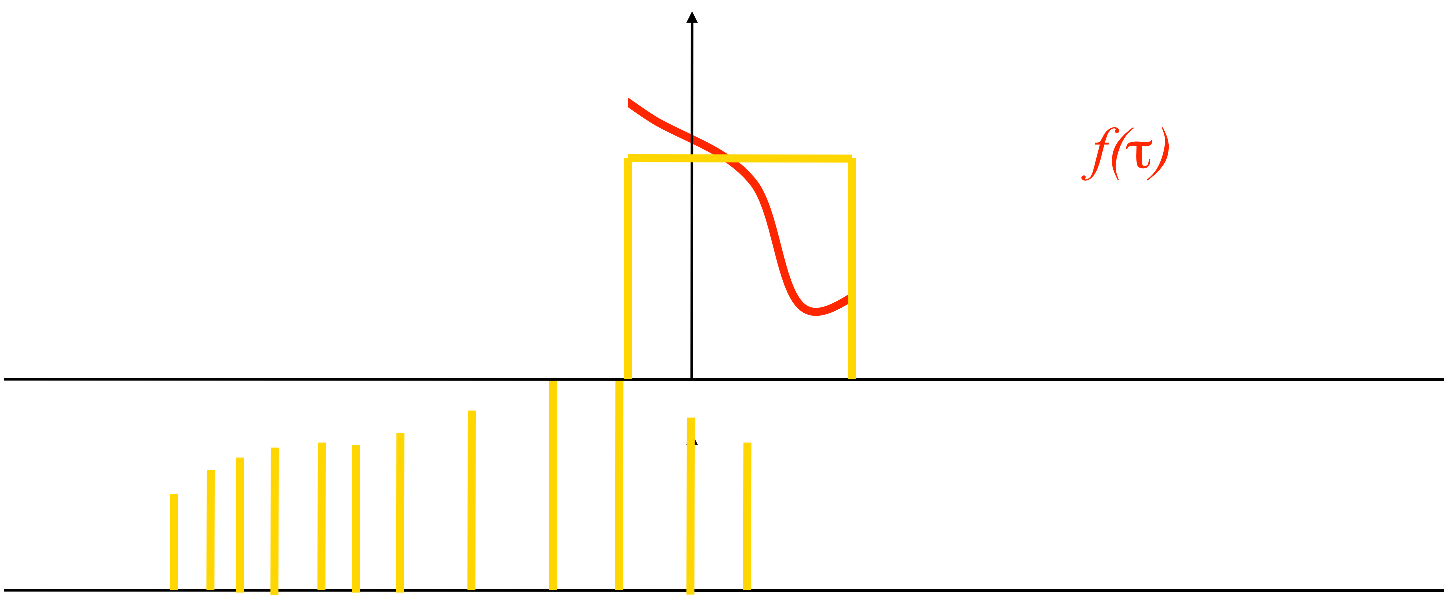

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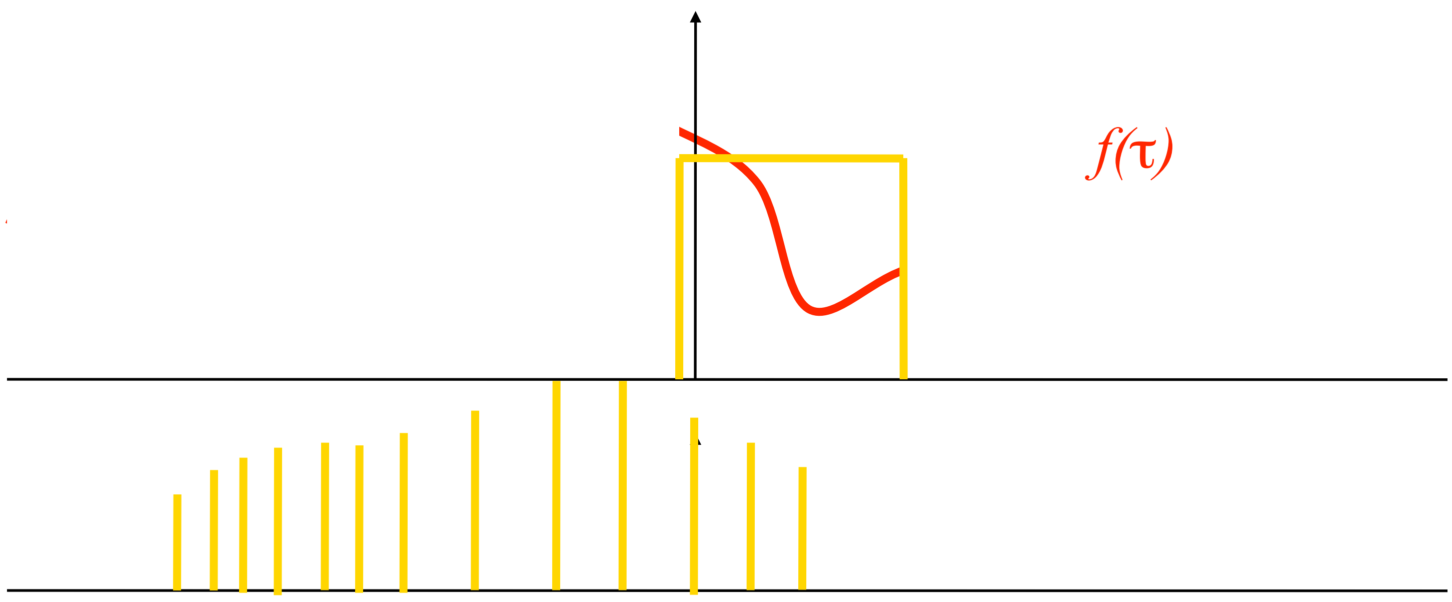

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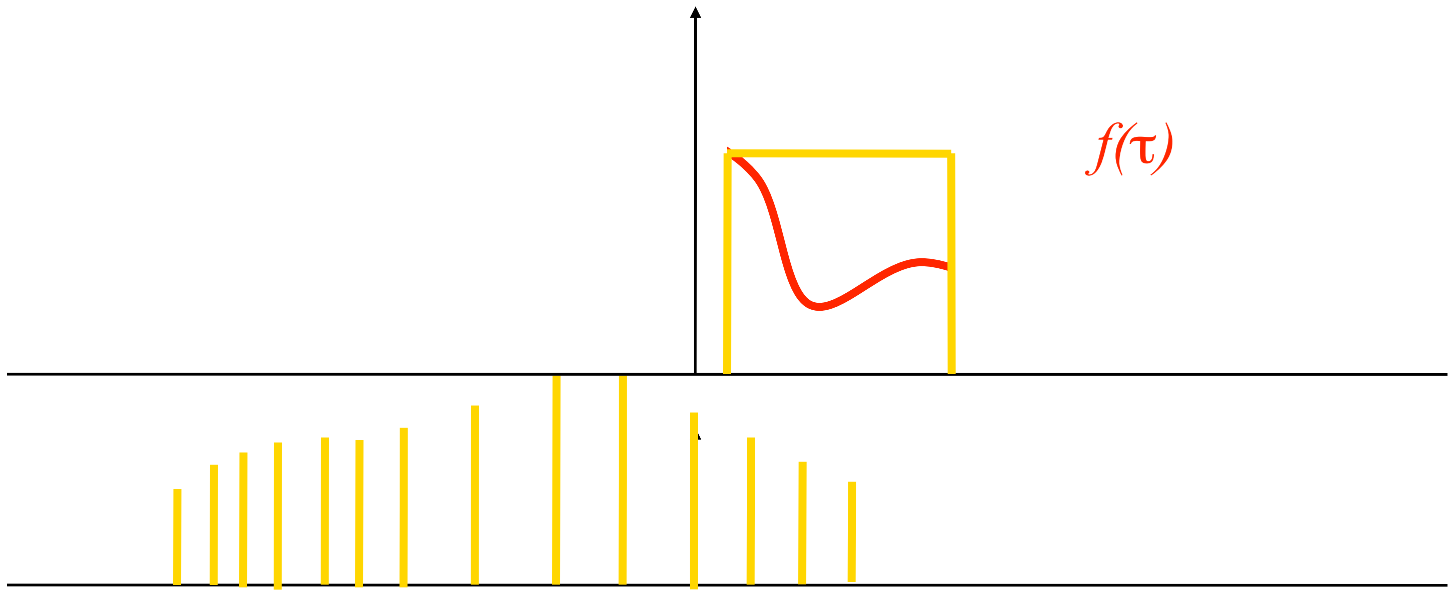

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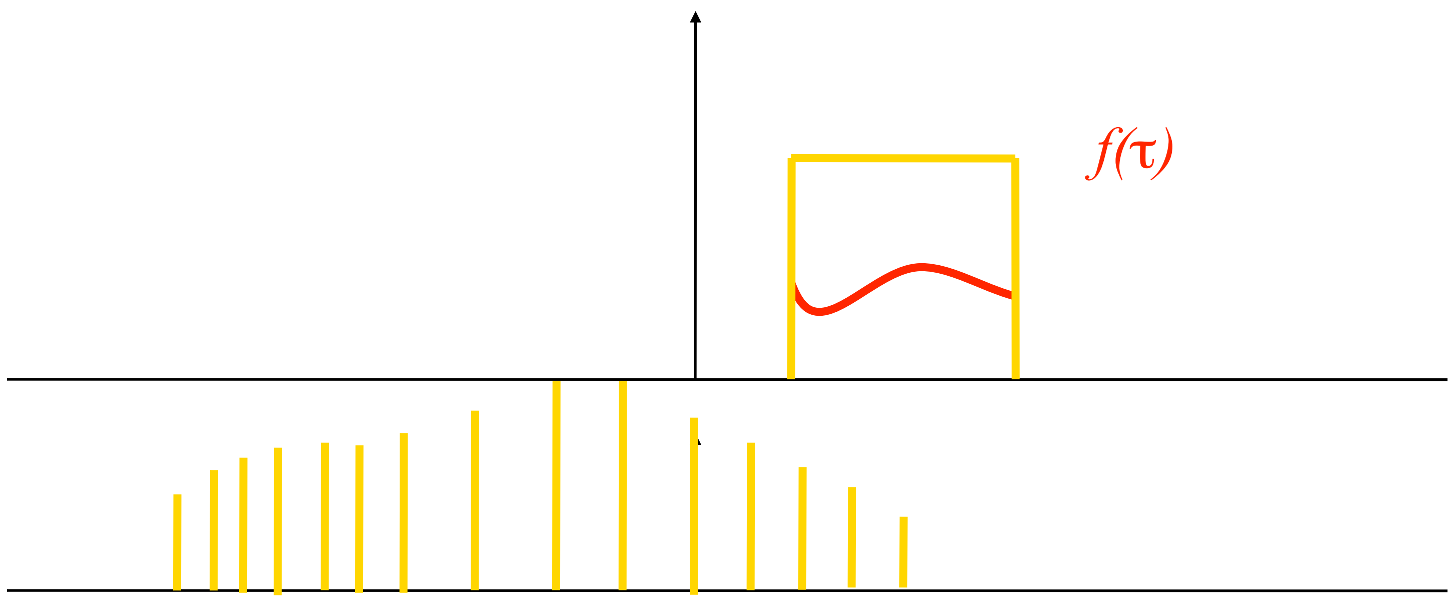

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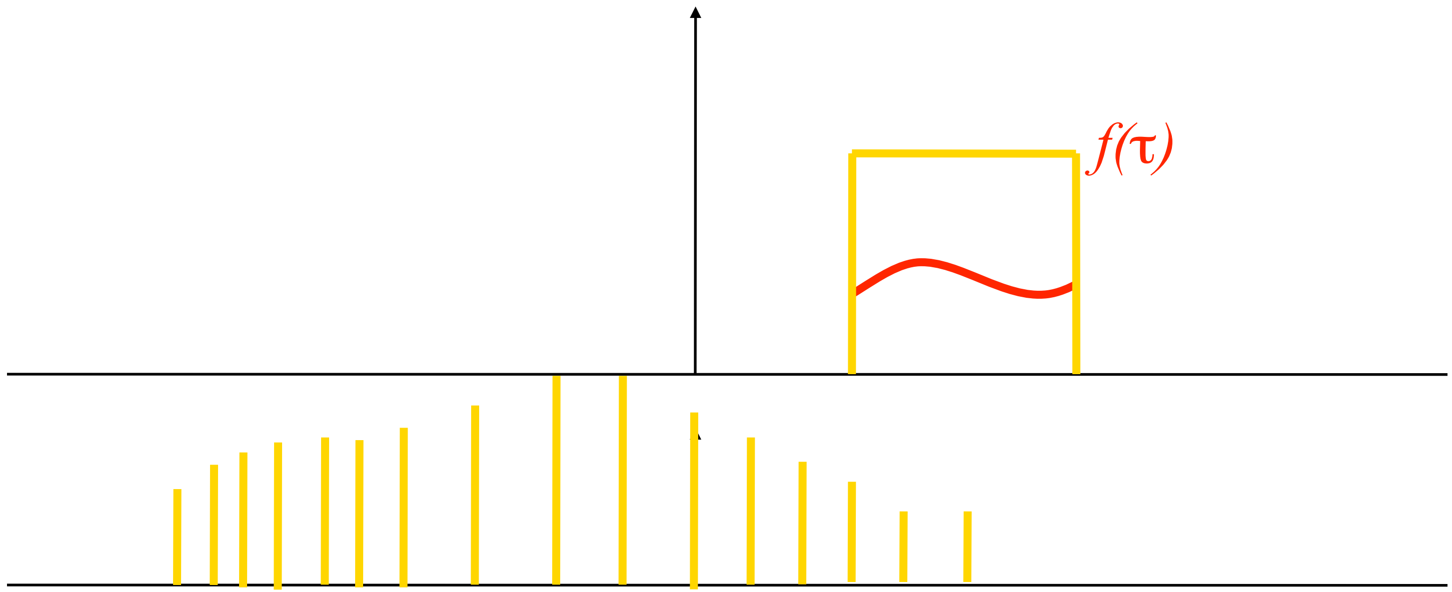

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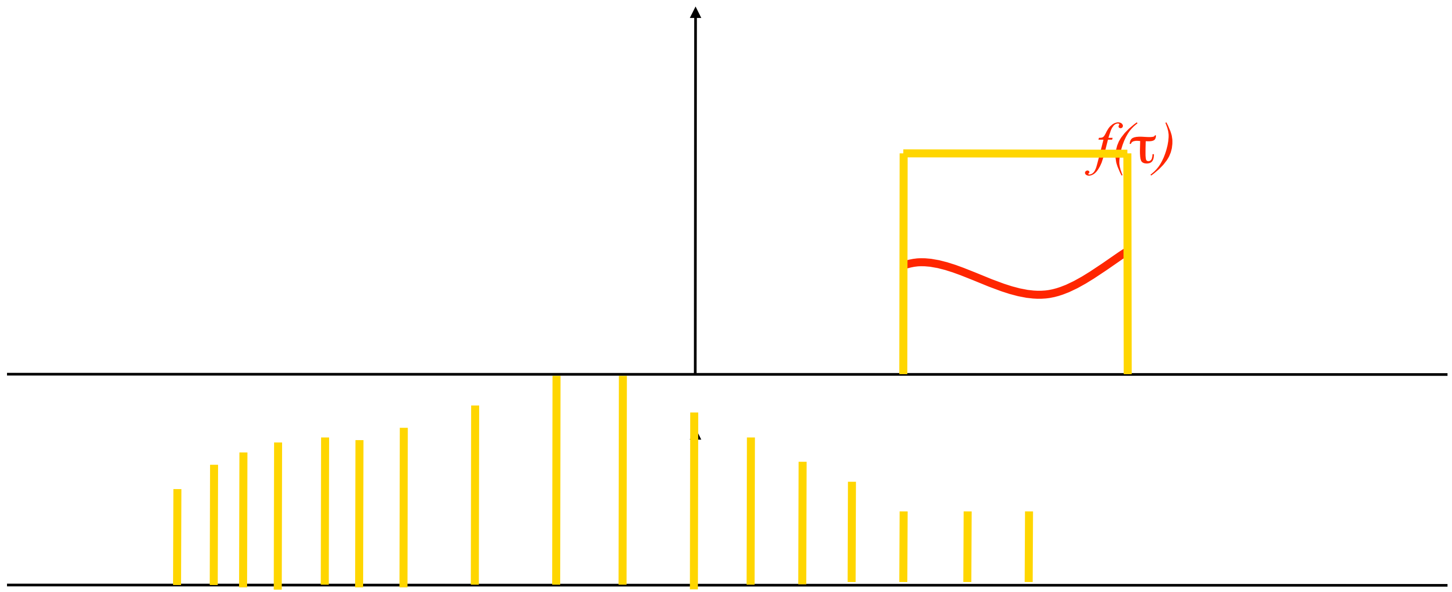

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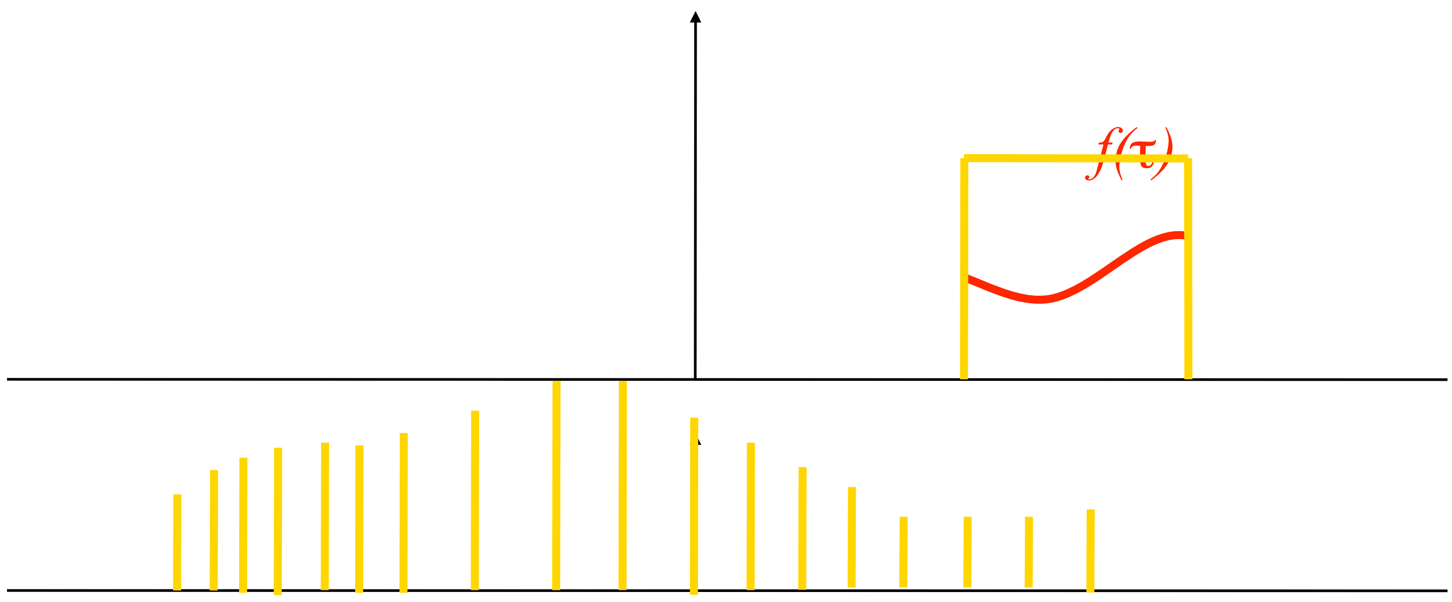

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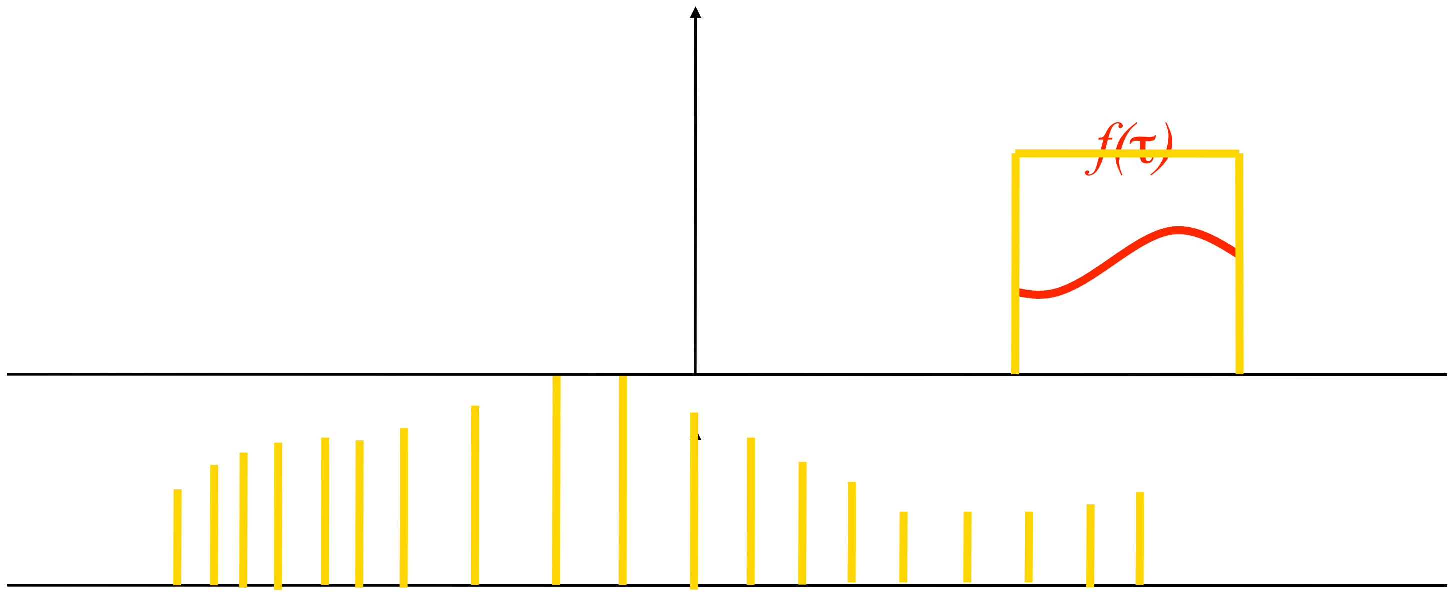

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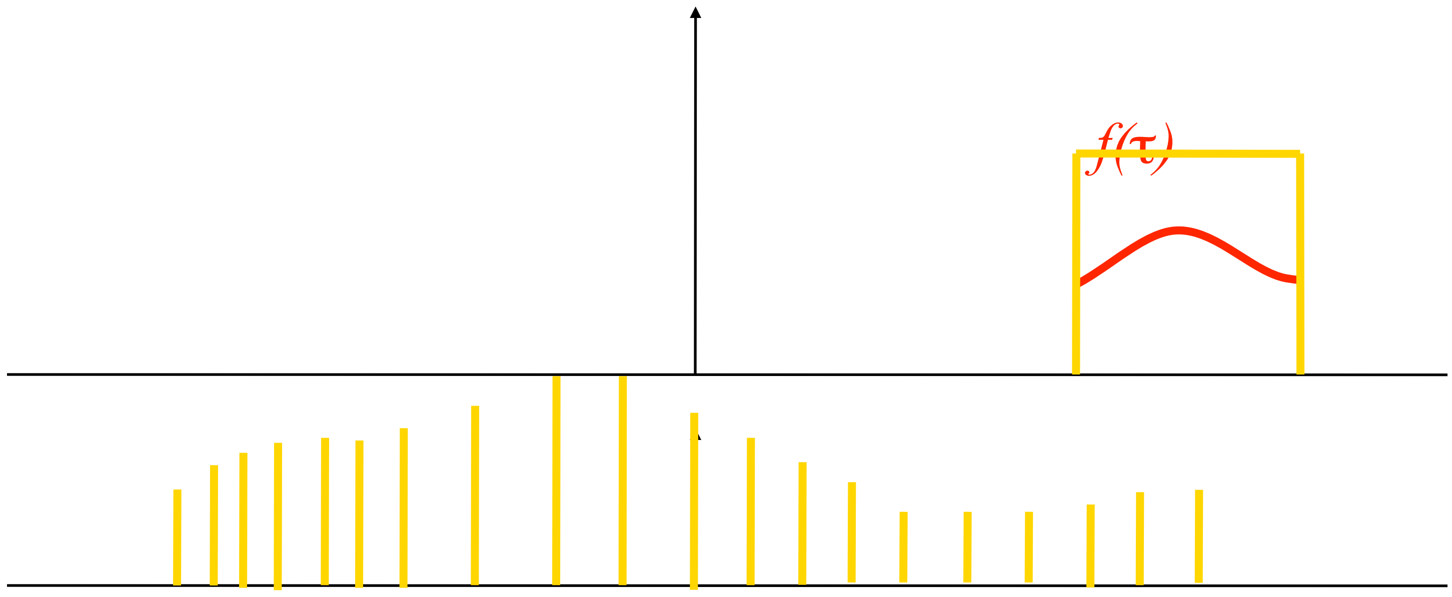

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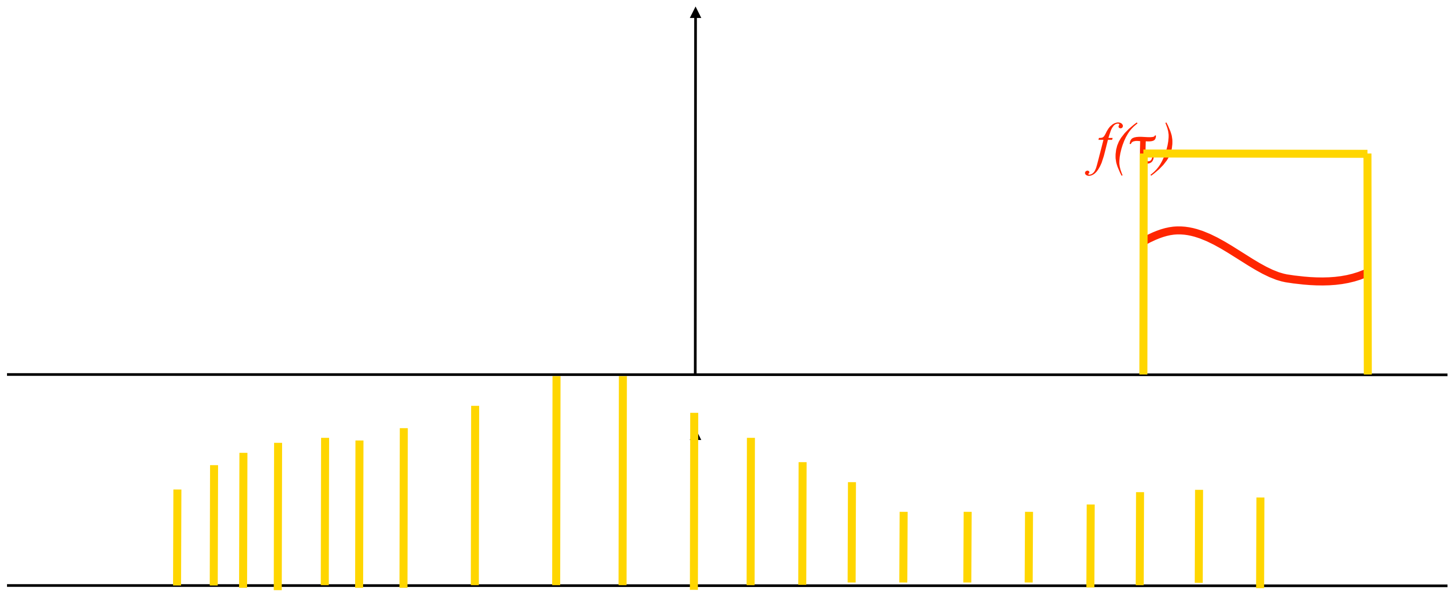

Convolution

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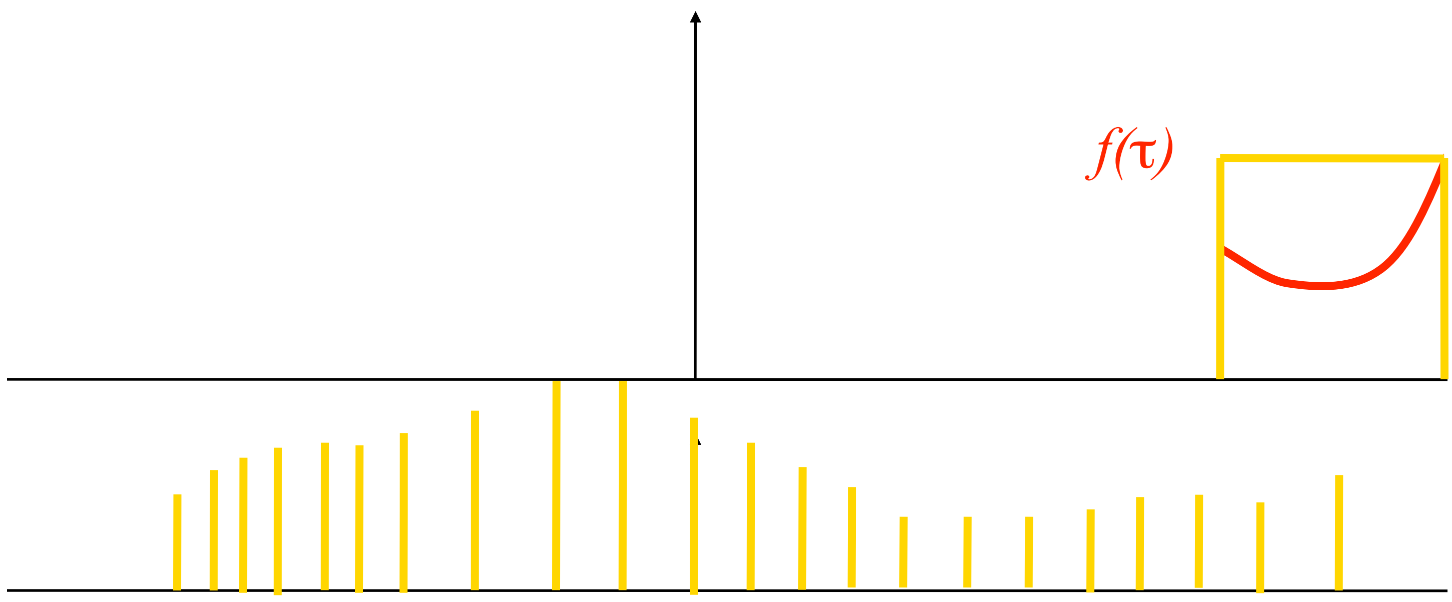
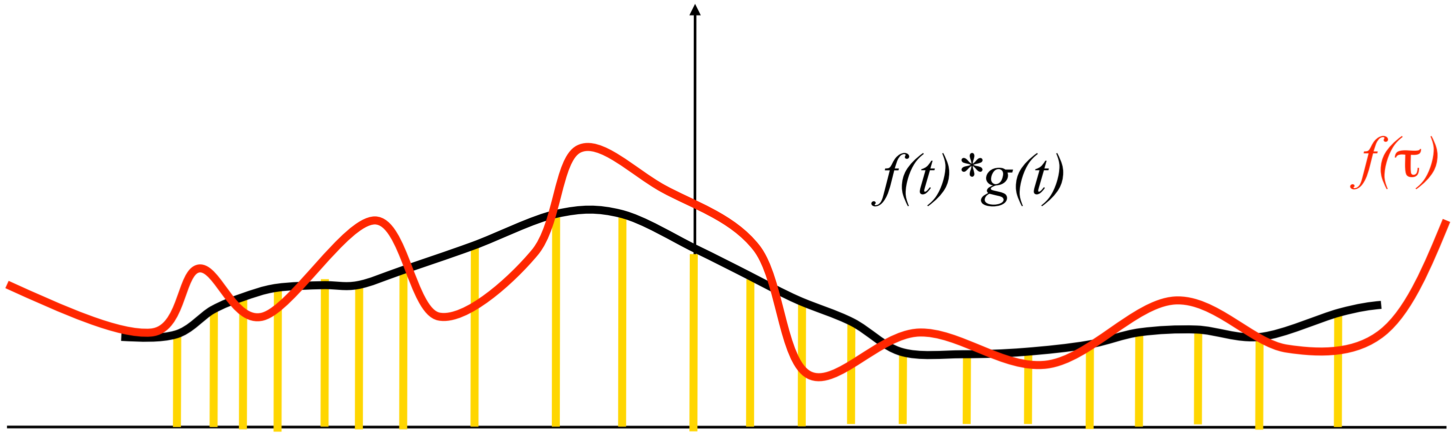
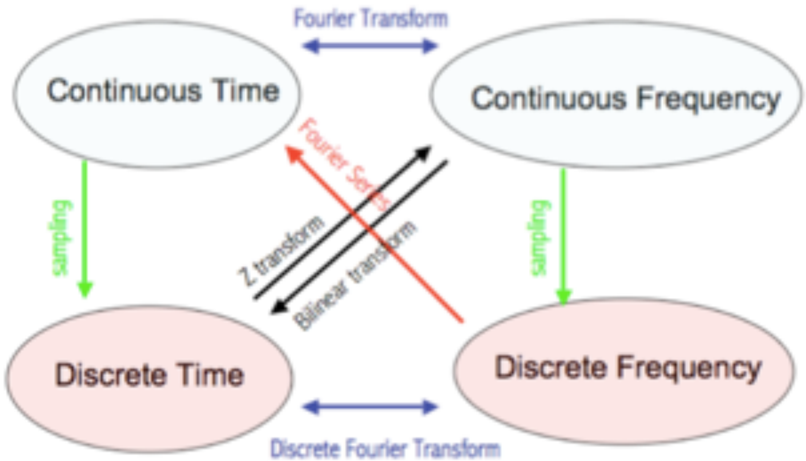


Convolution

- This particular convolution smooths out some of the high frequencies in f(t).

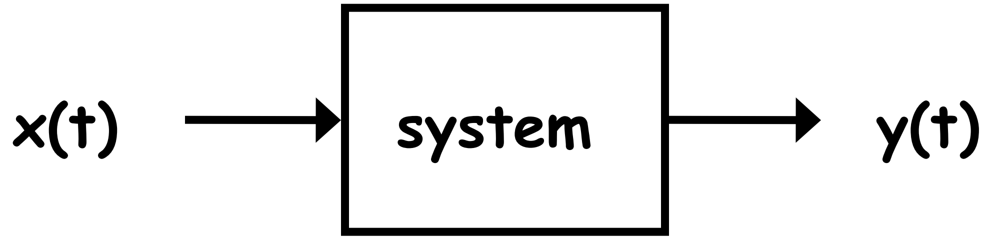


Various spaces and transforms



Signal type	Continuous time	Discrete time	Transform Domain
Finite duration	Laplace	z	Continuous complex frequency (s-plane)
Finite duration	Fourier	Discrete-time Fourier (DTFT)	Continuous real frequency
Periodic	Fourier Series	Discrete Fourier Series (DFS)	Discrete real frequency

Transforms for Continuous Systems Analysis



Differential Equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \cdots + b_0 x$$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad \text{and} \quad x(t) = \frac{1}{j2\pi} \oint_{contour} X(s) e^{st} ds$$

Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{and} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + \cdots + b_0 x$$

Transfer Function

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \ldots + a_1 s^1 Y(s) + a_0 s^0 Y(s) = b_m s^m X(s) + b_{m-1}s^{m-1}X(s) + \ldots + b_0 X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

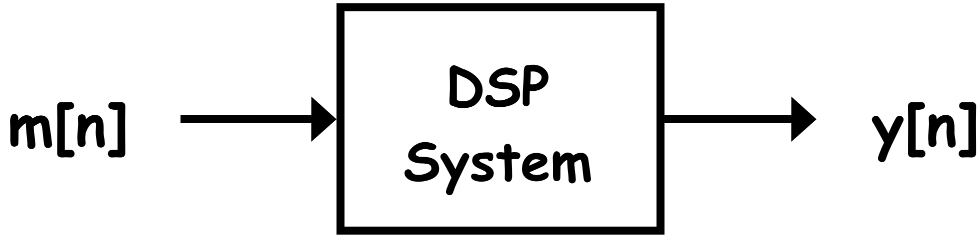
Frequency Response

$$(j\omega)^n Y(j\omega) + a_{n-1}(j\omega)^{n-1}Y(j\omega) + \ldots + a_0 Y(j\omega) = b_m (j\omega)^m X(j\omega) + \ldots + b_0 X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_m (j\omega)^m + b_{m-1}(j\omega)^{m-1} + \ldots + b_1 (j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1 (j\omega) + a_0}$$

The values of where the numerators is zero are referred to as **zeros** , as the response is zero at this frequency, regardless of the amplitude of the input signal. Conversely, frequencies for which the denominator is zero are called **poles**, as the response becomes very large at these frequencies.

Transforms for Discrete Systems Analysis



Difference Equation:

$$y[n] + a_1 y[n-1] + \ldots + a_k y[n-k] = b_0 m[n] + b_1 m[n-1] + \ldots + b_m m[n-l]$$

z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{and} \quad x[n] = \frac{1}{j2\pi} \oint_{\text{contour}} X(z) z^{n-1} dz$$

Discrete Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{and} \quad x[n] = \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$y[n] + a_1 y[n-1] + \ldots + a_k y[n-k] = b_0 m[n] + b_1 m[n-1] + \ldots + b_m m[n-l]$$

Transfer Function - z transforms

$$z^n Y(z) + a_{n-1} z^{n-1} Y(z) + \ldots + a_1 z Y(z) + a_0 Y(z) = b_m z^m X(z) + b_{m-1} z^{m-1} X(z) + \ldots + b_0 X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

Frequency Response

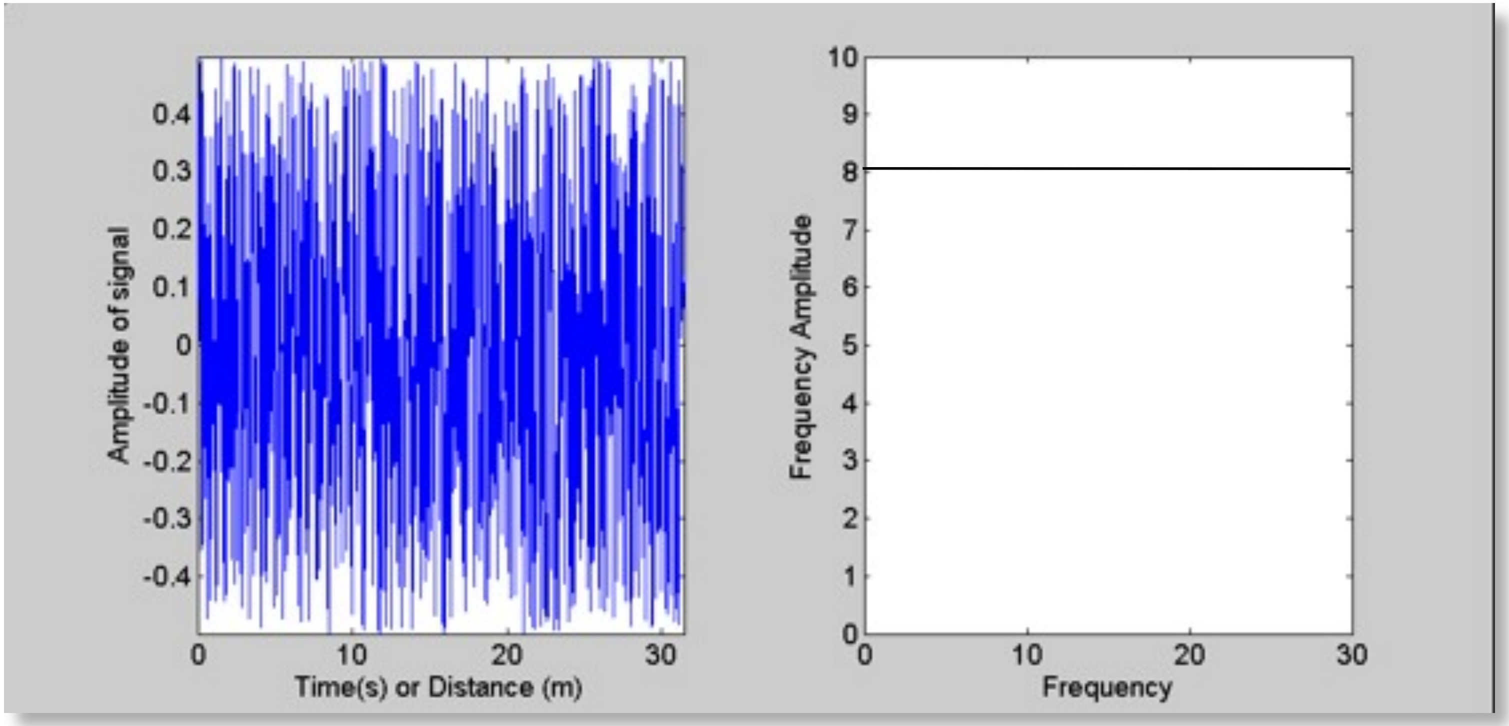
$$e^{j\omega n} Y(e^{j\omega}) + a_{n-1} e^{j\omega(n-1)} Y(e^{j\omega}) + \ldots + a_0 Y(e^{j\omega}) = b_m e^{j\omega m} X(e^{j\omega}) + \ldots + b_0 X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_m e^{j\omega m} + b_{m-1} e^{j\omega(m-1)} + \ldots + b_1 e^{j\omega} + b_0}{e^{j\omega n} + a_{n-1} e^{j\omega(n-1)} + \ldots + a_1 e^{j\omega} + a_0}$$

The values of where the numerators is zero are referred to as **zeros** , as the response is zero at this frequency, regardless of the amplitude of the input signal. Conversely, frequencies for which the denominator is zero are called **poles**, as the response becomes very large at these frequencies.

Fourier Spectra: Main Cases

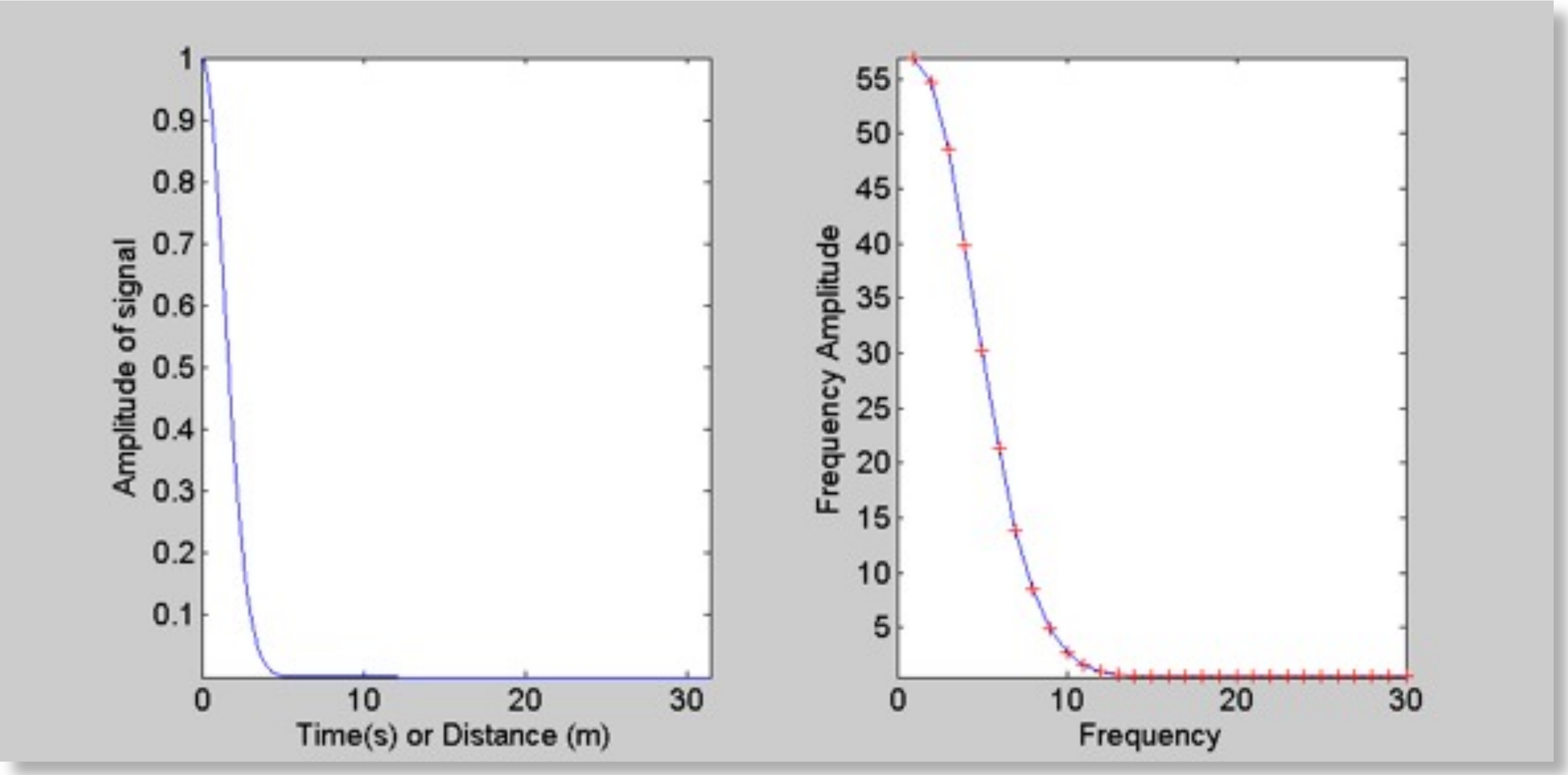
random signals



Random signals may contain all frequencies. A spectrum with constant contribution of all frequencies is called a white spectrum

Fourier Spectra: Main Cases

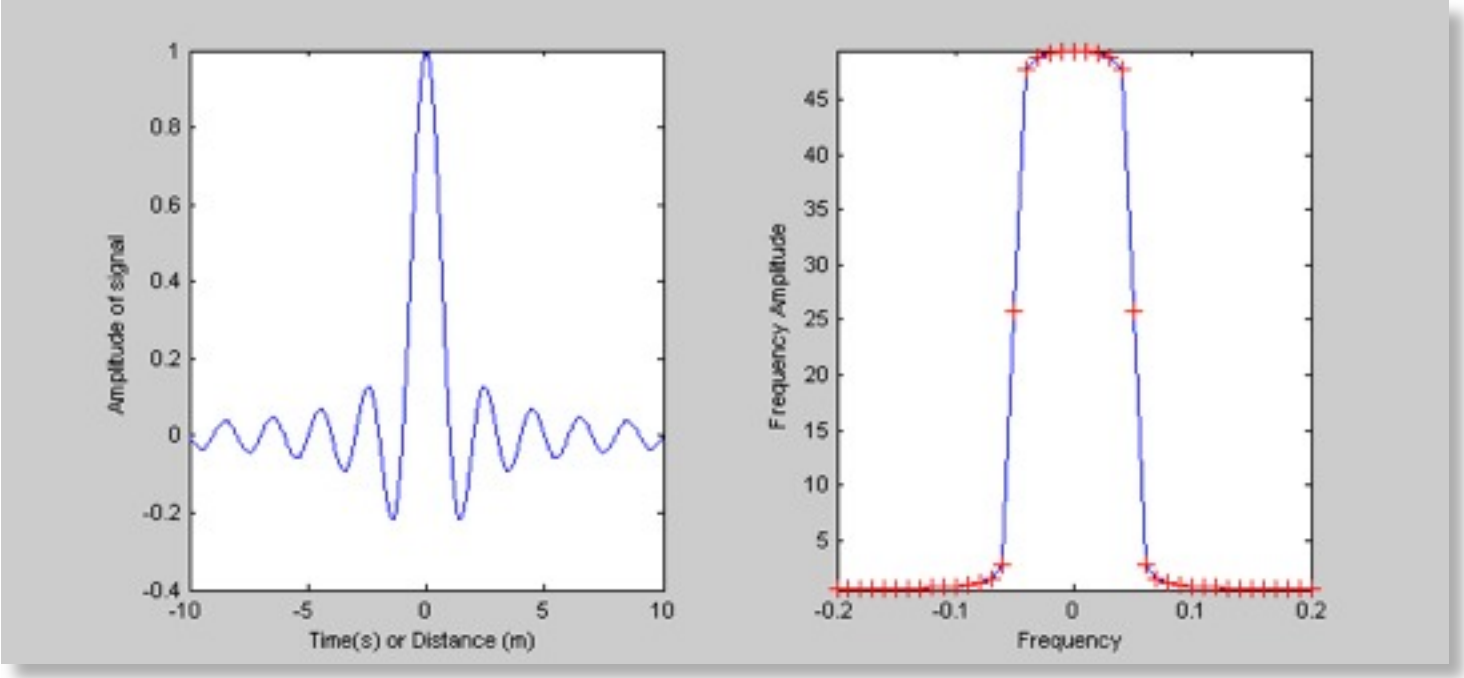
Gaussian signals



The spectrum of a Gaussian function will itself be a Gaussian function. How does the spectrum change, if I make the Gaussian narrower and narrower?

Fourier Spectra: Main Cases

Transient waveform

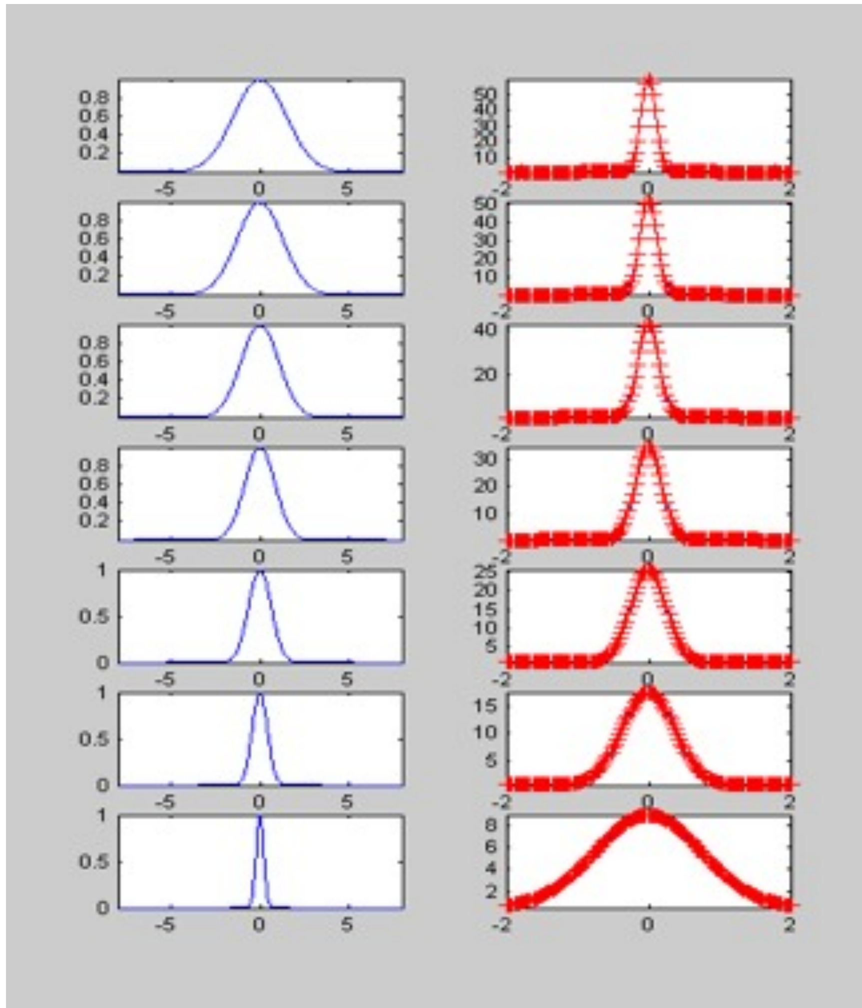


A transient wave form is a wave form limited in time (or space) in comparison with a harmonic wave form that is infinite

Pulse-width and Frequency Bandwidth

time (space) spectrum

Narrowing physical signal



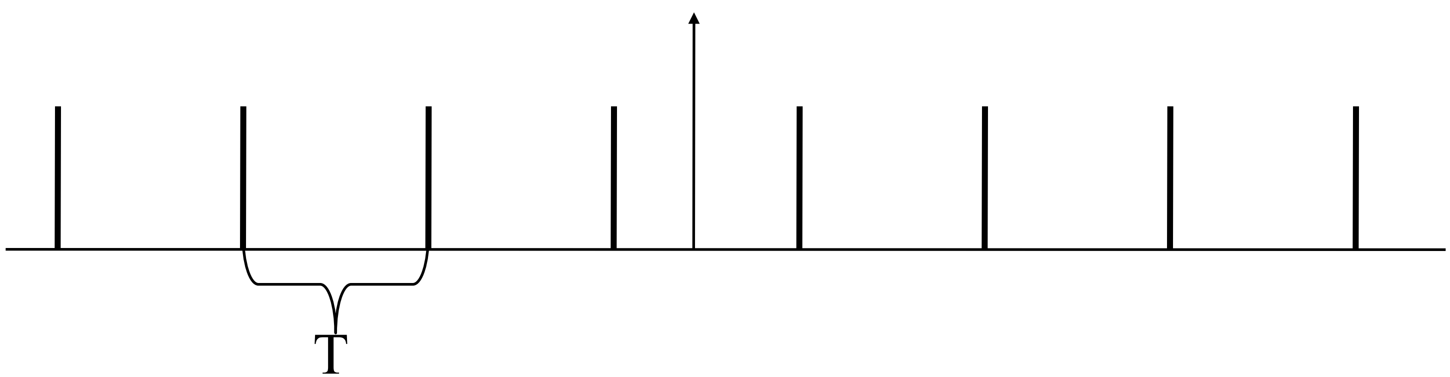
Widening frequency band

Sampling Function

- A Sampling Function or Impulse Train is defined by:

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

where T is the sample spacing.



Sampling Function

- The Fourier Transform of the Sampling Function is itself a sampling function.
- The sample spacing is the inverse.

$$S_T(t) \Leftrightarrow S_{\frac{1}{T}}(\omega)$$

Convolution Theorem

- The convolution theorem states that convolution in the spatial domain is equivalent to multiplication in the frequency domain, and viceversa.

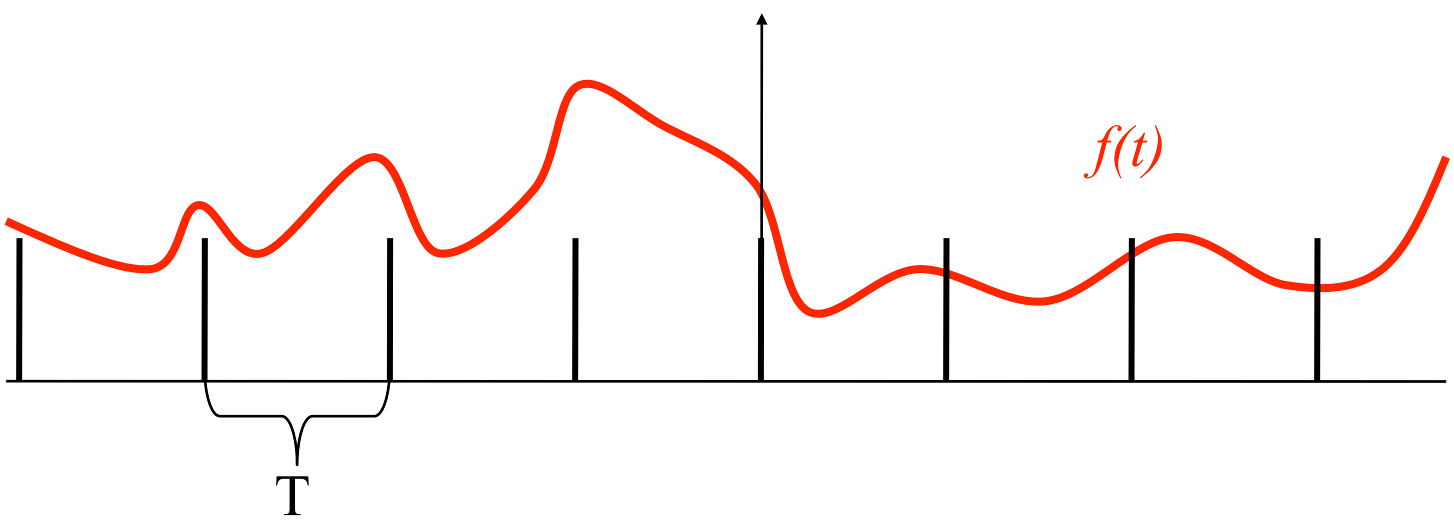
$$f(t) * g(t) \Leftrightarrow F(\omega)G(\omega)$$

$$f(t)g(t) \Leftrightarrow F(\omega) * G(\omega)$$

Convolution Theorem

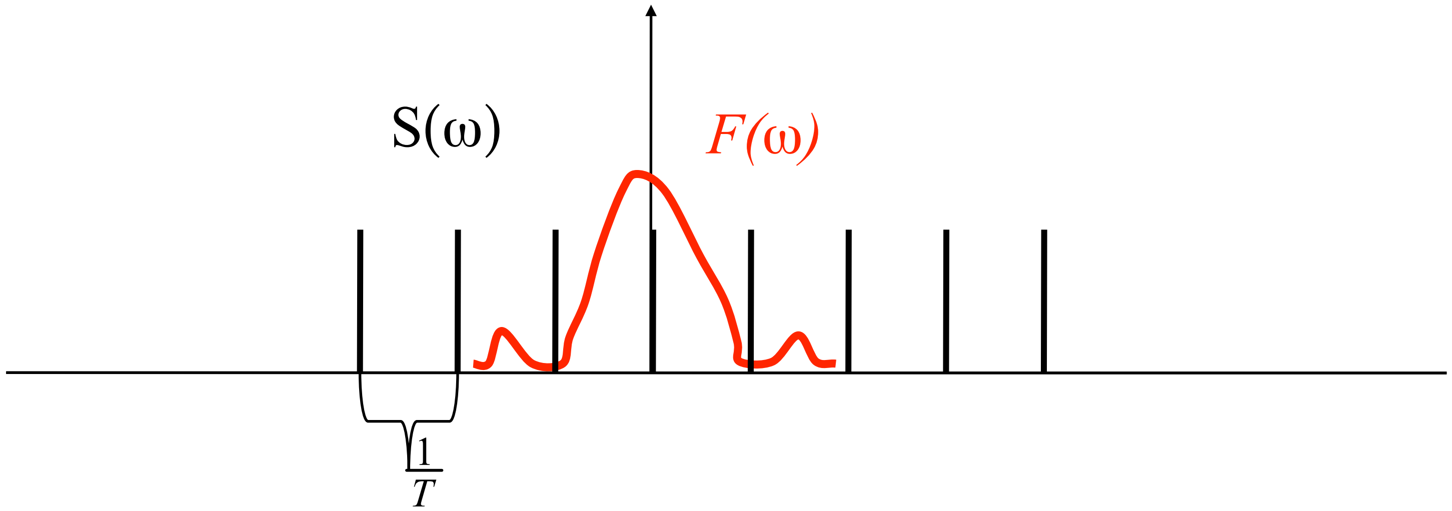
- This powerful theorem can illustrate the problems with our point sampling and provide guidance on avoiding aliasing.

- Consider: $f(t)\ S_T(t)$



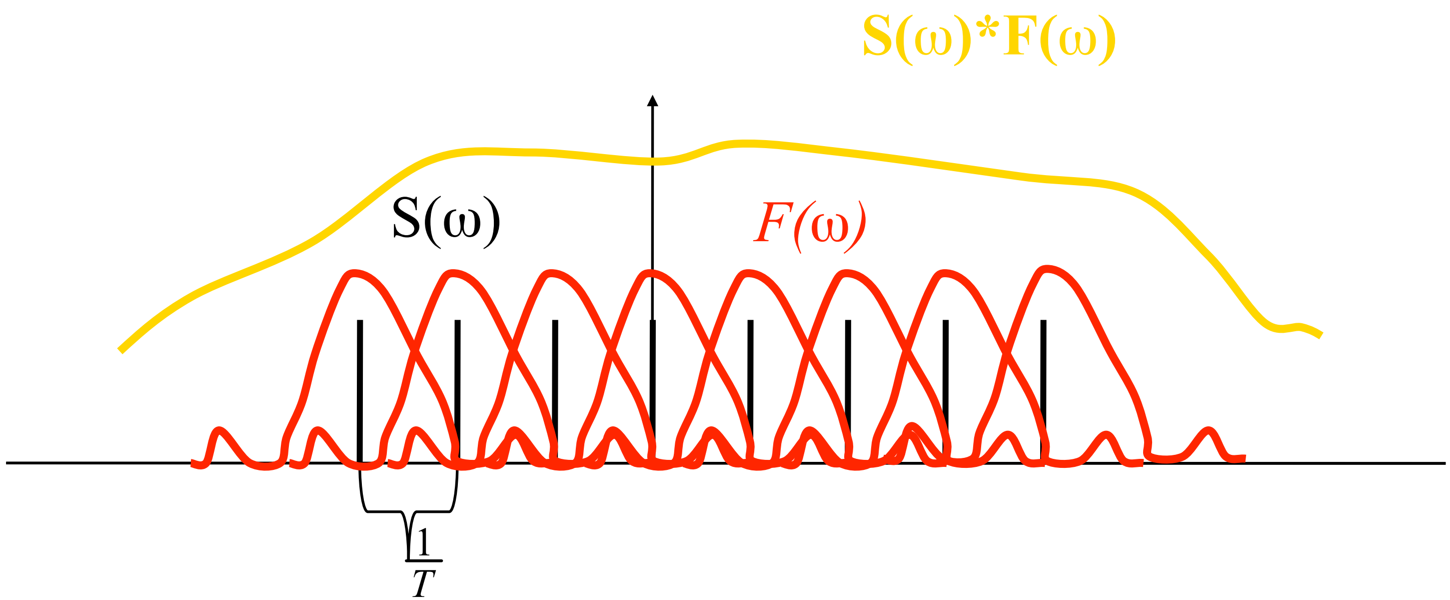
Convolution Theorem

- What does this look like in the Fourier domain?



Convolution Theorem

- In Fourier domain we would convolve



Aliasing

- What this says, is that any frequencies greater than a certain amount will appear intermixed with other frequencies.
- In particular, the higher frequencies for the copy at 1/T intermix with the low frequencies centered at the origin.

Aliasing and Sampling

- Note, that the sampling process introduces frequencies out to infinity.
- We have also lost the function f(t), and now have only the discrete samples.
- This brings us to our next powerful theory.

Sampling Theorem

- The Shannon Sampling Theorem
  - A band-limited signal f(t), with a cutoff frequency of λ, that is sampled with a sampling spacing of T may be perfectly reconstructed from the discrete values f[nT] by convolution with the sinc(t) function, provided:

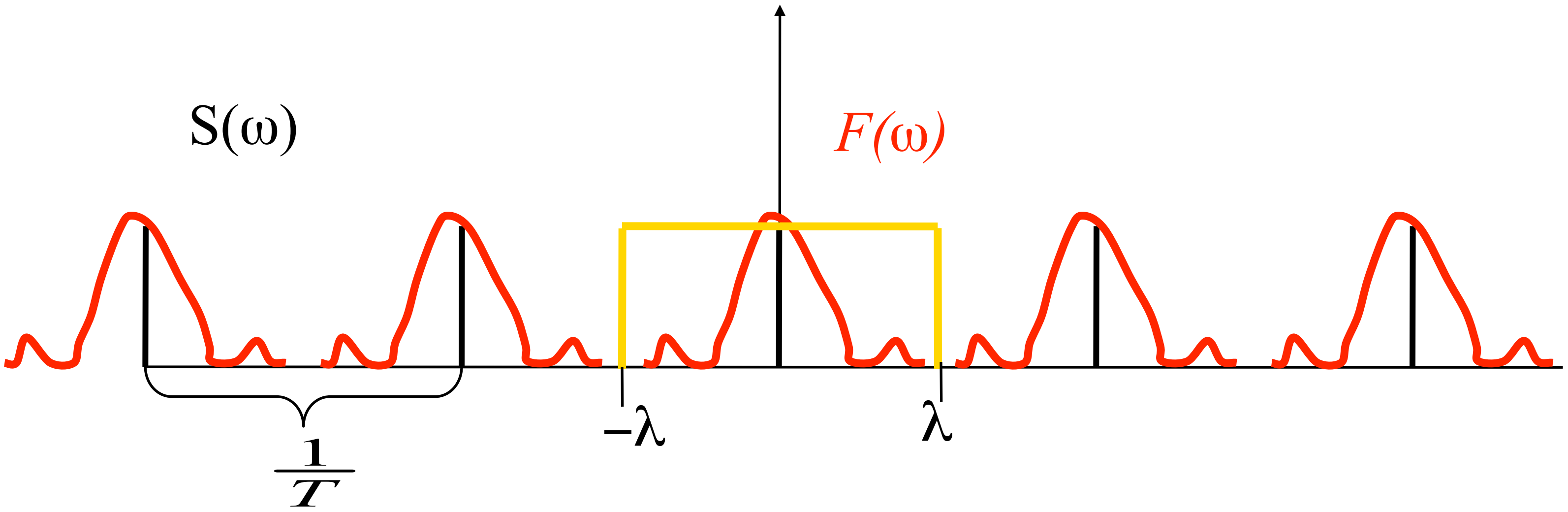
$$\lambda < \frac{1}{2T}$$

Sampling Theory

- Why is this?
- The Nyquist limit will ensure that the copies of F (ω) do not overlap in the frequency domain.
- I can completely reconstruct or determine f(t) from F(ω) using the Inverse Fourier Transform.

Sampling Theory

- In order to do this, I need to remove all of the shifted copies of F(ω) first.
- This is done by simply multiplying F(ω) by a box function of width 2λ.



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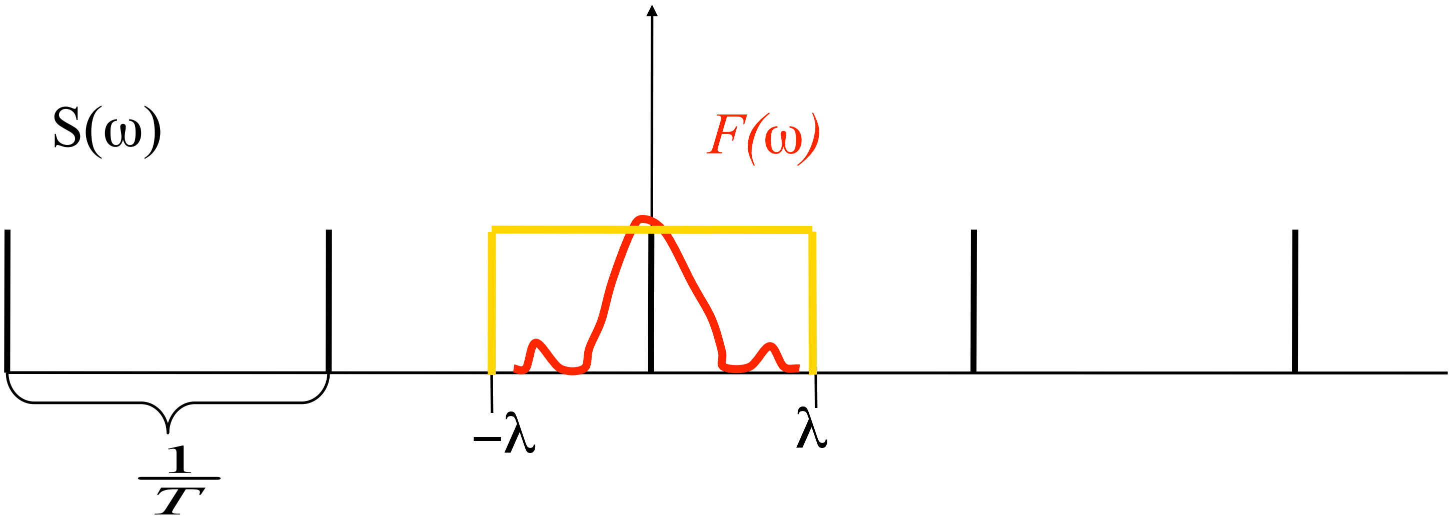
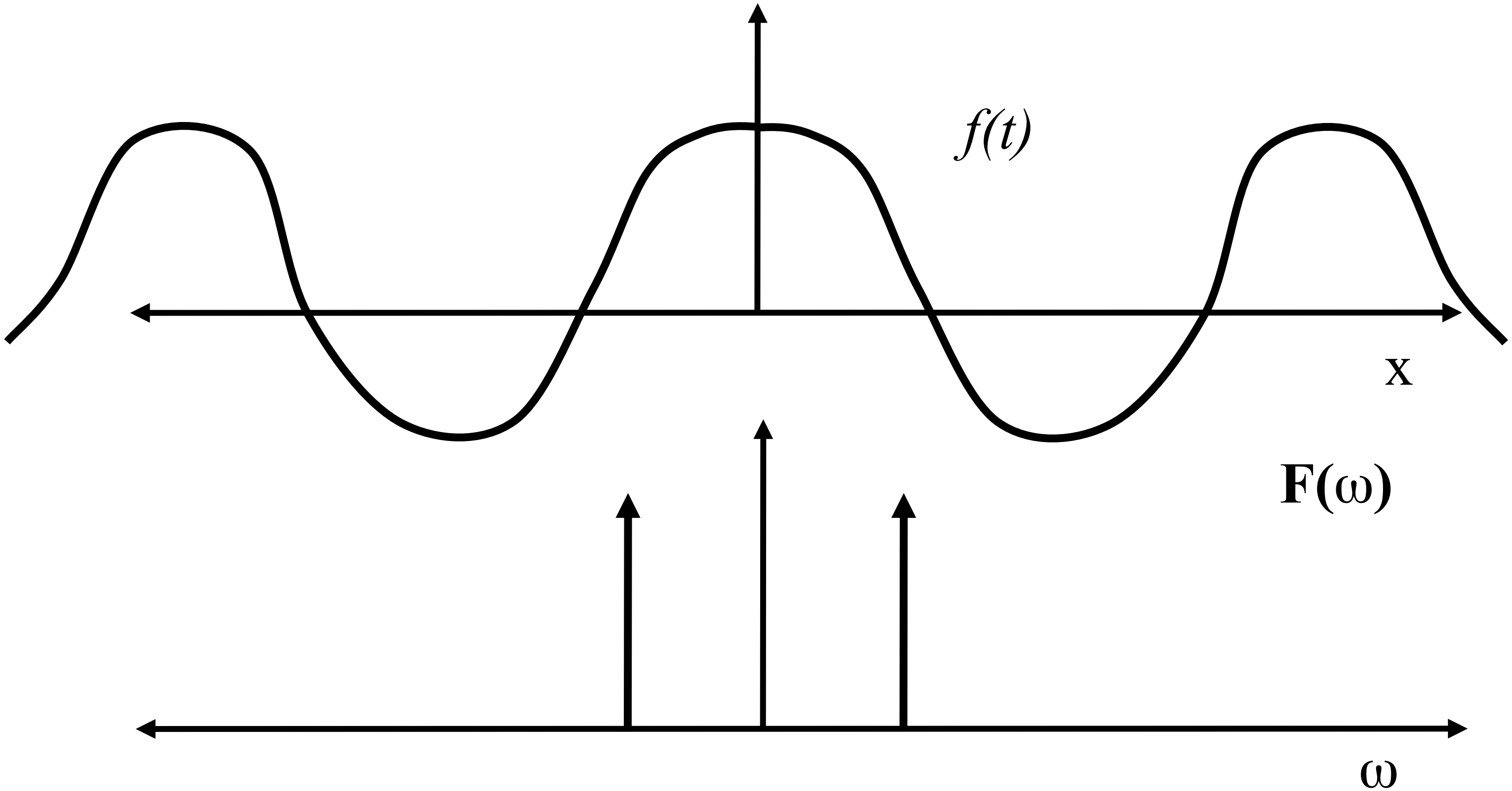


A Cosine Example

- Consider the function f(t)=cos(2πt).



Sampling Theory

- So, given f[nT] and an assumption that f(t) does not have frequencies greater than 1/2T, we can write the formula:

  $$f[nT] = f(t)\, S_T(t) \Leftrightarrow F(\omega) * S_T(\omega)$$

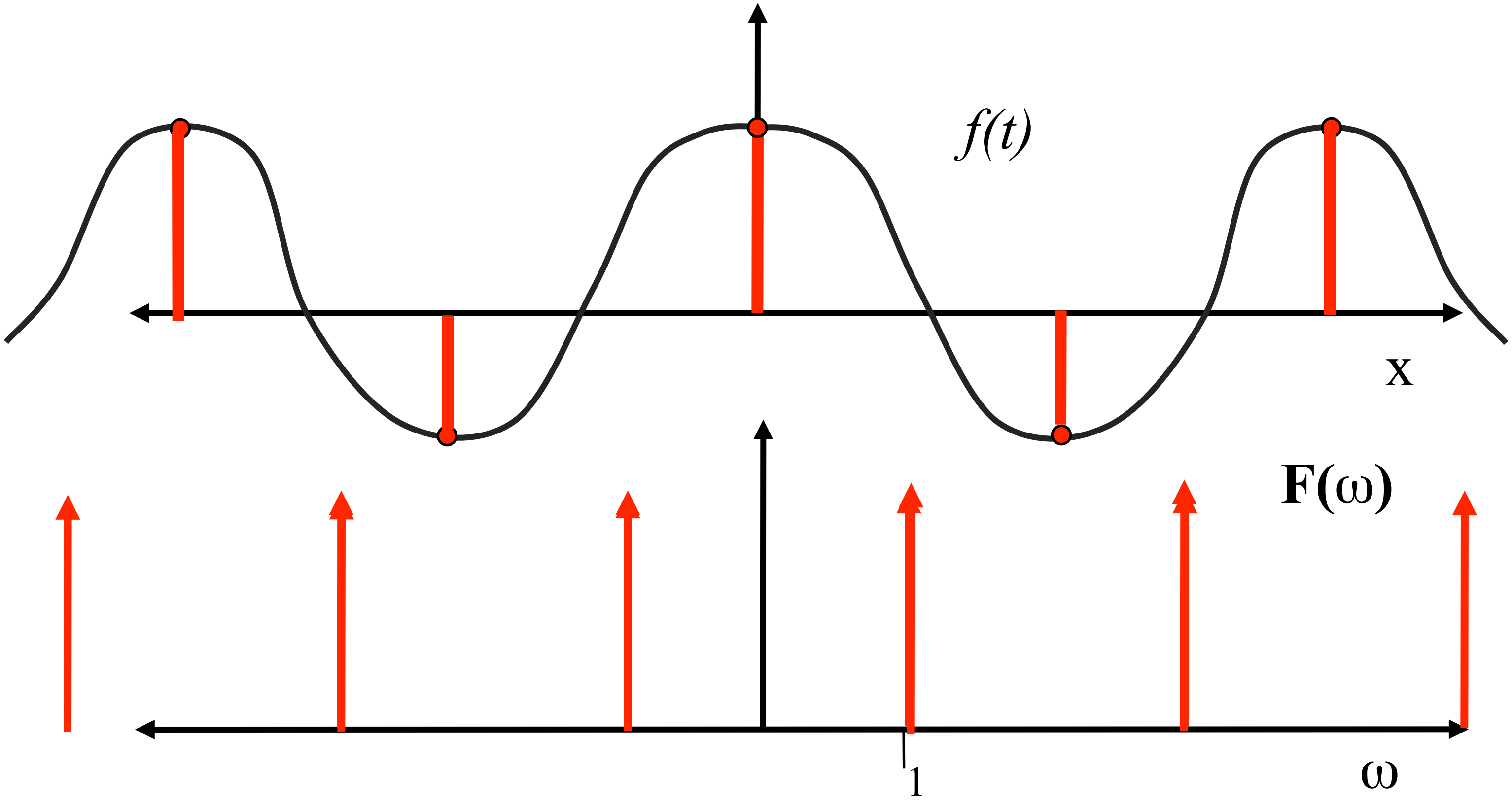
  $$F(\omega) = (F(\omega) * S_T(\omega))\, Box_{1/2T}(\omega)$$

  therefore,

  $$f(t) = f[nT] * \mathrm{sinc}(t)$$

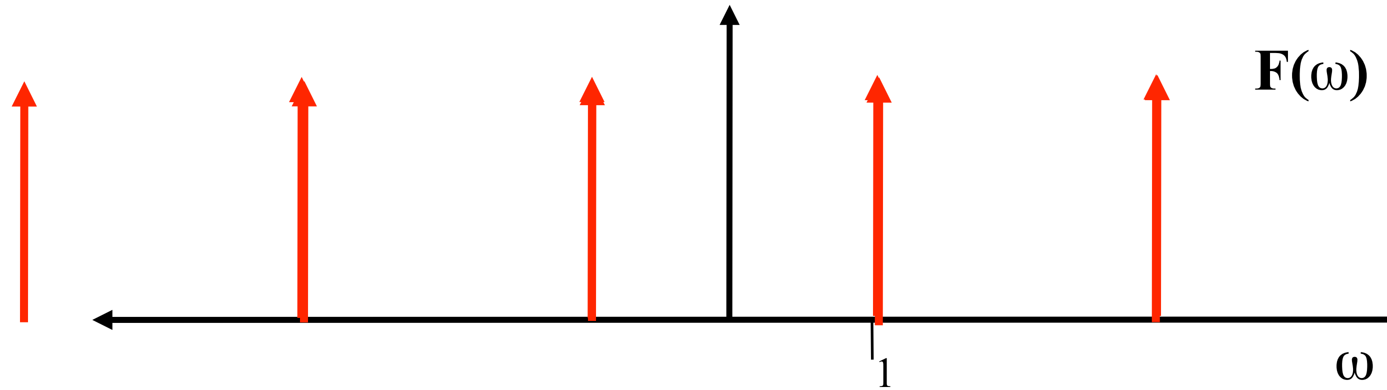
A Cosine Example

- Now sample it at T=1/2



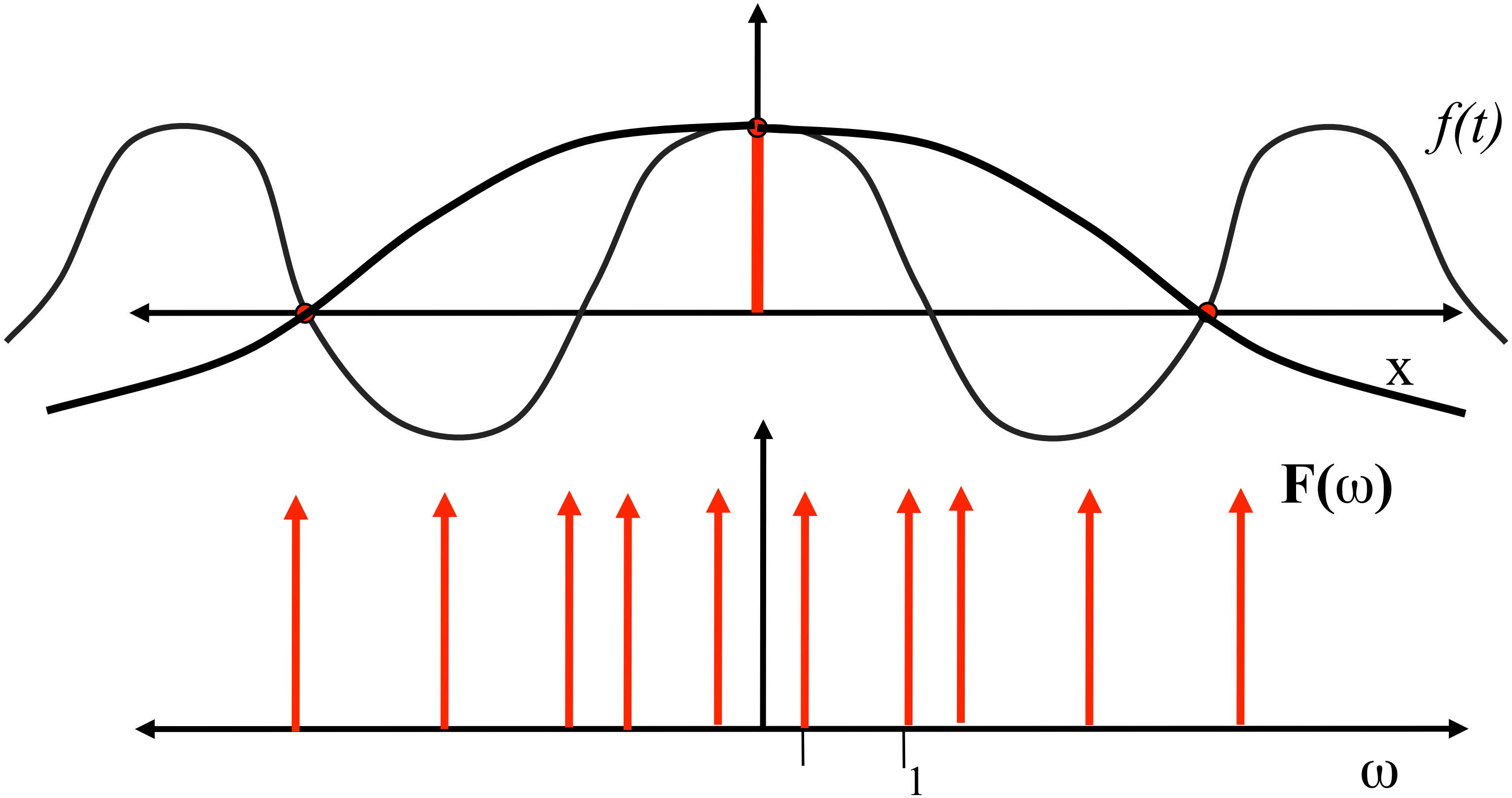
A Cosine Example

- Problem:
  - The amplitude is now wrong or undefined.
- Note however, that there is one and only one cosine with a frequency less than or equal to 1 that goes through the sample pts.



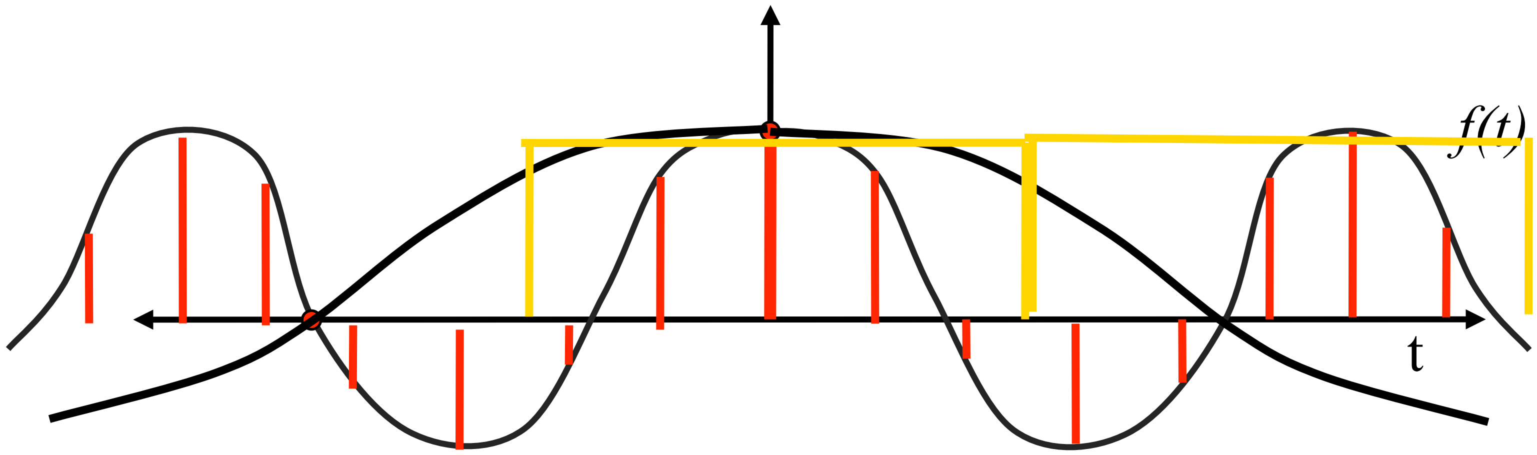
A Cosine Example

- What if we sample at T=2/3?



Supersampling

- Supersampling increases the sampling rate, and then integrates or convolves with a box filter, which is finally followed by the output sampling function.



Sampling and Anti-aliasing

- The problem:
  - The signal is not band-limited.
  - Uniform sampling can pick-up higher frequency patterns and represent them as low-frequency patterns.