# Chapter 1
# Fundamental Fluid Mechanics

**Abstract** This chapter of the book introduces the basic elements of fluid mechanics constituting the essential background for understanding the blood flow phenomena in the cardiovascular system. It discusses the physics of flow and its implications. This chapter is aimed to provide an intuitive understanding, accompanied by an essential mathematical formulation that ensures a rigorous reference ground.

## 1.1 Fluids and Solids, Blood and Tissues

The most definitive property of fluids, which include liquids and gases, is that a fluid does not have preferred shape. A fluid takes the shape of its container regardless of any geometry it had previously. In contrast, a solid consists of constituting elements with a predefined shape. When the relative position of these constituent elements is infinitesimally changed, internal stresses develop to restore the elements to their original, stress-free state. This distinctive property of solids is called elasticity. An elastic deformation typically is completely reversible as the energy stored in the deformed elements is totally released when the deformation ceases.

Fluids, on the other hand, do not share this feature of the solid materials. They have no preferred geometry; thus they possess infinitely independent, stress-free states. Nevertheless, fluids exhibit an internal resistance during their relative motion. This resistance is due to the development of internal stresses in response to a "rate of deformation". This behavior is due to *viscosity*. Therefore, a fluid experiences a viscous resistance during the motion, which is caused due to sliding fluid elements on each other. Given that the viscous stresses represent a frictional phenomenon that appears during motion, when the motion is ceased, no internal stress returns the fluid to its original state, as in the solids. The mechanical energy that deforms the fluid elements is not being stored anywhere; it dissipates due to internal viscous friction, which is transformed into heat and

A. Kheradvar, G. Pedrizzetti, *Vortex Formation in the Cardiovascular System*, DOI 10.1007/978-1-4471-2288-3_1, 

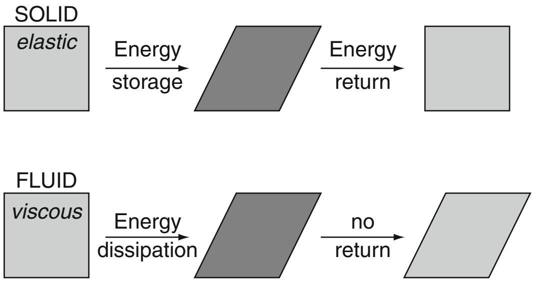


**Fig. 1.1** Solid materials are characterized based on their elastic behavior. Due to elastic deformation of materials, the elastic potential energy is stored in the elements of that solid, which is released when the solid returns back to its original shape. Alternatively, the viscous behavior of fluids appears as an internal resistance during the deformation process, which is due to internal shear-stresses that are friction-driven, and are associated with the dissipation of energy

dispersed away. This energetic difference between elastic and viscous behaviors is sketched in Fig. 1.1. However, the distinction between fluids and solids is not as sharp. Most materials present both elastic and viscous behaviors. Some materials can behave either as fluids or solids in some respects. For example, a glacier is a solid if one can walk on it, yet it flows like a fluid during its slow motion over the years.

Blood is composed of deformable cells (elastic elements) immersed into plasma (fluid element). Therefore, blood is not a simple material; rather it is a mixture of heterogeneous elements. If the dimension of the cells is comparable with the size of the container, the corpuscular nature of the blood takes a fundamental role in the physical processes occurring at such a scale. One example is the blood flow in the capillaries where red blood cells (RBCs) as biconcave disks with a diameter of about 7–8 μm must deform to pass through the vessels with diameters as small as 5 μm. Flow in the arterioles as well as venules is also directly influenced by the corpuscular nature of blood. As the diameter of the blood vessel increases, the influence of individual RBC progressively decreases. Every 1 $mm^3$ of blood contains about 2 million RBCs. Therefore, it is estimated that blood flow in vessels with diameters larger than 1 mm is rather continuous than granular. This representation of blood allows employing a rich theoretical background of continuum mechanics and differential mathematics to solve problems involving biological flows.

Once assuming blood as a continuum, its corpuscular nature is represented by viscosity, which cannot be considered constant. In fact, the apparent blood viscosity is not an intrinsic material property, and thus changes its value depending on the type of blood motion at different sites. For example, blood viscosity is reduced in regions with high shear rates when the blood cells are separated away and the observed friction is mostly due to the plasma. Conversely, the viscosity is increased in the central part of a rotating duct due to denser population of RBCs there. Therefore, as a general rule, the viscosity of blood is a function of the percent concentration of RBCs in blood or local hematocrit. Such variability is influenced by several factors, and is usually small. However, evaluation of such small variations is difficult particularly for three-dimensional flows with whirling motion. Therefore, flow in large vessels is usually treated as a Newtonian fluid, which is a continuous fluid with constant viscosity whose value is about three times greater than the viscosity of water.

Occasionally, when friction is negligible compared to other existing factors, blood behavior may be approximated as an ideal fluid with no viscosity. Such approximation is the basis of the Bernoulli theorem (to be discussed in Sect. 1.4). The Bernoulli theorem is useful in computations involving brief tracts or situations where blood elements are away from the vessels' boundaries. However, viscous forces are never negligible adjacent to the solid boundaries.

## 1.2 Conservation of Mass

The first physical law governing the mechanics of blood as a continuum is the *conservation of mass*, or *law of continuity*. The conservation of mass states that the difference between the flow that enters and leaves a certain container is equal to the variation of the volume of fluid in that container. In general, this principle also accounts for the variation of the fluid density due to either compression or dilation. Blood is essentially incompressible under physiological conditions, meaning that its density cannot vary appreciably.

When applied to systems with rigid walls, continuity states that the flow that enters through a rigid vessel is identical to the flow that exits at the same instant. In other terms, the discharge flow inside a rigid vessel is the same when measured at any cross-section of the vessel, independent of the vessel's geometry. The discharge or flow-rate, $Q$, is given by the product of the area of the cross-section, $A$, and the blood velocity herein, $U$,

$$Q = U \times A; \tag{1.1}$$

where, $U$ is the longitudinal velocity averaged over the whole cross-section. The continuity law states that $Q$ is constant along a rigid vessel. Therefore, if the cross-sectional area $A$ decreases, the velocity $U$ should necessarily increase to keep their product constant. As a result, the flow moves faster where the diameter is less and slower where it is more.

The concept of continuity is also valid for the flow entering into a compliant chamber. If the chamber volume, $V$, varies during time, the entering flow-rate, $Q_{in}$, is not necessarily equal to the exiting one, $Q_{out}$, and their difference corresponds to the fluid stored in the compliant chamber

$$Q_{in} - Q_{out} = \frac{dV}{dt} \tag{1.2}$$

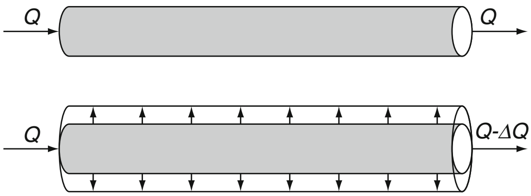


**Fig. 1.2** In a rigid vessel (*above*), the flow-rate cannot change along the vessel, in an elastic vessel the flow-rate is reduced downstream during the vessel expansion because part of the incoming fluid is stored laterally

As an example, for the case of the left ventricle, during systole when the mitral valve is closed and $Q_{in}=0$, the difference between end-diastolic and end-systolic volumes corresponds to the volume flown into aorta. Similarly, the same volume variation corresponds to the transmitral flow during diastolic filling.

A compliant vessel is able to store part of the incoming fluid during expansion. Therefore, the discharge is not constant along a vessel with elastic walls. The flow reduces downstream during vessel expansion and increases during contraction. This mechanism is used by compliant vessels to smooth out the sharp flow accelerations, accumulate the blood volume when the vessels expand and release volume during contraction. The law of continuity (1.2) states that the difference in the flow-rate between the two consecutive cross-sections must balance the change of the internal volume in that segment of the vessel. For example, the volume stored laterally in a segment with length $L$ of an expanding vessel is $\Delta V = \Delta A \times L$, where $\Delta A$ is the change in the vessel area. Continuity (Eq. 1.2) also states that the rate of change of volume $dV/dt$, which is equal to $dA/dt \times L$ balances the difference of the flow rate, $\Delta Q$, between the two ends of the vessel segment. As a result, the (negative) flow gradient along the vessel, $\Delta Q/L$, is balanced by the rate of expansion of the cross-sectional area of vessel, $dA/dt$ (Fig. 1.2).

In general, by considering an arbitrarily brief segment, the flow gradient $\Delta Q/L$ along the vessel is expressed by $dQ/dx$, and the law of continuity for the vessel describes as

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{1.3}$$

Equation 1.3 shows how a sharp peak of flow decreases downstream ($dQ/dx < 0$) along an elastic (compliant) vessel due to vessel expansion ($dA/dt > 0$) based on a synchronous peak of pressure (Pedley 1980, Sect. 2.1.1; Fung 1997, Sect. 3.8).

The law of continuity has been expressed in Eq. 1.2 in terms of a balance for the entire fluid volume, while Eq. 1.3 describes a balance across a cross-section of a duct. The same law holds for any arbitrary small volume of a moving fluid. Intuitively, conservation of mass implies that the fluid cannot move away from a given point along all directions unless additional volume of fluid replaces for it. In other words, the flow cannot *diverge* from a point. As an example, consider a microscopic volume as sketched in Fig. 1.3, continuity requires that the flow diverging from such volume along one direction will be balanced by the flow converging to

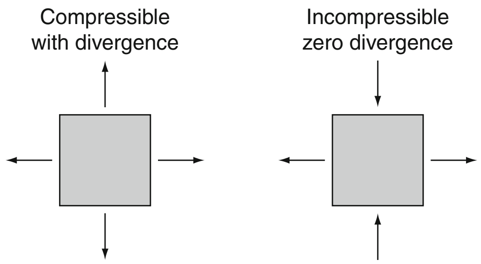


**Fig. 1.3** The divergence of flow creates a deficit of mass inside the generic volume that can be accommodated only by a change of the flow density. In an incompressible fluid, the flow density cannot vary, and the contributions to mass along the different directions must balance and result in a total zero divergence of the flow

the same volume from another direction. In general, at any point in a flow field, the total *divergence* of the fluid motion must be equal to zero.

This concept introduces an important constraint on the physically realizable patterns of flow motion. Such a constraint facilitates the development of coherent vortices in the flow. When the velocity is constant it automatically satisfies the continuity since divergence can only occur in the presence of spatial velocity gradients (see Panton 2005, Sect. 5.1). In Fig. 1.3, the small volume presents a mass deficit along the direction $x$ due to an increase in the velocity along that direction. This is a positive velocity gradient $\partial u_x/\partial x$. When the gradient is zero, the velocity at the two faces is equal and the same flow that enters one side exists from the other. Such a gradient measures the total flow-rate per unit volume, entering across the two surfaces facing the direction $x$. Similarly, the gradient, $\partial u_y/\partial y$, is the total flow-rate across the surfaces facing the direction $y$. In an incompressible fluid, the total flow-rate contributing to a fixed volume must be equal to zero. Positive gradient must be balanced by opposite-sign gradient along the other directions. In Equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{1.4}$$

the velocity vector field has zero divergence. Equation 1.4 is often written in a more general and compact form as

$$\nabla \cdot u = 0 \tag{1.5}$$

using the pseudo-vector operator *nabla*.[1]

The zero-divergence characteristic is very important for the velocity field in cardiovascular system as it drastically reduces the adverse behavior of blood trajectories. Due to this property, vortices can be only developed along the regions where the flow is in contact with a wall.

[1] The generalization of equations in three dimensions often makes use of the vector-operator ∇, called *nabla*, for simpler compact writing. Nabla is a three-component derivative operator that can be seen as a vector of derivatives whose components in Cartesian coordinates, are given by [$\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$].

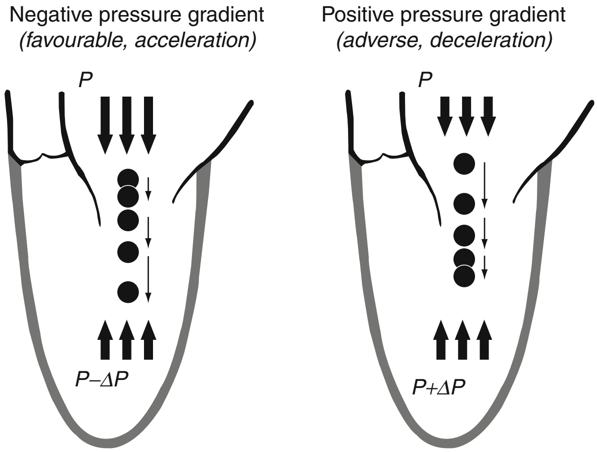


**Fig. 1.4** A fluid particle accelerates in presence of a negative pressure gradient (*left picture*) and decelerates in presence of an adverse pressure gradient

## 1.3 Conservation of Momentum and Bernoulli Theorem

Momentum is a vector quantity equivalent to the product of the mass and the velocity. The conservation of momentum states that the impulse of any system does not change unless an external force is applied to it. The conservation of momentum can be expressed for a fluid by considering that a flow particle accelerates only when there is a pressure gradient applied to it. As sketched in Fig. 1.4, a fluid particle accelerates along a streamline when there is a *negative pressure gradient* along such a direction. Newton's second law per unit volume of fluid is shown here:

$$\rho a = -\frac{\partial p}{\partial s} \tag{1.6}$$

where $\rho$ is the fluid density. However, the acceleration of a fluid particle, $a$, is not a realistically measurable quantity since individual particles cannot be followed during their motion. Therefore, it is only feasible to deal with quantities measured at fixed positions, rather than moving particles. As a result, the acceleration of a fluid particle is expressed in terms of velocity space-time variations at fixed positions.

Let us consider a particle that instantaneously passes through a fixed position $x$ at the time $t$. With reference to Fig. 1.5 (left), consider a flow field with velocity that is spatially uniform and increases only in time. The fluid particles in this field accelerate while they cross the position $x$ as the velocity increases everywhere. This acceleration given by the local velocity time-derivative $\partial u/\partial t$ is called *inertial acceleration* since it is associated with a change in the inertia of a volume of fluid. A particle may also accelerate in a steady flow when the velocity is constant everywhere in time, if it moves from a region of low-velocity toward a region where velocity is higher. The Fig. 1.5 (right) illustrates a particle that accelerates when

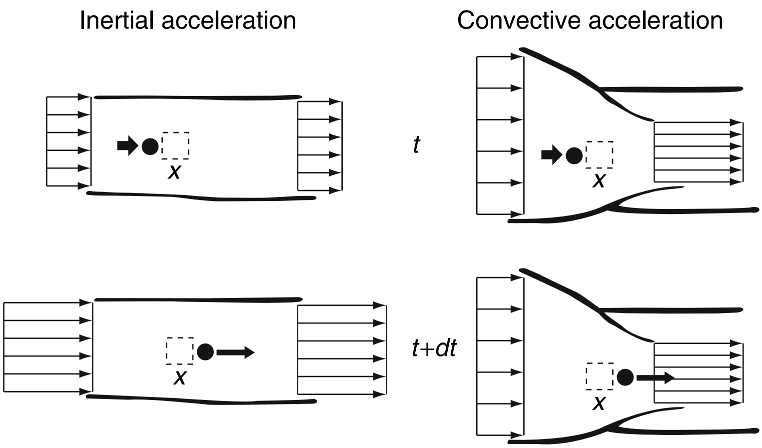


**Fig. 1.5** Acceleration of a fluid particle measured in terms of velocity variations around the fixed position *x*. A particle accelerates when its velocity increases while crossing the position *x*. This can occur for inertial acceleration (*left panel*) when velocity at all places is increasing during time. Acceleration can also occur due to convection (*right panel*) when a particle moves toward a region where velocity is higher

enters into a section with smaller diameter. In other words, a particle accelerates if the velocity increases along the direction of the particle motion. This is called *convective acceleration*, which considers a particle that crosses the position *x* traveling a small distance $ds = udt$ during the time interval $dt$. If the velocity increases along the length $ds$, the particle experiences an acceleration based on the rate of change of velocity ($du/dt=udu/ds$).

In general, acceleration occurs based on two possibilities: (a) inertial acceleration, a reflecting the local time increase of velocity, and (b) convective acceleration, reflecting an increase in velocity in the direction of motion. In mathematical terms, the Newton law of motion (1.6) zfor the fluids translates as:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s} \tag{1.7}$$

that is known as the *Euler equation*. This equation, although physically represents the balance of momentum for a moving particle, contains gradients of velocity and pressure, measurable at fixed points only. This equation requires some underlying assumptions; the first is that *s* is the direction of motion along a streamline, the second is that there is no frictional forces, and lastly no external force is being applied to the fluid.[2]

[2] This equation is also valid in presence of gravity (or any other *conservative* force).

The Bernoulli law is derived from the Euler equation. The Euler Eq. 1.7 can be rewritten in the alternate form of

$$\rho\frac{\partial u}{\partial t}+\frac{\partial}{\partial s}\left(\frac{1}{2}\rho u^{2}+p\right)=0 \tag{1.8}$$

which shows that the variation in the total energy along a streamline is balanced by the change of fluid inertia. Considering the integration rule for the sum of the variations along a line, integration of Eq. 1.8 between two arbitrary points of 1 and 2 along a streamline gives:

$$\frac{1}{2}\rho u_2^2+p_2=\frac{1}{2}\rho u_1^2+p_1-\rho\int_1^2\frac{\partial u}{\partial t}dx \tag{1.9}$$

which is called the general *Bernoulli equation*. The Bernouli equation represents the conservation of mechanical energy given by the sum of the potential energy, $p$, and the kinetic energy, $\frac{1}{2}\rho u^2$. The Bernoulli theorem states that variation of the mechanical energy from a point to another point along a streamline results in acceleration or deceleration of the fluid along that path, if there is no frictional (viscous) forces. The last term accounts for variation in the fluid inertia.

When inertial effect is negligible, the Bernoulli theorem reduces to a special case of the conservation of energy, $p+\frac{1}{2}\rho u^2$, along a streamline. The Bernoulli theorem is often employed to measure the pressure drop from a large cardiac chamber (when $u_1$ is negligible), at peak systole or at peak diastole when the inertial term is negligible. The same principle can be used for flow across the aortic valve or across a stenosis. In these cases, Eq. (1.9) simplifies to

$$p_2-p_1=\frac{1}{2}\rho u_2^2 \tag{1.10}$$

This formula is often utilized to obtain pressure from flow velocity, and if the velocity is expressed in terms of *m/s* and pressure in terms of *mmHg*, then $p_2-p_1=4u^2$.

The Bernoulli theorem represents the conservation of mechanical energy and allows evaluating the transformation of potential energy into kinetic energy and vice versa. It explains the underlying connection between variations in velocity and pressure. In its simplest form, flow that enters a stenotic segment increases its velocity due to conservation of mass and, because of the conservation of mechanical energy expressed by the Bernoulli theorem, reduces the pressure to balance the increase in kinetic energy.

In fact, the law of conservation of momentum implies that of conservation of energy and vice versa. In mechanics, momentum and energy balances are equivalent physical laws. The Euler equation (1.7) is an important simplified form, which is valid along a streamline only. For completeness, let us describe the three-dimensional Euler equation as:

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = -\nabla p \tag{1.11}$$

written in compact form using the operator ∇, where the velocity, $u$, is a three-dimensional vector. The Euler equation (1.11) is either a vector equation or a system of three scalar equations along each coordinate. The Euler equation describes the motion of a fluid under the fundamental assumption that any form of friction is neglected, thus it represents the conservation of mechanical energy, allowing transformations between different states with no energy loss. This equation is valid for ideal fluids or *inviscid* flows in which the friction due to viscous phenomena can be neglected. These dissipative phenomena are considered in the next sections.

## 1.4 Conservation of Momentum and Viscosity

Viscosity is an intrinsic property of a fluid that gives rise to the development of viscous shear stresses inside the flow. Energy dissipation due to viscous stresses is the only mechanism for energy loss in fluids. In fact, the viscous friction phenomenon transforms mechanical energy into thermal energy.

The viscous stresses are responsible for the resistances of the fluid against motion. These stresses develop in presence of a velocity difference among the adjacent fluid elements that slide on each other. One example on the development of viscous friction is the confined fluid between two parallel plates, as shown in Fig. 1.6, where one plate moves relatively to the other. This motion is steadily sustained once a constant force is applied to the upper plate. This force is not associated with the acceleration of fluid particles and is balanced by the viscous friction. In such an arrangement, a fluid element is subjected to a forward traction exerted by the faster moving fluid above, and a backward resistance from the slower fluid below. To achieve equilibrium, the two stresses must be equal and opposite, given that the element does not accelerate and no other forces is applied. Such a flow presents a constant shear stress across a fluid gap. This value is equal to the wall shear stress acting on the moving plate and corresponds to the force per unit surface applied to keep it in steady motion.

In a Newtonian fluid, this shear stress is proportional to the shear rate that is equal to the ratio of the wall velocity $U$ to the height of the fluid gap. The proportionality constant is the *dynamic viscosity*, commonly indicated with the symbol $\mu$, whose normal value in blood is about $3.5 \times 10^{-3}$ kg m$^{-1}$ s$^{-1}$. The shear stress $\tau$ on a slice of a moving fluid is proportional to the shear rate across the slice surface:

$$\tau = \mu \frac{du}{dn} \tag{1.12}$$

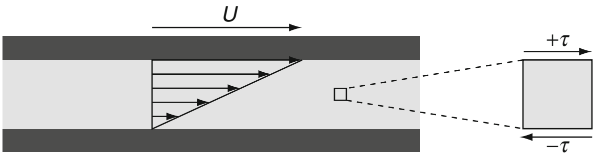


**Fig. 1.6** When the upper plate is in motion with velocity $U$ relative to the fixed plate below, shear stress is constant across the gap between the two plates because every fluid element is in equilibrium subjected to the shear forces given by the faster fluid above and the slower fluid below. In a Newtonian fluid, the velocity grows linearly from zero adjacent to the bottom plate to $U$ at the upper plate

In a non-Newtonian fluid, the viscosity is not constant and depends on the local state of shear rate of the fluid elements. However, as previously discussed in Sect. 1.1, non-Newtonian effects are typically negligible in large vessels due to the dominance of the inertia in that setting. As a result, the non-Newtonian corrections do not significantly affect the analysis involved in vortex dynamics.

Newtonian viscous stresses can be immediately accounted for the motion along a streamline. When shear stress is constant across a streamline, a forward shear stress acts on that streamline from above and an identical backward stress acts from below with no net force (Fig. 1.6). In general (e.g. Fig. 1.4), the motion of the fluid particle is subjected to a shear stress on one side and another shear stress on the opposite side with a negative sign. The total viscous force, per unit volume, is thus due to the variation of the shear stress across the streamline, in formulas ($\partial\tau/\partial n$), where $n$ indicates the direction across that streamline. This viscous force can be described in terms of the velocity, $\mu\partial^2u/\partial n^2$, based on Eq. 1.12. Considering the viscous forces in the equation of motion, it transforms the Euler equation into the Navier-Stokes equation:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial s}=-\frac{\partial p}{\partial s}+\mu\frac{\partial^2 u}{\partial n^2} \tag{1.13}$$

The two equations differ by the viscous term only.[3] This term is not easy to evaluate in most flow conditions, and often neglected when applying the Bernoulli balance. However, in three-dimensional flow, the viscous term must be properly accounted for all gradients of shear stress across a point. As a result, the three-dimensional Navier-Stokes equation describes as (see Panton 2005, Sects. 6.2 and 6.6):

$$\frac{\partial u}{\partial t}+u\cdot\nabla u=-\frac{1}{\rho}\nabla p+\nu\nabla^2 u \tag{1.14}$$

[3] This intuitive result is somehow simplified: the viscous term should include variations along all directions about the streamline. In particular for three-dimensional flow, it should include the other direction perpendicular to both the streamline and to $n$. It was simplified here to avoid unnecessary symbolic complications.

This fundamental equation describes all aspects of the fluid motion, under the assumption of incompressible, Newtonian fluid, in the absence of non-conservative forces. The *kinematic viscosity* $\nu$, which is introduced in Eq. 1.14 is given by the ratio of the dynamic viscosity to the fluid density, $\nu = \mu/\rho$, and its value for blood is about $3.3 \times 10^{-6}$ m$^2$/s (or 3.3 mm$^2$/s). The kinematic viscosity is a more common viscous coefficient when dealing with blood motion, including vortex dynamics, because incompressible motion is independent of the actual value of density. This is a value that enters into play only as a multiplicative factor when motion is quantified in terms of force, work, or energy.

The kinematic viscosity is a small coefficient by itself. Therefore, for the viscous term to show significant effect, presence of sharp velocity gradients are required. Indeed, viscous effects are often negligible with respect to potential-kinetic energy transformation along brief paths (e.g. when the Bernoulli equation is used). Alternatively, dissipation is going to have significant effect once summed up over long fluid paths and for circulation balances.

Presence of viscosity introduces a fundamental novel element to fluid dynamics. Viscosity implies the continuity of motion between adjacent slices that smoothly slide over each other due to the gradient of velocity. However, they cannot present a net velocity difference otherwise the shear rate and the viscous term would rise to infinity. This continuity also applies to the first fluid elements adjacent to a solid boundary, where it implies the *adherence* between the fluid and structure. This condition, normally referred as *no-slip condition*, is a result of viscosity and does not apply in an ideal fluid.

## 1.5 Boundary Layer and Wall Shear Stress

The adherence of fluid at the solid boundaries is a purely viscous phenomenon, which implies that the viscous effects can never be neglected close to the walls. Near the solid boundary, a layer of fluid exists whose motion is directly influenced by the adherence to the boundary. This influence of wall on the fluid is gradually reduced, and sometimes even disappears once moving away from the boundary toward the bulk flow. The *boundary layer* is the region where velocity rapidly grows from a zero value at the wall to reach values comparable to those found at the center of the vessel. The *wall shear stress*, $\tau_w$, is the stress exerted by the fluid over the endothelial layer, and is proportional to the wall shear rate, which is the gradient of velocity at the wall. This value is approximately the ratio of the fluid velocity away from the wall, $U$, to the boundary layer thickness, commonly indicated as $\delta$:

$$\tau_W = \mu \left.\frac{du}{dn}\right|_{wall} \approx \mu \frac{U}{\delta} \tag{1.15}$$

The higher the fluid velocity is, the higher the wall shear stress will be. Additionally, a thinner boundary layer results in higher wall shear stress.

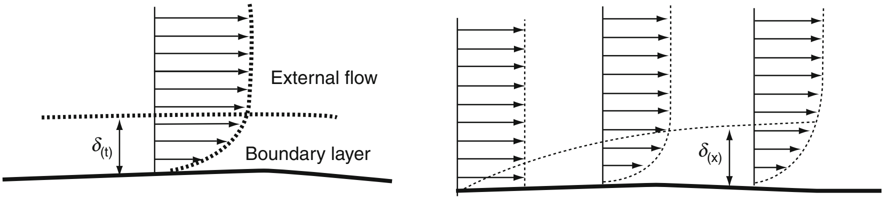


**Fig. 1.7** The boundary layer is the thin region between the wall and the external flow. It grows in time in a starting flow (*left*), and it grows downstream of the entrance of a vessel (*right*)

The thickness of the boundary layer, $\delta$, grows based on the tendency of shear to diffuse away from the wall due to the viscous friction between adjacent fluid elements. Let us consider a fluid that is initially at rest being impulsively set to motion; at the very beginning, the whole fluid volume is set into motion with the exception of the first layer next to the wall where the fluid particles remain adherent. As time proceeds, this viscous adherence slows down the adjacent layers and the boundary layer progressively grows, as shown in Fig. 1.7 (left). Its rate of growth is directly proportional to the kinematic viscosity $\nu$, and inversely proportional to the thickness of the layer ($d\delta/dt \approx \nu/\delta$), which results in a square root growth in time.

$$\delta(t) \approx 5\sqrt{\nu t} \tag{1.16}$$

The coefficient in front (here set equal to 5) may vary in different types of flows. Eq. 1.16 represents a general expression for estimating the length reached by a viscous diffusion process.

Based on the same reasoning, the boundary layer growth downstream the entrance of a vessel can be evaluated; as shown in Fig. 1.7. (Schlichting and Gersten 2000, Sect. 2.2; Fung 1997, Sect. 3.5)

$$\delta \approx 5\sqrt{\nu \frac{x}{U}} \tag{1.17}$$

The size of the boundary layer cannot grow indefinitely as discussed earlier based on the Eqs. 1.16 and 1.17. For example, in a vessel, the boundary layer can grow until it fills the entire vessel. Once its thickness is comparable to the vessel radius, it has no room for further increase. This is the particular case of the *Poiseuille flow* (see Sect. 1.6). Otherwise, for oscillatory cardiac flow, the boundary layer develops during one heartbeat and restarts from zero in the next one. Therefore, its size growth is limited to a fraction of the heartbeat.

Boundary layers are important because they are the locations where the shear stress develops due to the frictional forces on the surrounding wall. These boundary regions are often unstable and can detach from the wall, penetrating the bulk flow regions with high velocity gradients. In fact, the *boundary layer separation* is the only mechanism that generates vortices in incompressible flows.

## 1.6 Simple Flows and Concepts of Cardiovascular Interest

A simple but important type of flow motion is a flow inside a cylindrical vessel with circular cross-section. Considering that the flow is steady and uniform, both inertial and convective accelerations are zero (see Eq. 1.13). To be more precise, a cylinder of fluid with unit length and radius $r$ is pushed ahead by the negative pressure gradient $dp/dx$ acting on the cross-sectional area $\pi r^2$, and is subjected to the shear stress $\tau = \mu du/dr$ on the lateral surface $2\pi r$. These two forces must balance to ensure equilibrium. Therefore:

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx} \tag{1.18}$$

where the pressure gradient is constant over the cross-section because no cross-flow exists. The Eq. 1.18 is satisfied by a parabolic velocity profile $u(r)$ that ensures the adherence condition at the vessel wall (Fig. 1.8). Assuming a vessel with radius $R$, the solution is

$$u(r) = \frac{-1}{4\mu}\frac{dp}{dx}\left(R^2 - r^2\right) = \frac{2U}{R^2}\left(R^2 - r^2\right) \tag{1.19}$$

Equation 1.19 is the well-known *Poiseuille flow* (see Schlichting and Gersten 2000, Sect. 5.2.1; Fung 1997, Sect. 3.2). The *Poiseuille flow* can be equivalently expressed in terms of the pressure gradient or the mean velocity $U$ (second equality in Eq. 1.19), that are related based on $dp/dx = -8\mu U/R^2$ verifiable by taking an integral from Eq. 1.19. It features a maximum velocity at the center that is *twice* the average velocity, and a wall shear stress of $4\mu U/R$. Pressure loss, $-dp/dx$, is usually expressed with respect to the kinetic energy, $½\rho U^2$, per unit diameter length, $D = 2R$ of the tube. The *friction factor*, $\lambda$, is described as:

$$\lambda = \frac{-2D}{\rho U^2}\frac{dp}{dx} \tag{1.20}$$

that is a dimensionless quantity suitable for generalization under different conditions. In the Poiseuille flow, the friction factor is given by $64/Re$ where the *Reynolds number* is

$$Re = \frac{UD}{\nu} \tag{1.21}$$

and represents the ratio of inertial to viscous forces in the fluid. Flows with low values of the Reynolds numbers are highly viscous, smooth, with high pressure-

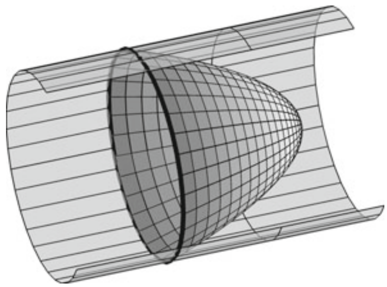

**Fig. 1.8** The parabolic velocity profile or Poiseuille flow, develops in a cylindrical duct with circular fixed walls. It is valid under the assumption of steady flow and represents a good approximation of the time-averaged flow

loss. Flows at higher *Re* exhibit less dissipation to the available kinetic energy, thus they are less smooth and more easily subject to instabilities.

The Reynolds number has a fundamental role in the stability of any fluid motion. Once its value increases above a certain critical threshold, the flow does not dissipate the incoming energy and is prone to instability toward a more complicated motion to achieve further dissipation. The critical value for Poiseuille flow is about $Re_{cr} = 2{,}300$. Above this value, the rectilinear flow becomes unstable. In that case, the flow is considered in transition to turbulence. However, fully turbulent motion is anticipated at *Re above* 10000. Normally, in the cardiovascular system, the Reynolds number is transitional only at mid-diastole in the LV and mid-systole in the aorta. The peak *Re* may even reach about 7000.

Cardiovascular flow is not steady but pulsatile, and the Poiseuille profile (1.19) is valid only where pulsatility is negligible. When studying uniform, unsteady flow, the momentum balance (1.18) must include the inertial acceleration due to the velocity variation in time. A featuring result of the pulsatile flow is that the unsteady boundary layer has a thickness (following Eq. 1.16) proportional to $\sqrt{\nu T}$ where $T$ is the duration of the pulse. When the oscillation is rapid, the inertial term is not negligible, and a thin boundary layer develops near the wall that gives rise to a solution that fundamentally differs from the parabolic profile seen in Poiseuille flow.

The *Womersley number* is the ratio of the vessel diameter, $D$, to the thickness of the unsteady boundary layer (see Fung 1997, Sect. 3.5):

$$Wo = \frac{D}{\sqrt{\nu T}} \tag{1.22}$$

This number is a useful parameter to discern the behavior of unsteady flows. When *Wo* is close to or smaller than 1, the viscous friction diffuses from the wall into the entire vessel. The unsteady flow is a sequence of parabolic profiles of Poiseuille type, as shown in Fig. 1.9. Given that the denominator of (1.22) cannot significantly vary in humans, with values little below 2 *mm*, the parabolic solution is found in all vessels whose diameter does not exceed this value. In larger arteries, *Wo* can be around or even above 10, for example in the aortic root. In these cases, as shown in Fig. 1.9, the oscillatory boundary layer is confined to a fraction of the vessel radius, and produces relatively sharp local variation in the velocity profile adjacent to the

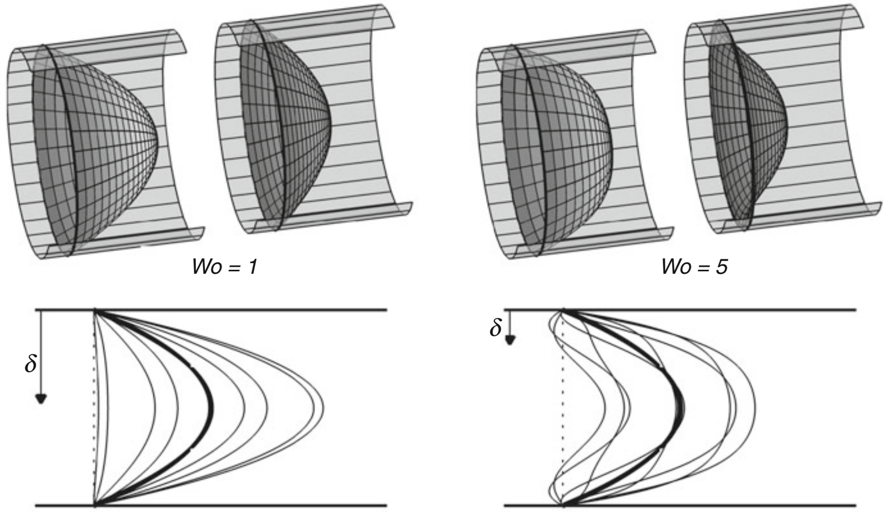


**Fig. 1.9** Examples of the unsteady velocity profile that develops in a cylindrical duct when the flow is pulsatile. The *left* side corresponds to a Womersley number $Wo = 1$, which means that the boundary layer has a thickness comparable to the vessel radius. The *right* side corresponds to $Wo = 5$, in which the oscillatory boundary layer is smaller and remains localized close to the wall. The spatial velocity profile is shown on top during peak flow deceleration. The flow profile at eight instants during one heartbeat are shown below, the thick line corresponds to the time-average Poiseuille profile

wall. The Poiseuille parabolic profile is a good approximation for the time-averaged flow characteristics only, while calculation of extreme values requires the evaluation of the oscillatory terms.

An analogous modification of the flow due to the presence of a boundary layer with limited thickness is found at the proximal region of vessels. Here, the parabolic, steady, or time-averaged solution does not establish. At the entrance of a vessel, the boundary layer starts from zero and grows downstream "the entry region" where the boundary layer has not yet grown enough to fill the entire vessel (Fig. 1.7). The velocity profile is flatter in the bulk region, and not reached by the boundary layer while the centerline velocity is initially equal to the average velocity $U$ and slowly increases toward the Poiseuille value, $2U$. The length of this "entry region" can be obtained from Eq. 1.17.

The previous simple examples of flow in straight vessels provide an insight on how the shear that develops in the boundary layer may influence the overall flow, and its interaction with the wall. The effect of the boundary layer is even more significant in presence of irregularly-shaped geometries. A simple example is the flow that crosses a brief constriction such as a stenosis or an orifice. At the constriction, the boundary layer does not develop other than along a very short length with a tiny thickness. However, the boundary layer exists to ensure the adherence condition, although hardly visualized. Such a small, intense boundary layer takes a fundamental role in the subsequent flow development.

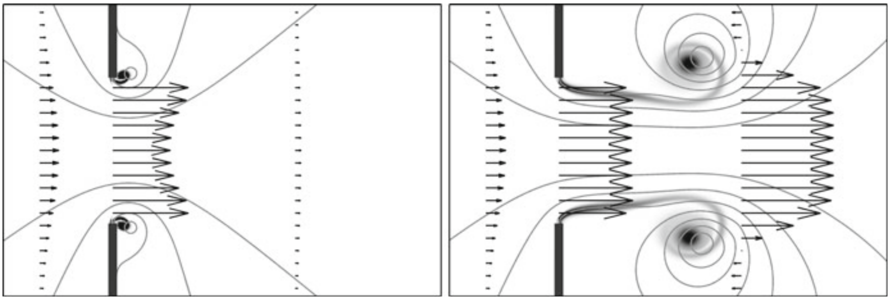

**Fig. 1.10** Two snapshots during the development of flow crossing a circular orifice. At the initial stage, *left panel*, the fluid velocity is higher near the edges because of the convergence toward the constriction. The boundary layer at the orifice is extremely thin because it has no development wall upstream. As time proceeds, the jet extends downstream the core region bounded by a shear layer. The pictures show the velocity profile at three positions (*same scales*), the streamlines (*gray lines*), and the shear region (*shaded gray*) developing due to the boundary layer

To further clarify this concept, let us consider the flow for an extreme case of the irregular vessel. At the initial instant of a starting flow across an orifice, as depicted in the left panel of Fig. 1.10, the flow converges from the chamber, upstream into the orifice and then diverges in an approximately symmetrical manner. Nevertheless, the tiny boundary layer is convected downstream the orifice with the moving fluid. Initially, such a shear layer travels a very short distance (Fig. 1.10, left side), and its tip rolls-up because of the velocity difference between the two sides. As time proceeds, the original boundary layer gives rise to an elongated free *shear layer* that surrounds the jet core and separates it from the ambient fluid (Fig. 1.10, right side). The forefront jet is composed of the initially separated shear layer that has rolled-up into the jet front.

The phenomenon described here represents a typical mechanism for formation of vortices. It is a consequence of viscous friction that develops from the adherence at the wall, as discussed earlier in Sects. 1.4 and 1.5. The viscous effects are confined to the shear layer only, and the other flow regions that are not directly influenced by the shear layer elicit negligible viscous effects. This fact describes how very small regions with dominant shear phenomena may dramatically change the overall flow field. This behavior is commonly observed, and is evident in vortical flow patterns where vortices cover limited regions in the space at particular times.

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