Vortex Dynamics

Abstract This part introduces the reader to the understanding of fluid motion in terms of vortex dynamics. First the conceptual background on vortex dynamics is drawn, developing an intuition why most fluid phenomena involve vortices and why they are so relevant. Then the reader is accompanied along the life of vortices: where they come from, why they form as vortices, their interaction with other vortices, up to the final dissipation. During this, vortices reveal to have a major influence on the wall shear stress along nearby tissues, and the process of vortex formation is associated to the development of forces on surrounding boundaries. Finally, an account is worthy of turbulence in terms of vorticity.

2.1 Defnitions

Vortices are fundamental performers in fluid mechanics and they develop in almost every realization of fluid motion (as extensively shown in the beautiful book by Lugt 1983). The presence of vortices dominates the corresponding fluid dynamics and the associated energetic phenomena. Vortices that develop in the large vessels of the cardiovascular systems play a fundamental role in the normal physiology and bring about the proper balance between blood motion and stresses on the surrounding tissues.

The fluid velocity is commonly assumed as the principal quantity describing fluid motion. However, velocity is not able to evidence the underlying dynamical structure of a flow field, like stresses, mixing, or turbulence, that depend on velocity gradients. The weakness of a description based on velocity alone is particularly critical when the fluid motion features the presence of vortex structures. In general, *vorticity* is the preferable fundamental quantity for the analysis of incompressible fluid dynamics. Vorticity, which represents the local rotation rate of fluid particles, allows emphasizing the structure that hides behind the flow field; it also represents a complete description of the flow and allows recovering the whole velocity field.

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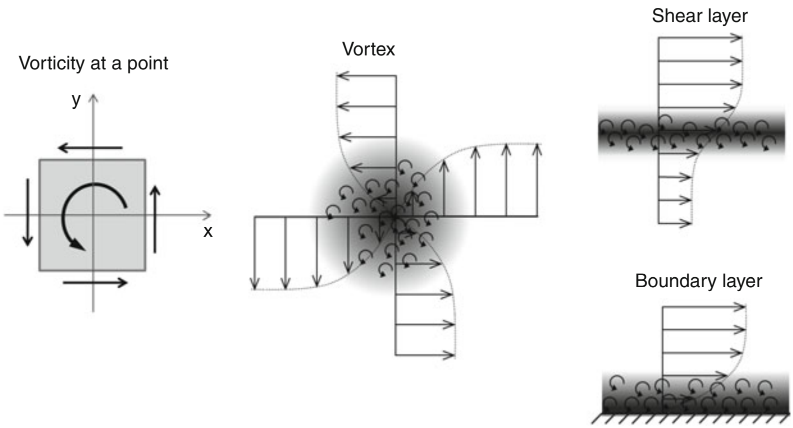


Fig 2.1 Vorticity corresponds to the local rotation of a fluid particle. The spatial distribution of vorticity gives rise to different flow structures. An accumulation of vorticity in a compact region corresponds to a vortex; an elongated distribution of vorticity corresponds to a shear layer that, when it is adjacent to the wall, is a boundary layer

In mathematical terms, the vorticity is a vector, indicated with $\omega(t,x)$ and defined as the *curl* is of the velocity field $u(t,x)$, normally expressed as the internal product between the *nabla* operator and the velocity field

$$\omega(t,x) = \nabla \times u \tag{2.1}$$

that in Cartesian coordinates components reads

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} \end{bmatrix} \tag{2.2}$$

The interpretation of vorticity is particularly intuitive in a two-dimensional flow field, when only the x and y components of the velocity field exist. In this case vorticity has only the component z, perpendicular to the plane of motion, $\omega = \partial u_y/\partial x - \partial u_x/\partial y$, and physically corresponds to (twice) the local angular velocity of a fluid particle. In fact, a positive vorticity corresponds to a vertical velocity, u_y, increasing horizontally, along x, and a horizontal velocity u_x, decreasing vertically. It is easy to understand, see Fig. 2.1 (leftmost sketch), that this type of velocity differences about a point represents a rotational motion (Panton 2005, Chap. 3).

The relevance of vorticity is not limited to local rotation. The spatial distribution of vorticity gives rise to different possible fluid structures; that is why vorticity is commonly considered the skeleton of the flow field and the fundamental quantity to define the flow structure. A *vortex* can be loosely described as a fluid structure that possesses circular or swirling motion; it actually is a region of compact vorticity, a circulatory motion surrounds a region where vorticity has accumulated. In addition to vortices, the vorticity map allows recognition of any basic flow structure. A shear layer, that is an elongated layer of friction between streams with differential motion, is actually a layer of vorticity, a *vortex layer*. The boundary layer discussed in the previous chapter is a vortex layer adjacent to the wall that develops because of the velocity difference between the outer flow and the fluid attached to the wall for viscous adherence. The correspondences between velocity and vorticity distributions are sketched in Fig. 2.1. The intensity of a vortex is normally measured by its *circulation*, normally indicated with Γ, that is the integral of the velocity along a closed circuit surrounding the vortex, and is equivalent to the sum of all the vorticity (mathematically, the integral) within the vortex area contained inside the circuit. The intensity of a vortex layer is measured by the difference of velocity, the *velocity jump* Δu, between the flow above and below the layer, equivalent, again, to the sum of the vorticity (mathematically, the line integral) across the layer. Vortices and vortex-layers are the fundamental vorticity structure in flow fields. Their different three-dimensional arrangements and combinations give rise to the complexities of all evolving flows.

The significance of vorticity can be best appreciated by the decomposition, due to Helmholtz and Stokes (Panton 2005, Chap. 17), of the complete velocity field into two distinct contributions: a rotational component u_{rot} that accounts for the whole vorticity in the flow field, and one *irrotational* component u_{irr} that is independent from the vorticity content

$$u = u_{rot} + u_{irr} \tag{2.3}$$

This decomposition allows building an easier path to the intuitive understanding of the physical phenomena concurring in a complex flow field.

The irrotational component of the velocity field is a particularly simple field, in incompressible flows. It will be shown below that it follows from the conservation of mass only (continuity constraint), and does not involves the equation of motion. The irrotational flow helps to satisfy the instantaneous balance of mass without any evolutionary mechanism, without fluid dynamics, only kinematic congruence. A flow without vorticity thus gives rise to an irrotational velocity field only, and does not depend on the balance of momentum. When required, the equation of motion can be then employed to derive the pressure distribution from the velocity. In the case of irrotational flow, this can be performed with the simple Bernoulli equation for an ideal flow because energy dissipation is absent in an irrotational flow. In fact, the viscous term of the Navier-Stokes Eq. 1.14, $\nabla^2 u$, which can be written for an incompressible flow as $\nabla \times \omega$, is identically zero for a flow without vorticity.

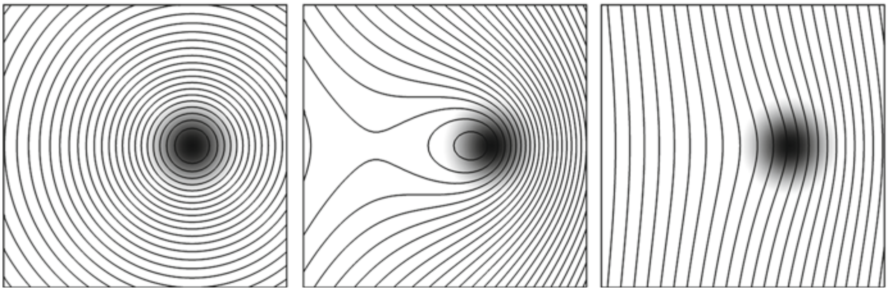


Fig 2.2 A vortex is a region where vorticity has accumulated; it is not necessarily a region exhibiting circulatory motion. A flow made of a vortex only is made of circular streamlines (*left panel*). The streamlines are modified when a uniform vertical flow of moderate (*centre*) ad high intensity (*right panel*) is added. In the three panels the vortex is unchanged, and so is shear in the flow

These considerations on the trivial character of the irrotational component of the flow stem, mathematically, from the fact that any irrotational vector field can be expressed as the gradient of a scalar field, the so-called potential φ, as $u_{irr} = \nabla\varphi$. In an incompressible flow, the continuity Eq. 1.5 applied to such a gradient field becomes the well-known Laplace equation for the flow potential, $\nabla^2\varphi = 0$. This is a linear equation whose solutions are easily obtained by several methods based on the values at the boundaries. As a further property, given the linearity of the Laplace equation, the irrotational velocity component can be expressed as a direct superposition of several elementary irrotational flows.

The velocity decomposition is the key tool to recognize the presence and the role of vortices in a flow. A vortex, as said, is a region where vorticity has accumulated; *a vortex is not necessarily a region exhibiting circulatory motion*. It may appear as such or the circulatory pattern may remain hidden behind an irrotational contribution that covers its rotary features. The velocity field corresponding to an isolated vortex is purely rotational; its streamlines, shown in Fig. 2.2 (left), rotate about the vortex and describe a circulatory motion. When an irrotational contribution adds on top of the same vortex flow, it may modify the apparent vortex signature in terms of streamlines. To explain this point, let us consider the same vortex of Fig. 2.2 (left) with an additional uniform flow, a rigid translational motion from top to bottom that is evidently an irrotational component and does not affect the value of vorticity, of shear rate anywhere. The resulting flow fields are shown in Fig. 2.2 for increasing values of the uniform motion (central and rightmost panels). The three fields of Fig. 2.2 present exactly the same vortex, the same gradients of velocity at all points; nevertheless from a superficial qualitative view in terms of streamlines the underlying vortex may not be equally recognizable.

Fluid dynamics phenomena related to evolutionary dynamics, friction, dissipation, forces, boundary layer, vortex formation etc., are dominated by the rotational part of the velocity field, while the irrotational contribution may have a role in terms of transport and mass conservation only. Therefore, a flow field can be evaluated from the dynamics of the vorticity, plus an irrotational contribution to adjust mass conservation. This is why vorticity, and vortices in which vorticity organizes, are the fundamental elements of the flow: the skeleton and the sinews of fluid motion (Moffat et al. 1994).

2.2 Dynamicsof Vorticity

The vorticity is the fundamental quantity that describes a fluid flow. From the knowledge of the vorticity field only, the entire flow field inside a given geometry can be reconstructed (technically, by inversion of Eq. 2.1). It is therefore tempting to analyze the dynamics of a fluid motion following the dynamics of the vorticity itself. This is often useful because vorticity occupies only a small fraction of the flow field, and takes standard shapes that allow an immediate characterization of the whole flow field.

The vorticity field has the further simplifying property that it obeys the same constraint of the velocity in an incompressible fluid previously expressed in Sect. 1.2 and Eq. 1.4. Mathematically, vorticity is a field with zero divergence (simply because the divergence of a curl is zero by mathematical definition)

$$\nabla \cdot \omega = 0 \tag{2.4}$$

This means that the vorticity field cannot take arbitrary geometric shapes. Therefore vorticity typically develops in terms of vortex tubes (whose associated velocity circulates around the tube) or of vortex layers (associated with a difference of velocity, a shear rate, across the layer). Moreover, the total vorticity contained inside a vortex tube is conserved like the discharge in a tube of flow: a vortex tube cannot terminate abruptly, and must either be a closed ring or terminate by spreading into a vortex layer.

Vorticity is an evolving field that follows deterministic evolutionary laws. Their mathematical expression can be immediately derived from the conservation of momentum, the Navier-Stokes Eq. 1.14, rearranged in terms of vorticity. By taking the curl of the Navier-Stokes equation, the *vorticity equation* is obtained (Panton 2005, Sect. 13.3)

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \nabla^2 \omega \tag{2.5}$$

that expresses the law of motion in terms of vorticity. Despite the apparent mathematical complexity, the simple qualitative inspection of this equation permits to extract some important concepts regarding vortex dynamics. For example, it can be immediately recognized that the vorticity equation does not contain the pressure (or any conservative force like gravity). In fact, the distribution of pressure has no direct influence on vortex dynamics; on the contrary, however, pressure strongly depends on vorticity that rules friction and energy losses.

A first property of the vorticity evolution is that if vorticity is zero at one instant it remains zero afterward. This is seen by inspection of Eq. 2.5 where all terms are identically zero when vorticity is zero, and vorticity cannot change in time. Given that vorticity cannot be created inside the fluid, thus it can only be generated at the interface between the fluid and the boundary. This apparently simple fact is a fundamental element for the study of vortex dynamics: in incompressible flows vorticity does not appear spontaneously within the fluid, *the only place where vorticity can be created is at the boundary between fluid and tissue*. Indeed, the issue of the generation of vorticity, and vortex formation in particular, is a key one and it will cover several of the following chapters.

After vorticity is generated, it is subjected to few possible evolutionary phenomena. The primary one is that vorticity is transported with the flow as if it were a passive tracer (although not effectively passive, because velocity is related to vorticity itself). This is represented by the two terms on the left hand side of (2.5) that simply describe the variation of vorticity over a particle moving with the flow. The first term is the time variation of vorticity at the fixed position crossed by the particle; the second term gives an increase of vorticity when a particle points in a direction along which vorticity grows (i.e., when velocity is aligned with a positive gradient of vorticity). They take a form analogous to, for example, the first two terms in Eqs. 1.11 or 1.14, describing the acceleration on a moving particle. Therefore, vorticity moves with the local fluid velocity, like a tracer, and can further change its value in virtue of two additional phenomena.

The first, corresponding to the first term on the right hand side of (2.5), represents the phenomenon of increase of vorticity by *vortex stretching*. Consider a small cylinder of fluid along whose axis the velocity increases, thus velocity is lower at the base and higher at the cylinder top; as time proceeds, the cylinder elongates, it is stretched by the velocity gradient (and shrunk in the transversal direction for the conservation of mass). Well, the vorticity vector behaves in the identical manner as material fluid, when fluid is stretched the vorticity vector is stretched as well and the vorticity value increases (Batchelor 1967, Sect. 5.2). This term represents the stretching and turning of vortex lines (Panton 2005, Sect. 13.5) as if they were lines of fluid. A further important aspect of this term is that it is exactly zero in a two-dimensional flow. In a two-dimensional flow, the vorticity is perpendicular to the plane of motion and there is no velocity gradient out of plane: vorticity stretching is intrinsically a three-dimensional effect.

Before turning the attention to the last term containing the viscous effects, let us recapitulate the dynamics of vorticity in the absence of viscous effects. First, an element of fluid that contains no vorticity remains without vorticity afterward. This is the first of the three Helmholtz's laws for inviscid flow (Panton 2005, Sect. 13.9). Then, the vorticity is a vector that behaves like a small string element of fluid.

It moves with the flow and it is stretched and tilted with it. This is essentially the second Helmholtz's law. The third law follows from the fact that vorticity is a field with zero divergence and the total vorticity contained inside a vortex tube (or a vortex filament, when it is thin enough) is conserved along the filament while it moves with the flow.

The picture becomes extremely simple and intuitive in a two-dimensional flow, or in a motion that is locally approximately two-dimensional. In this case, the vorticity vector has a unique nonzero component perpendicular to the plane of motion, therefore it loses its vector character and can be considered as a scalar. Stretching is absent and vorticity is simply transported with the flow. The value of vorticity is stuck onto the individual fluid particles; vorticity simply accumulates into vortex patches, redistributes into vortex layer, accordingly to the motion of fluid particles.

The last, viscous term in the vorticity Eq. 2.5 introduces the effects of friction and energy dissipation in terms of vorticity. The action of viscosity on vorticity introduces a phenomenon analogous to that of heat diffusion or diffusion of a tracer like ink or smoke (Panton 2005, Sect. 13.4). The distribution of vorticity is smoothed out by viscosity; a sharp vortex reduces progressively its local strength while it widens its size in a way that the total vorticity is conserved. In general, the diffusion process is of a simple interpretation. Like in any diffusive process, the rate of diffusion is higher in presence of sharp vorticity gradients, therefore the magnitude of viscous dissipation become increasingly relevant where vorticity presents changes over short distances. This leads to the most important aspect of energy losses in fluid motion: *viscous dissipation is most effective at small scales*. The vector property of viscous diffusion is evidenced when it produces the annihilation of close patches of opposite sign vorticity. This has a peculiar consequence in three-dimensions. When two portions of vortex filaments get in contact, the opposite-sign vorticity locally annihilates; this accompanies the reconnection of the cropped, oppositely pointing vortex lines (that cannot terminate into the flow). The viscous reconnection phenomenon is the underlying mechanism leading to topological changes, metamorphoses of three-dimensional vortex structures, and increased dissipation by turbulence (see Sects. 2.7 and 2.8).

In summary, the dynamics of vorticity is made by its transport with the fluid elements, intensification by three-dimensional straining of such fluid elements, and smoothing by viscous diffusion. A dynamics that sees vorticity arranged into tubular and sheet-like structures ensuring a continuity of vortex lines.

2.3 BoundaryLayer Separation

As said above, in incompressible flows, vorticity cannot be generated within the fluid. Vorticity can only develop from the wall in consequence of the viscous adherence between the fluid and the bounding tissue. Vorticity is produced because of the no-slip condition at the interface between the fluid and the solid surface; it then progressively diffuses away from the wall through the viscous diffusion mechanism to produce a layer of vorticity at the boundary. The boundary layer thickness corresponds to the length at which the viscous diffusion penetrates into the flow, which is proportional to $\sqrt{\nu t}$ as taught from Eqs. 1.16 and 1.17. The boundary layer was introduced in Sect. 1.5 as the region adjacent to the wall where the velocity rises from the zero value that it takes at the boundary to a finite value away from it. However, its interpretation as a vorticity layer is more intuitive for addressing vortex formation processes.

The boundary layer has a fundamental importance in fluid mechanics as it represents the *unique* source of vorticity in a flow field. It can be easily verified that the value of vorticity at the wall also corresponds to the wall shear rate and, after multiplication with viscosity, to the wall shear stress

$$\tau_W = \mu\omega \tag{2.6}$$

therefore the wall vorticity is often employed as synonymous of wall shear rate (sometimes, given the constancy of viscosity, also of wall shear stress).

In small vessels, the thickness of the boundary layer is comparable to the diameter and fills the entire flow field. At such small scales, in arterioles and capillaries, viscous diffusion is the dominant phenomenon; vorticity smoothly diffuses into the whole flow and vortices, with rare exceptions, are absent. On the contrary, in large blood vessels or inside the cardiac chambers, the boundary layer often remains thin and is capable to penetrate for diffusion over a small fraction of the vessel size. Indeed, until it remains attached to the wall, it has a minor influence to the flow and only represents a viscous slipping cushion for the outside motion. However, under many circumstances, it happens that such a thin boundary layer detaches from the wall and is ejected into the bulk flow. This is the process of *boundary layer separation*, when thin layers of intense vorticity enter into the flow and give rise to local accumulation of vorticity and eventually to the formation of compact vortex structures.

Boundary layer separation is normally a consequence of the local deceleration of the flow (Panton 2005, Sect. 20.11; Batchelor 1967, Sect. 5.10). The whole process of boundary layer separation is sketched in Fig. 2.3. When flow decelerates, the boundary layer is subjected to deceleration as well and, because of incompressibility, a local stream-wise deceleration associates with a growth of the thickness at the same location. This tongue of vorticity is lifted and strained by the outside flow while the vorticity value at the wall below decreases. As this process progresses, opposite sign wall vorticity appears and a secondary boundary layer develops below the separating shear layer. The separation point at the wall, from where the separation streamline departs, corresponds to the place where vorticity is zero. The secondary vorticity is itself decelerated in its backward motion and is lifted up. Eventually, it cuts the connection between the original boundary layer and the separating vorticity that detaches and enters into the flow. It should be also reminded that vorticity is not a passive tracer, it is made of velocity gradients and

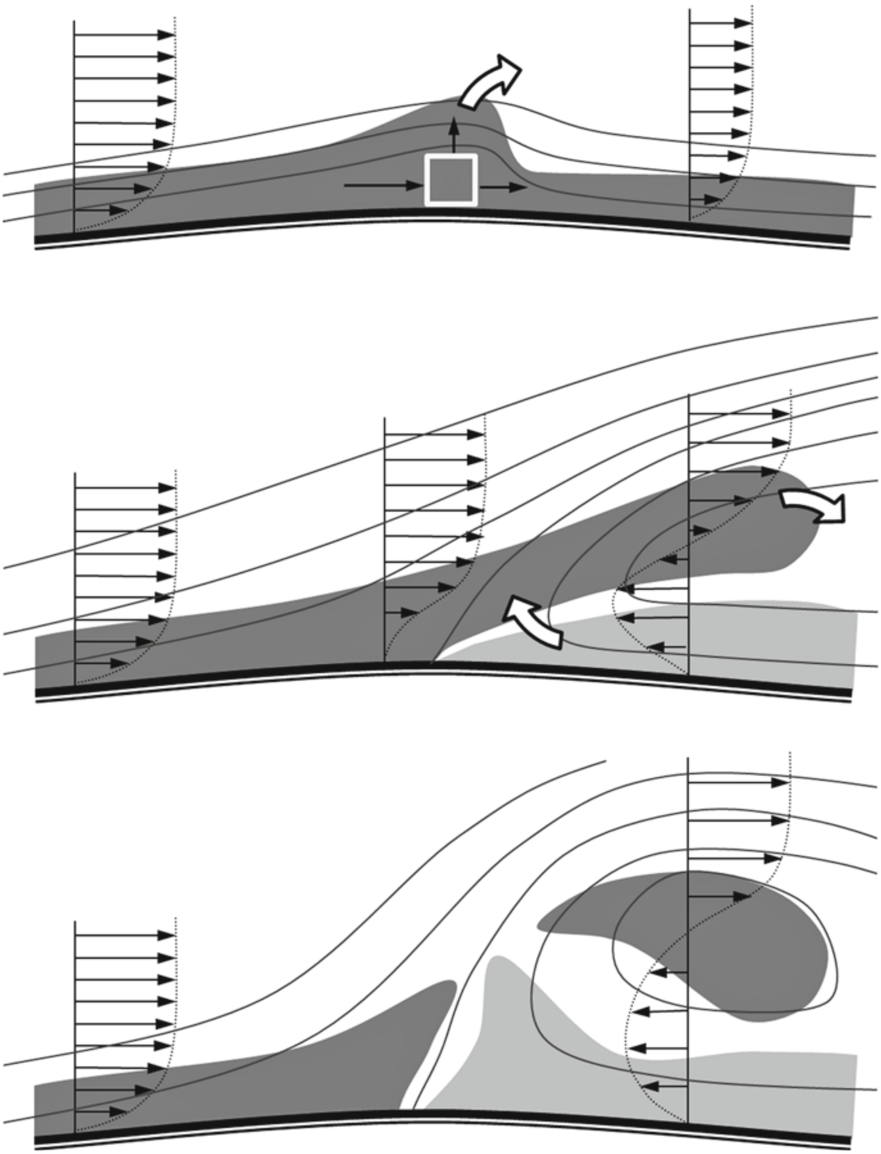


Fig 2.3 Sketch of the boundary layer separation process. The dark gray indicates layers with clockwise vorticity, the light gray is counter-clockwise; streamlines and velocity profiles are drawn. The deceleration of the flow produces a local thickening of the boundary layer due to mass conservation balance (*upper panel*). Such emerging vorticity is therefore lifted and transported downstream by the external flow (see *arrows*). A shear layer then extends away from the wall and produces a secondary boundary layer, with oppositely rotating vorticity (*mid panel*). The separated clockwise vorticity tends to roll-up while the secondary layer lifts up for the same initial mechanism, because it backward motion is decelerating (see *arrows*). Eventually, the separating vortex layer detaches from the boundary layer and becomes an independent vortex structure

it represents the underlying structure of the flow itself. Fig. 2.3 shows the qualitative velocity profiles and streamlines that develop in correspondence of the separating vorticity field.

Boundary layer separation is thus a consequence of the local deceleration of the flow. In other terms, separation develops in presence of an *adverse pressure gradient* (pressure growing downstream) that pushes from downstream and decelerates the stream. The most common way to have an adverse pressure gradient is that of a geometric change: a positive curvature of the wall, like an enlargement in a vessel. In this case, the velocity decreases, for mass conservation, kinetic energy decreases, and the value of pressure increases for the Bernoulli balance. Therefore, boundary layer separation develops behind a stenosis, or at the entrance of an aneurism. An extreme case of geometric change is that of a sharp edge, this is often found at the entrance of a side-branching vessel, and certainly on the trailing edge of the leaflets of the cardiac valves. In the case of sharp edges, the flow deceleration is so local that the position of boundary layer separation is definitely localizable at the edge. The vorticity that developed on the upstream side detaches at the sharp edge and leaves the tissue tangentially.

Geometric changes are not the unique possible sources for the development of a flow deceleration. Immediately downstream of branch sucking fluid away from a main vessel, the velocity reduces and an adverse pressure gradient develops. Similarly, boundary layer separation develops for the so-called *splash* effect, when a jet reaches a wall and produces high velocity streamlines that decelerate when they are deflected along the wall. Finally, the local flow deceleration is often produced by previously separated vortices. A vortex that gets close to a wall gives rise to a localized increase (or reduction, depending on its circulation) of the flow velocity at the wall below, and a corresponding deceleration immediately downstream (or upstream). The vortex-induced boundary layer separation is a frequent phenomenon that may become particularly critical in some applications. In fact, the area of principal separation is often localizable and properly protected, whereas an unexpected separation induced downstream due to a previously separated vortex may occur at unexpected locations.

2.4 Vortex Formation

The separation of the boundary layer represents the starting phase of the vortex formation process. The featuring property of any shear layer is the difference of velocity between its two sides: the farther side of shear layer that detaches from the wall moves with a speed that is higher than the side closer to the wall. Therefore, the separating shear layer curves on itself and eventually rolls-up into a tight spiral shape. Now, during the rolling-up process, the distance between two successive turns of the vortex layer progressively reduces, with the closest neighboring turns at the center of the spiral. The viscous diffusion process smears out this tight spiraling structure into by a compact inner core with a smooth distribution of vorticity (Wu et al. 2006, Sect. 8.1).

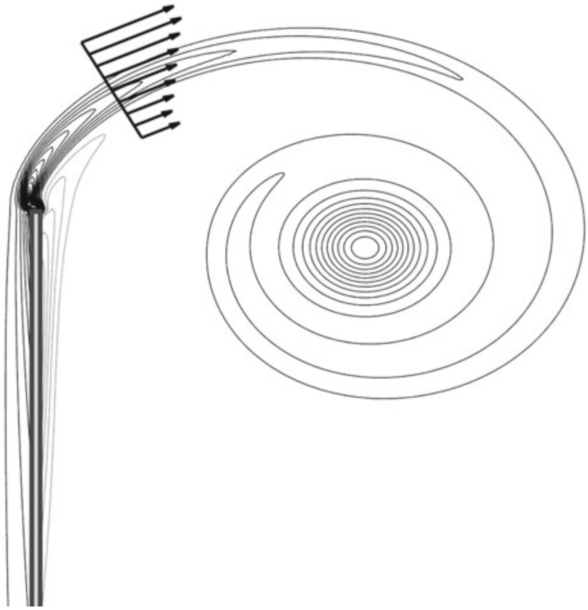


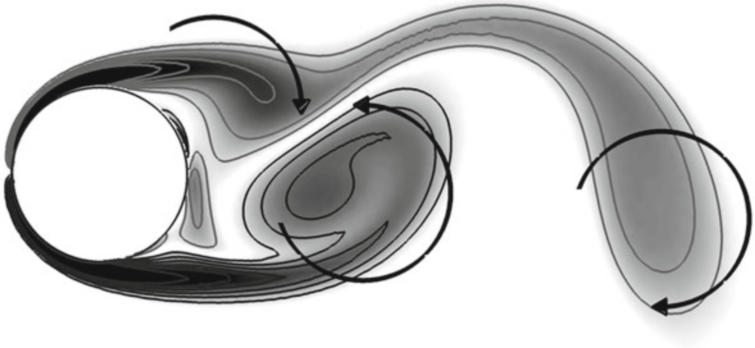
Fig 2.4 Vortex formation from a sharp edge obstacle. The shear layer separates from the upstream "wetted" wall and rolls-up into a spiral. The tight turns in the inner part of the spiral spread for viscous diffusion into the inner core of the formed vortex

[AU1]

The roll-up and the formation of an isolated vortex behind a sharp edge obstacle are shown in Fig. 2.4. In the case of such a sharp geometric change the boundary layer separation localizes at the edge, the boundary layer from the upstream "wetted" face of the obstacle leaves the edge tangentially and immediately rolls-up into a spiral. The vorticity viscous diffusion acts with higher strength in the tighter inner spiral branch and gives rise to a smooth vortex core.

An isolated forming vortex grows with a self-similar shape until there are no external disturbances that can influence its formation. Thus until the vortex size is small enough in comparison to the size of the surrounding geometry. The properties of the initial self-similar growth can be obtained by simple dimensional arguments. Assume that the bulk velocity grows proportionally to t^{α} and that separation occurs from an edge of internal angle β, such that $\beta=0$ is a diaphragm and $\beta=90$ is a corner. It follows that the flow velocity around the edge is given by $At^{\alpha}r^{\lambda-1}$ where r is the distance from the edge, A is a dimensional coefficient, and $\lambda = 180/(360 - \beta)$ (see for example Batchelor 1967, Sect. 6.5; Pullin 1978; Saffman 1992, Sect. 8.5). Then, from dimensional arguments (based on the fact that A and t are the only dimensional quantities available), the typical length size of the vortex increases in time as t^{n}, with $n = (1 + \alpha)/(2 - \lambda)$, and the vortex intensity, its circulation, grows like t^{2n-1}.

These estimates allow evaluating the force acting on the surrounding walls due to the vortex formation, which turns out to be proportional to t^{2n-2} (see Sect. 2.6). Vice versa, they allow evaluating the flow corresponding to a given force (or pressure difference). For example, in an orifice with zero internal angle, $\beta=0$ ($\lambda = 1/2$), a flow time-profile t^{α} associates with a force going like $t^{(4\alpha-2)/3}$; this shows that a flow that increases faster than square root growth ($\alpha < 1/2$) requires an unrealistic, theoretically infinite, effort.



[AU1]

Fig 2.5 Formation of vortices behind a circular cylinder. Oppositely rotating vortices separate from the two sides of the body in an alternating sequence. The previously separated clockwise vortex detached from the upper wall translated downstream, a counter-clockwise vortex has been formed from the lower wall, and a novel clockwise vortex is under formation from the wall above

This typical roll-up phenomenon can be disturbed during its development in particular when the shear layer is particularly thin. A curved shear layer is, in fact, subjected to an intrinsic instability that gives rise to the birth of wavy disturbance and the following roll-up of multiple small double spirals along the shear layer itself (Pullin 1978; Luchini and Tognaccini 2002; Pedrizzetti 2010). This instability has the origin in the Kelvin-Helmholtz instability for an infinitely thin vortex sheet (Batchelor 1967, Sect. 7.1) and it is more effective the thinner the vortex layer is. The thickness of the shear layer is given by the boundary layer thickness prior to separation, as given by Eqs. 1.16 or 1.17. In general, a separating shear layer is relatively thin in large vessels, especially when it detaches in a sharp enlargement after a short converging section, where the boundary layer is kept attached by the spatially accelerating flow. This is the case of the trailing edge of cardiac valves.

The vortex formation from a smooth surface is still described by the picture given above, where a few additional elements of complexity can be emphasized. First, the actual position of separation depends on the local flow structure; it cannot be preliminarily identified and may even change during time (Pedrizzetti 1996). Furthermore, the separation from a smooth surface is inevitably accompanied by a more direct interaction between the forming vortex and the nearby wall when the viscous dissipation effects normally support the formation of smoother vortex structures.

One typical example of the external separation from the smooth surface of a bluff body is shown in Fig. 2.5 featuring the formation of oppositely rotating vortices from the two sides of a circular cylinder. In such an example such vortices interact and influence the opposite separation process eventually producing a sequence of alternating vortices known as the von Karman street that is usually found behind bluff bodies (Panton 2005, Sect. 14.6). The development of alternating vortices is quite a common phenomenon when previously separated vortices may influence vortex formation in nearby regions. It is also present, with some differences, in internal flows when a vortex formed on one side of a vessel creates a vortex-induced separation on a facing wall. That, in turn, may induce a weaker further separation in a sort of wavy pattern extending and decaying downstream.

[AU1]

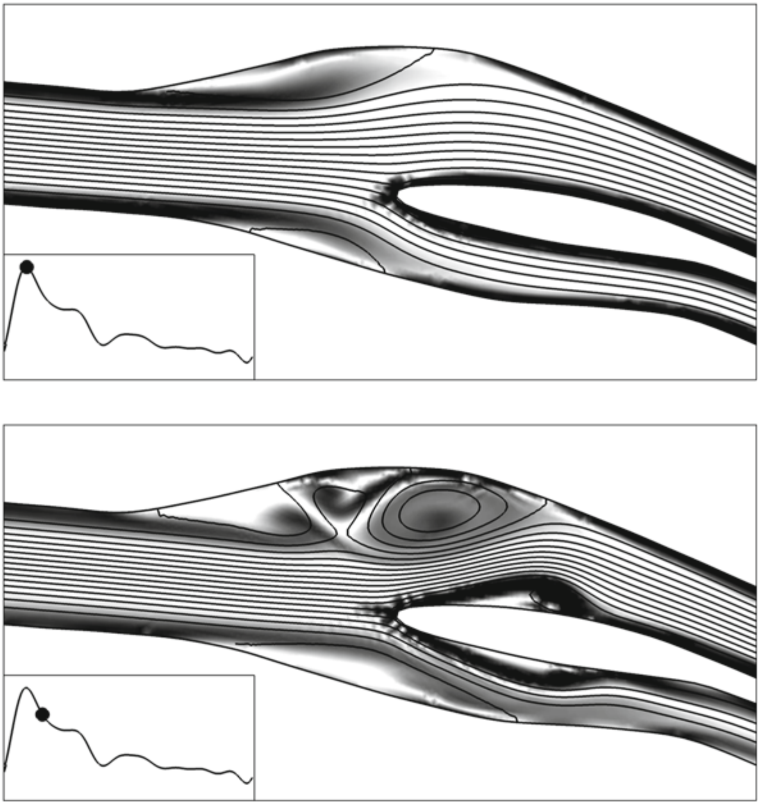


Fig 2.6 Formation of vortices at a carotid bifurcation. The accelerating systolic flow (*upper panel*, at peak systole) leads to a smooth boundary layer separation at the carotid bulb. After the peak (*lower panel*) the vortex just formed at the bulb either interacts with the bulb boundary layer creating multiple small vortices, and gives rise to a vortex-induced secondary separation in the oppositely facing wall of the internal carotid artery. The same phenomena in a much weaker version are noticeable also on the opposite side at the entrance in the external carotid artery

The internal separation, with the following formation of a vortex inside of a vessel is in general a smoother phenomenon, because the presence of confining walls does not allow vortices to grow into large structures, keeps vortices more constrained within smaller scales, and is more influenced by viscous diffusion. Nevertheless, the presence of a vortex inside a vessel may change the entire flow. It has a blocking effect that locally deviates the streamlines modifying the wall shear stress distribution, possibly producing further separations. It changes the unsteady pressure drop, and in a branching duct it may affect the relative flows division in the daughter vessels. An example is given in Fig. 2.6 that reports the vortex formation in the bulb of a carotid bifurcation. During the systolic acceleration, the boundary layer separates tangentially from the common carotid artery and develops a smooth roll-up within the bulb close to the nearby wall. During deceleration, the formed vortex locally affects the wall shear stress inside the bulb with multiple opposite sign wall vorticity.

It has a blocking effect that deviates the streamlines at the entrance of the internal carotid artery into a faster jet. It produces secondary vortex-induced separation inside the internal carotid that eventually (not shown in the picture) gives a secondary vortex formation and a further small separation little downstream.

A general analysis of the vortex formation process can be outlined when the flow enters from a small vessel into a large chamber forming a jet whose head is the forming vortex. Here, after the very initial roll-up phase, a measure of the length of such a jet is given by the product of Vt, where V is the velocity at the opening and t is the time. In this case it is enlightening to define a dimensionless *formation time*, *FT*, introduced by Gharib et al. in 1998, as the ratio of the jet length with respect to the diameter of the opening D

$$FT = \frac{V \times t}{D} \tag{2.7}$$

The formation time represents a dimensionless number that characterizes the progression of vortex formation. As such, it allows the unitary description of the vortex formation processes as they occur under different conditions. In reality, the definition of formation time has a more profound physical meaning (Dabiri 2009). The separating shear layer has a strength given by the jump of velocity between its two sides, given approximately by V, and translates downstream with a velocity that is again proportional to V, thus it feeds the circulation Γ of the forming vortex at a rate $d\Gamma/dt \cup V^2$. The formation time thus also represents the dimensionless measure of the vortex strength, the circulation Γ, normalized with VD. The definition (2.7) can be extended to the case when either V or D vary during time, by integration of the ratio V/D during the period of vortex formation.

The generality of the formation time concept permits to uncover general properties of the vortex formation that are common to the different cases. These, which do not appear in the simple description of vortex formation given above, will be revealed in the next chapter.

2.5 Three-Dimensional Vortex Formation

The vortex formation process described in the previous chapter is given in terms of two-dimensional pictures. It allows an immediate and intuitive understanding of the fundamental phenomenon. Actually, the initial phase of any vortex formation process is, with rare exceptions, always two-dimensional and the three-dimensional influence enters into play at some later stages.

The two-dimensional description implicitly treats the vorticity as a scalar quantity; this is a common practice because vorticity vector has the unique component that is perpendicular to the two-dimensional plane of motion. On the opposite, in order to understand the three-dimensional features of vortex formation, it must be reminded that vorticity is a three-dimensional vector. In particular, that vorticity is a solenoidal vector field, with zero divergence as dictated by the condition (2.4). It means that vorticity behaves like an incompressible flow: *vortex lines*, lines everywhere tangent to the vorticity vector, must be continuous and cannot originate or terminate in the flow. In a viscous flow they are always closed lines, although sometime complicated.

A consequence of the continuity of vortex lines is the concept of *vortex tube*, sometime called *vortex filament* when the tube is thin enough. A vortex tube is a thick collection of vortex lines, a tube whose lateral surface is made of vortex lines. The solenoidal nature of vorticity imposes that a vortex tube typically maintains its individuality during flow evolution. It is transported by the local flow, deformed and stretched by the velocity gradients, and enlarges because of vorticity diffusion, while it maintains its individuality. There are, however, situations when a vortex tube must abandon its individual life. When a tube approaches another tube, the nearby vortex lines belonging to different tubes begin to wrap one around the other, with a strong local stretching that is ultimately smoothed out by viscosity. In such "close encounters" a tube may fuse with another tube, filament, or vortex lines.

Another concept that enters in three-dimensional vortex dynamics is that of *self-induced velocity*. Let us remind that a two-dimensional vortex is actually a rectilinear vortex tube that does not vary along the third direction; the corresponding velocity field is a rotation on and around the tube. A three-dimensional vortex tube, in general, is not rectilinear and presents a curvature that may change along its length. Well, because of the relation between velocity and vorticity, a curved vortex tube corresponds to a velocity field made of a rotation around the tube plus a translation, or self-induced velocity, of the tube itself.

This is easily visualized: assume that the rotation velocity around the curved vortex tube is such that the velocity is externally upward and internally downward; this means that every small portion in the curved tube also pushes downward the nearby elements (that are in the internal side with respect to the tangent), with an overall result of a downward translation. The self-induced velocity of a curved vortex filament is (Saffman 1992, Sect. 11.1) proportional to the vortex circulation Γ, its curvature $1/R$, where R is the radius of curvature, and also weakly influenced by the ratio R/c where c is transversal size of tube. In formulae, the velocity has an intensity $\Gamma/4\pi R \log(R/c)$ (remind that the logarithm is a very slowly growing function) and is directed perpendicular to the plane that locally contains the filament. Therefore the tighter is the curve the higher is the self-induced velocity, with the additional element that thin filaments are some time faster than fat ones.

With these concepts in mind we can consider and describe vortex formation in three-dimensional flows.

The simplest case of three-dimensional vortex formation is that from a circular orifice, in that case the forming vortex tube has the shape of a ring. Vortex rings are well known objects of fluid dynamics (Shariff and Leonard 1992) that are easily generated using a piston-cylinder apparatus. A vortex ring is a stable vortex structure, it has an axial symmetric and vortices with a shape close to a ring also tend to the axisymmetric shape by an internal homogenization. Because of their stability, vortex rings are often encountered in nature, including when puffing smoke out of the mouth.

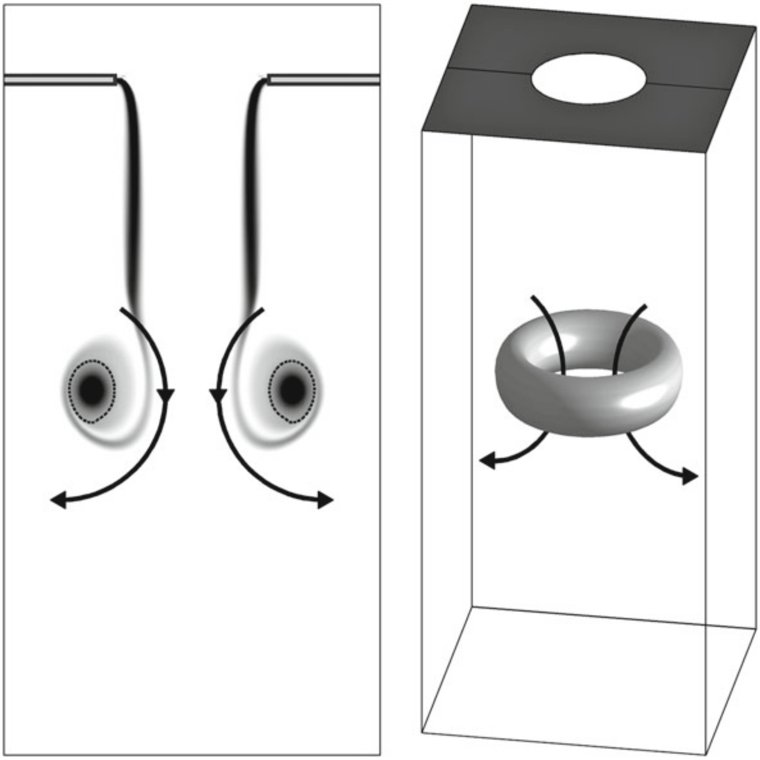


Fig 2.7 Formation of a vortex ring from a circular sharp orifice. The orifice velocity is here constant during time, this snapshot corresponds to a formation time $FT = 3$. *Left panel*: distribution of vorticity on a transversal cross-cut; the vortex core is indicated with a *dashed line*. *Right panel*: three-dimensional view of the vortex ring corresponding to the core of the forming vortex visualized by the λ_2 method (see text). The shear layer separating from the edge rolls-up into a vortex ring that corresponds to the jet head. The vortex has a curvature, the self-induced velocity is directed downstream and adds on top of the background velocity

Figure 2.7 shows one instant, corresponding to a formation time equal to 3, during the formation of a vortex ring behind a circular orifice. The vorticity distribution on a transversal section (left panel) shows the shear layer separating from the orifice that eventually rolls-up into the jet head; however it must be kept in mind that this planar picture corresponds to a three-dimensional vortex structure that is more difficult to represent on paper. The vortex ring corresponding to the vortex core is shown (right panel) to emphasize the main element of the three-dimensional vortex. In general, however, there is some ambiguity on the effective delineation of a vortex boundary. This is not a big issue in two-dimensional systems when the entire vorticity field can be shown in color scale on the picture plane and the different elements of the vortex structure are immediately recognized, from the separating shear layer, to the rolling-up spiral, to the vortex core (see the left panel of Fig. 2.7). However, this case is particularly simple because the vorticity has an axial symmetry and only the azimuthal component: this flow is conceptually planar. Nevertheless, its three-dimensional representation, on the right panel of Fig. 2.7, certainly contains less complete information, and the choice of the vortex core boundary severely influences the three-dimensional structure that is eventually visualized.

The definition of a vortex structure is a critical issue in three-dimensional flows, when vorticity is a vector field arbitrarily extended in three-dimensional space. A generally accepted definition of a vortex is still lacking. The level of vorticity cannot be sufficient because the highest magnitudes are typically found in the separating vortex layer, or in the boundary layer, structures made by vorticity that have not yet become a vortex. Loosely speaking, a vortex is a region where vorticity levels are higher than in the surrounding, where the vorticity distribution presents some coherence, and also where the relative motion of surrounding fluid elements follows circular paths or, more precisely, where the fluid is subjected to a centripetal acceleration. There are several tentative definitions; currently, unless the specific problem suggests an *ad-hoc* definition, the most accepted technique to identify a vortex structure is the so-called λ_2 method introduced in by Jeong and Hussain (1995). There the vortex boundary is identified by the constant level of a scalar quantity, λ_2, evaluated from the properties of velocity gradient.[1] Specific details are contained in (Jeong and Hussain 1995) where a review of other existing methods is also given. In brief, physically, this quantity identifies the regions where fluid pressure is minimum with the further care that only pressure gradients imputable to vorticity (rotational motion) are accounted, leaving aside the influence of the irrotational part of the velocity. Therefore the scalar quantity, λ_2, takes minimum values at the center of a vortex where pressure is low because centrifugal forces push the flow away. This technique has been successful in several applications, ranging from simple vortex flows to turbulence, and it allows extracting and visualizing the coherent vortex structures and then to build interpretations schemes of their dynamics.

Vortex rings are the simplest vortex structures that exhibit phenomena that are typical of three-dimensional vortex dynamics. A vortex ring presents a self-induced velocity proportional to its circulation and its curvature. Such a self-induced velocity gives rise to a peculiar limiting process of three-dimensional vortex formation that was first reported by Gharib et al. in 1998. During its formation, the vortex ring is continuously fed by the rolling-up shear layer separating from the orifice edge, therefore its circulation grows and the self-induced vortex translation velocity increases as well. The self-induced translation velocity of the vortex ring rises until it exceeds the velocity of the separating shear layer. At this point, the primary vortex detaches from the layer behind with a phenomenon known as *pinch-off*. At the same time the newly separated vorticity cannot reach the escaped vortex and eventually rolls-up in its wake. In one sentence, *vortex ring pinch-off occurs when the velocity of the trailing jet falls below the celerity of the leading vortex ring* (see Dabiri 2009 for a review).

[1] The scalar λ_2 is the intermediate eigenvector of the tensor build by the sum $D^2+\Omega^2$, with D and Ω the rate of deformation and rotation tensors, respectively, defined also as the symmetric and asymmetric parts of the velocity gradient. This sum corresponds to the contribution to pressure curvature imputable to rotary motion alone, when this contribution to pressure has a minimum two out of three eigenvalues must be negative, otherwise two of them are positive. Therefore monitoring the intermediate eigenvalue allows verifying the presence of such vorticity-induced pressure minima.

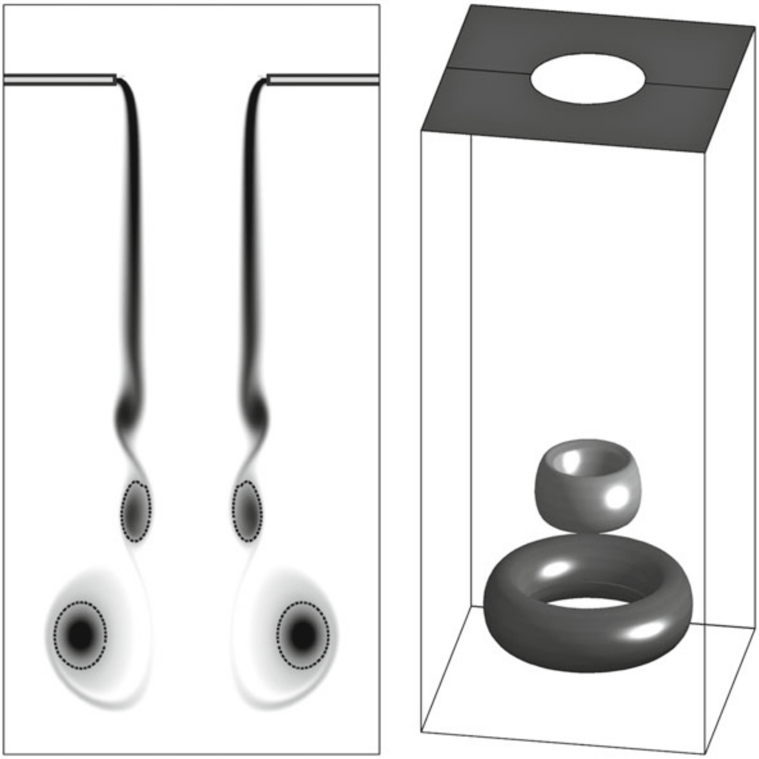


Fig 2.8 Formation of a vortex ring from a circular sharp orifice. The orifice velocity is here constant during time, this snapshot corresponds to a formation time $FT = 5$. *Left panel*: distribution of vorticity on a transversal cross-cut; vortex cores are indicated with dashed lines. *Right panel*: three-dimensional view of the vortex rings corresponding to the core of the forming vortices. The primary vortex has grown until its self-induced velocity has become larger than that of the shear layer behind, afterward the principal vortex escaped downstream and the shear layer produces smaller vortices in its wake

The timing of this limiting process has been found to be well described in terms of the formation time introduced at the end of the previous chapter. It revealed the existence of a critical value for the formation number, about $FT \approx 4$. Above this limit, the vortex ring cannot grow as a unique structure and multiple vortices develop in its wake. Indeed, the self-induced velocity of a curved vortex is proportional to Γ/D and the formation number can also be interpreted as the ratio between the velocity of the vortex ring and that of the shear layer. One example of the vortex ring formation process for a formation time larger than the critical value is shown in Fig. 2.8.

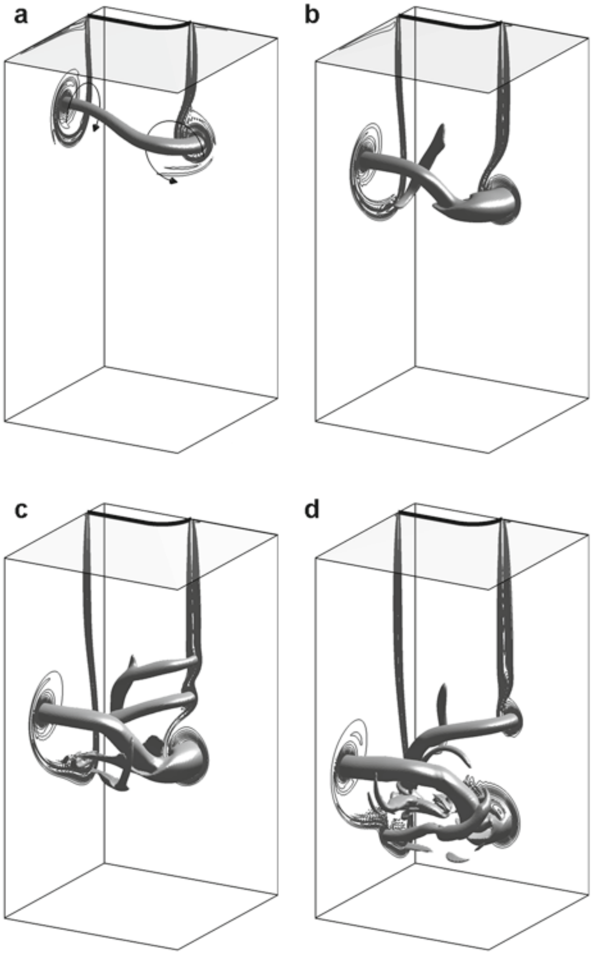
The case of vortex ring formation after a circular opening represents a preliminary conceptual basis for the interpretation of the more complex phenomena involved in the three-dimensional vortex formation from general geometries. Let us move forward and consider the flow across sharp edge orifices with a slender shape. In this case, the opening has a variable curvature and the separating vortex filament shall initially present a variable curvature along its axis as well. These differences give rise at least to two potential effects in sequence. First, the self-induced velocity, which is proportional to the curvature, will be different along the vortex filament and will progressively deform it. Secondly, the formation time depends on the local curvature and the vortex will reach the limiting critical value at different times along the different portions of the filament. These phenomena give rise to the deformation and eventual three-dimensional metamorphoses of the vortex structure separating from a non-circular orifice. Furthermore, once the vortex loses its stable regular shape, the three-dimensional interactions give rise to a progressive destruction of the vortex into smaller elements, which in turn deform into even smaller ones, until they are dissipated for viscous effects. In general, vortex formation after three-dimensional geometry may give rise to irregularly shaped structures, these become unstable and undergo to a rapid energy dissipation. One exemplary case of the three-dimensional vortex formation from a slender orifice is shown in Fig. 2.9 (Domenichini 2011).

The concept of limiting vortex formation, previously described for vortex rings, presents a renewed role in the prediction of the transition toward the breaking and dissipation of three-dimensional forming vortices. In fact, the major disturbance to the stability of a vortex loop occurs when part of it, the most curved portion, reaches the limiting formation time and develops small vortices. The limiting formation time should be computed along the vortex loop with the local value of the orifice curvature. Its variability gives the differential timing of the limiting process and represents a measure of the instability of the formed vortex structure. Its smallest value represents the earliest reach of the limiting process and the beginning of the transition toward three-dimensional disturbance.

The three-dimensional vortex formation from smooth surfaces, after a constriction like a stenosis or in a vessel enlargement, introduces additional elements of complexity that do not allow drawing a simple unitary picture of the involved phenomena. The initial instants following boundary layer separation and initial roll-up are essentially two-dimensional with a moderate influence from the three-dimensional structure. Afterward, the differential vortex formation leads to developments of widely different results depending on the differences in the separating geometry, in the interaction with the nearby walls and with other surrounding elements. Specific examples of such vortex formation phenomena as they realize in sites of interest for the cardiovascular circulation are provided through this volume.

An important source of complexity comes from the potential variability in the localization of the separation. In the two-dimensional description, the separation departs from the smooth wall at a point that, as said earlier, may change over time. In three-dimension, when the transversal span is included, the separation point transforms into a *separation line*. Until this line has a smooth geometry, it reflects into the formation of a smooth vortex loop. More often, the separation line is irregular along its length and deforms in time giving rise to an irregular vortex structure that immediately develops into a dissipative, chaotic, three-dimensional structure.

It is altogether common that the separation line is not even a closed line but a segment of finite length. In this case, the separation is localized and remains



[AU1]

[AU2] **Fig 2.9** Three-dimensional vortex formation from a slender orifice, made of two half circles connected by straight segments. The orifice velocity is here constant during time, the four snapshots correspond to a time sequence where the value of the formation time are $FT = 2.5$, $FT = 3.75$, $FT = 5$, $FT = 6.25$. One quarter of the entire space is shown for graphic clarity (allowed by symmetry); the vorticity contours are reported on the side planes to help understating the three-dimensional arrangement of the principal vortex filaments. In the initial phase, the formed vortex loop presents a variable curvature and deforms because of the higher self-induced translation speed in the more curved parts; such a deformation leads to further changes in the three-dimensional curvature and further deformations. Later on, the vortex reaches the limiting formation phase behind the circular part of the orifice only, where smaller vortices appear. Afterwards the vortex structure loses its individuality and becomes a set of entangled three-dimensional elements that rapidly dissipate for viscous stresses

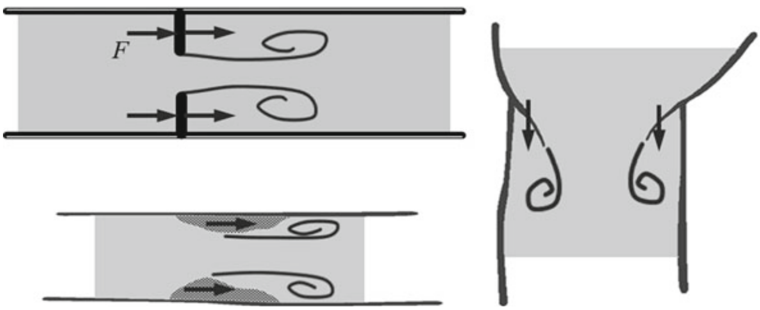


Fig 2.10 A vessel that presents an obstacle partially obstructing the fluid flow, may give rise to vortex formation that, in turn, generates a force on the obstacle. The picture reports example of a duct with a diaphragm (*top-left*), a vessel with a stenosis (*bottom-left*), and a trans-valvular flow (*right*)

incomplete in its three-dimensional formation. For example, a vessel can present, as it typically does, a constriction on one side only of the wall, and the vortex develops and rolls-up only locally. Nevertheless, the solenoidal constraint for the vorticity fields requires that any vortex line cannot terminate into the flow. Therefore the incompletely formed vortex loop, locally separated from the boundary layer, remains connected with the smooth vorticity distribution contained in the remaining boundary layer. Such type of vortex structure, with a reverse U-shape, normally called *vortex hairpin*, are evidently not stable, their tip is slower than the flow, and rapidly undergo a three-dimensional dissipative evolution.

In conclusion, three-dimensional vortex formation from circular geometries produces ring-like vortex loops that are subjected to a limiting process, and gives rise to multiple rings when this limit is overcome. Formation from irregular geometries normally leads to unstable vortex structures that are rapidly dissipated. Under relatively simple conditions, the limiting process still allows to evaluate the presence and the timing of the eventual vortex breakdown. The complexity of the actual phenomena involved should, however, be evaluated in the different specific conditions.

2.6 Energy Loss and Force of Vortex Formation

The previous chapters have evidenced how the vortex formation process dramatically influences the flow motion. In addition to this, formation also reflects in the generation of dynamical actions on the surrounding tissues and in energetic losses along the vessel.

Consider an obstacle, like a diaphragm, a valve, a stenosis, that partially obstructs the otherwise free flowing of the fluid inside a vessel, as shown in Fig. 2.10. In absence of any vortex formation, pressure would change as dictated by the Bernoulli balance. However, the development of a vortex provokes an additional pressure drop (or energy loss) due to the transformation of energy into vortex inertia, and an additional longitudinal force on the obstacle wall.

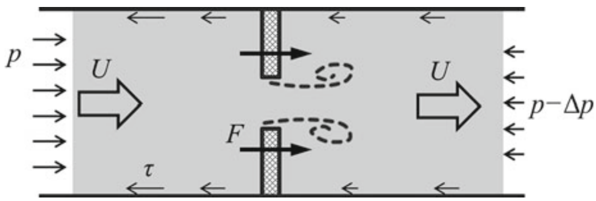


Fig 2.11 A schematic portion of a straight duct with an obstacle. The process of vortex formation gives rise to an irreversible pressure drop Δp, and to a vortex-force on the obstacle. These are typically estimable from global balances without the need to evaluate local details of the flow field

Let us evaluate the pressure drop in the simple configuration of a rectilinear duct with a diaphragm inside, as sketched in Fig. 2.11. This example allows evidencing the pressure drop due to vortex formation only, leaving aside the transformation between kinetic energy and pressure caused by variation of the duct size. The evaluation of pressure losses in a partially obstructed straight vessel can be performed by the equation of motion (like the Bernoulli theorem, in Eq. 1.9). Using the equations with particular care (written along the streamlines and summed-up for all stream-tubes connecting the inlet to the outlet) the pressure drop can eventually be expressed as

$$\Delta p = \rho L \frac{dU}{dt} + \frac{\rho}{A}\frac{dI}{dt} \tag{2.8}$$

where any viscous dissipative effects have also been here neglected. Both the two terms on the right side derive from the inertial part of the Bernoulli theorem integrated over all stream-tubes. The former the inertia of the average motion, this is a reversible pressure loss that sums to zero in a time-periodic motion when the acceleration and deceleration and a net sum equal to zero. The latter represents the additional change of fluid inertia that is imputable to the formation of the vortex. And it is measured by the variation of the impulse, $I(t)$, of the vortex system. The vortex impulse defined in general as a vector quantity (Saffman 1992, Sect. 3.2)

$$I = \frac{1}{2}\int \omega \times x dV \tag{2.9}$$

of which only the longitudinal component enters in (2.8). The apparently complicated definition of impulse (2.9) can be made more explicit when considering the formation of an individual vortex of growing strength $\Gamma(t)$ from a constriction of area A_0. The impulse is then given by the circulation multiplied by the area surrounded by the vortex $I(t) \cup A_0 \Gamma(t)$, for example the impulse of a vortex ring of radius R is $I = \pi R^2 \Gamma$, and from the pressure drop simplifies to (see also Saffman 1992, Sect. 3.8)

$$\Delta p = \rho L \frac{dU}{dt} + \rho \frac{A_0}{A} \frac{d\Gamma}{dt} \tag{2.10}$$

The pressure loss caused by vortex formation, expressed by the second term of Eqs. 2.8 or 2.10, is an inertial effect. It is a consequence of the adherence of the fluid onto the wall because of viscosity, although, it is independent from the actual value of viscosity. In summary, *vortex formation gives an irreversible transformation of energy into inertia, due to viscous adherence, but independent from the value of viscosity.*

The presence of an obstruction in the otherwise rectilinear duct deviates the flow and provokes the development of a force on the obstacle whose strength increases during a vortex formation process. This force follows from the difference of pressure between the front and the back faces of the obstacle. The high pressure in the front face is due to the flow that impacts on it to be deviated (in Bernoulli terms, the velocity decreased approaching the front face of the obstruction and the pressure rises). The dramatically lowered pressure on the back face, instead, is very much a consequence of the vortex generation. It is due to the sharp pressure decrease across the separating shear layer that connects the wall to the growing vortex. The total force made by the fluid onto the obstacle can be evaluated by integrating the value of pressure and shear stresses all over its solid surface. This direct calculation is often unfeasible because it requires a very accurate knowledge of the fluid properties very close to the interface with the solid boundary. An alternative approach follows from the integral balance of momentum (the product of mass and velocity integrated over the entire volume). It states that the rate of change of momentum within any arbitrary region of fluid, cleared from the net flow of momentum across the bounding surfaces, can only be imputable to the forces that act on the fluid contained inside that region (see, for example, Panton 2005, Sect. 5.14). This is again Newton's second law, here expressed in integral form for an entire region that reads

$$\frac{dM}{dt} + M_{flux} = G + P \tag{2.11}$$

where the left side contains the rate of change of momentum *dM*/*dt* (the acceleration of the fluid) and the flux of momentum M_{flux}, (positive when exiting). On the right hand side, *G* indicates the volume forces, like gravity, and the symbol *P* represents the forces acting on all the surfaces bounding the chosen region. These include both the contribution of pressure difference at the open ends of the region and the total stresses acting between the fluid and the obstacle, which is equal and opposite to the unknown force made by the flow onto the obstacle.

Let us consider again the simple duct with constant cross-section sketched in Fig. 2.11, with the objective of evaluating the longitudinal force, *F*, exerted on the solid obstacle because of the vortex formation process. In this case, the total flux of momentum is zero because the same amount that enters from the inlet exists at the outlet, and the volume force *G* due to gravity is simply the static vertical weight of the fluid volume weight that can be ignored. The longitudinal balance (2.11) thus expresses a dynamic equilibrium between the change of momentum, *dM*/*dt*, and the surface forces, P. First, the product of the fluid density with the velocity gives the momentum M over the whole volume. It is easy to understand that, in a duct, the whole volume can be spanned by a sequence of cross-section slices, and that the integrated velocity is the total fluid discharge, $Q = U \times A$, across that section. When the lateral walls do not move (or their velocity is negligible) the discharge does not vary along the length of the duct, and the rate of change of momentum is proportional to the rate of change of the discharge. Mathematically, it can be expressed as $\rho \times L \times dQ/dt$, where ρ is the fluid density, and L is length of the considered portion of the duct. The surface force, P, presents two different contributions. First, the forces acting on the open ends: the forward pushing pressure at the inlet section and the backward pushing pressure at the outlet, which sum up to $\Delta p \times A$. Second, the total longitudinal force made by the entire solid boundaries, $-F$. This includes both the actual force across the contour of the obstacle and the viscous shear stress on the lateral surfaces that are normally negligible along short tracks. Summing up, Eq. 2.11 becomes

$$\rho L \frac{dQ}{dt} = \Delta p A - F \tag{2.12}$$

Insertion of the expression for the pressure drop that was previously evaluated in Eq. 2.8 shows that the force generated on an obstacle by vortex formation can be primarily expressed by

$$F = \rho \frac{dI}{dt} \tag{2.13}$$

neglecting the additional contributions that may come from viscous losses. In general, when the tissues involve a more complicated geometry and possibly movable sidewalls, further terms should be included. These, however, do not relate directly with the vortex formation and are typically estimable from geometry and average flow properties.

In the simpler case of a single vortex of circulation $\Gamma(t)$ across an area A_0, when the pressure drop is expressed by Eq. 2.10, the force simplifies in

$$F = \rho A_0 \frac{d\Gamma}{dt} \tag{2.14}$$

This expression permits to associate a force to those elementary vortex formation processes when the intensity growth of the vortex is somehow estimable. For example, the initial strength of a starting vortex can be evaluated from general vortex formation concepts, like those introduced in Sect. 2.4. Therefore, when a flow accelerates across a sharp orifice, the resulting force during the starting phase turns out to be proportional to $\rho A_0 (U^2/t)^{2/3}$. On the other long-time extreme of a jet, when its head vortex is far downstream, the separating shear layer, of intensity U, translates straight downstream with a velocity proportional to U. In this case, see the end of Sect. 2.4, the emitted circulation rate is $d\Gamma/dt \cup U^2$ and the force about $\rho A_0 U^2$.

Recapitulating, the phenomenon of vortex formation generates an additional force on the obstacle that is proportional to the orifice area and the rate of growth of the vortex strength. *Vortex formation is associated with force generation*, an effect that may be beneficial or detrimental depending on the specific. For example, vortex formation behind a valvular leaflet supports and regulates its opening/closing dynamics; vice versa, when a stenosis is subjected to vortex formation it suffers a streamlined hammering at every heartbeat.

2.7 Vortex Interactions

When two or more vortices come nearby each other, they likely interact in an intense and irreversible manner. The interaction of vortices involves many different and very complicated phenomena, the principal are outlined here.

Let us first consider two-dimensional vortices. A vortex is associated with a rotating flow about it whose velocity is proportional to the vortex circulation, Γ, and inversely proportional to the distance, r, from the vortex center: $v = \Gamma/2\pi r$. Two vortices that come in close encounter reciprocally induce such a rotation velocity each other. When such vortices have the same sign they rotate together one around the other; in addition, the differential velocity within each individual vortex deforms it and makes the vortices winding up one over the other and eventually merge into a single larger one made by the sum of them. This process is associated with little energy dissipation. On the contrary, two vortices with opposite circulation, a vortex pair, translate together for the self-induced velocity (similarly to what a vortex ring does) along a straight or curved path depending on the relative strengths. Again, the differential velocity inside each single vortex produces the winding up of one's vorticity strip on the other; however, such vorticity strips are of opposite sign and do not merge rather they annihilate each other and reduce the individual vortices' strength.

The close encounter of three-dimensional vortex loops begins with the local interaction between the closest tubular elements of the two vortices that, initially, is nearly two-dimensional. Let us first remark that a close encounter between tubular elements of the same sign is an extremely rare event because the overall self-induced velocity of the corresponding vortex loops would tend to separate the vortices. Thus, three-dimensional interaction begins prevalently between two oppositely rotating portions of a vortex tube. One example of the interaction between two identical vortex rings is shown in Fig. 2.12. Initially, the local interaction is approximately the two-dimensional process described above: the nearby oppositely rotating tubular elements induce the velocity each other and try to translate away. This produces a local stretching of the three-dimensional vortex tube, a stretching that accelerates while the tubes become closer and, in a non-symmetric case, would locally wind up one another. The interacting structures develop increasingly small scales until viscous diffusion becomes a dominant effect, at this point the *reconnection of vortex*

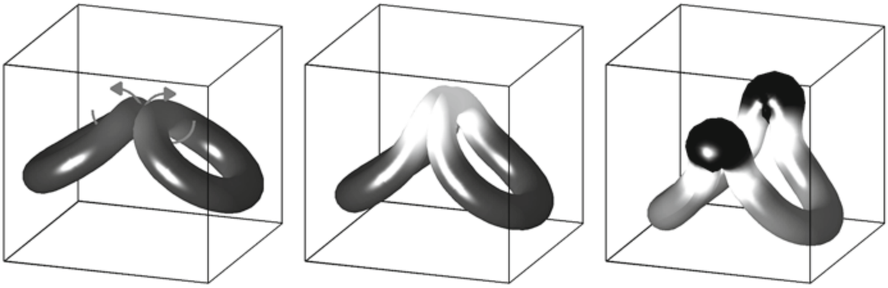


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Fig 2.12 Vortex reconnection, and topological metamorphosis between two impacting vortex rings of equal circulation; the brightness of the filament indicates the strength of the corresponding vorticity. When oppositely rotating vortex tubes get close, they produce a local vortex stretching due to the self-induced velocity (from *left to central panels*). During stretching, the boundary between the vortices becomes locally sharper until the filaments fuse one into the other for viscous effect (from *central to right panels*). After vortex reconnection a new structure is formed, typically its geometry is irregular, the vortex is often unstable and short lived

lines occurs: adjacent opposite vorticity is annihilated by dissipation and the vortex tubes tend to fuse one onto the other (Kida and Takaoka 1994).

The interaction between two identical vortices, like that shown in Fig. 2.12, may result into a complete vortex reconnection and a relatively simple new vortex tube. More often, however, one vortex is stronger than the other is, only part of its tubular structure can reconnect with the other weaker vortex, and the incomplete reconnection gives rise to new vortices with a complex branched geometry. In general, the vortex structure resulting from the fusion of previous interacting vortices, typically presents a very irregular geometry. Differential curvatures, which give sharply variable self-induced velocity and local motion, and differential vorticity strength, which gives axial flow along the tube, tend to rapidly further deform the vortex, produce further reconnections, and give rise to smaller vorticity structures. In other terms, an irregular three-dimensional vortex structure is overall unstable, tends to destroy itself, and is short lived. The more a vortex is regular, like a vortex ring, the more it remains coherent and lasts longer.

A special case of vortex interaction, which is particularly relevant in closed systems like cardiovascular vessels, is the *interaction with a nearby wall*. The vortex-wall interaction can be divided into two different phenomena: the *irrotational interaction*, that is a consequence of the wall impermeability; and the *viscous interaction* with the vorticity in the boundary layer. Let us consider the two effects separately.

First, an isolated vortex induces a rotary motion where streamlines are circular. When such a vortex approaches an impermeable wall, the streamlines must deform to avoid crossing the boundary. With reference to Fig. 2.13 (left panel), the modification of the flow field that satisfies the impermeability condition can be immediately constructed simply by symmetry considerations. It is the irrotational flow that would be induced by an *image vortex* of opposite circulation placed symmetrically below the wall. Such an image vortex gives a velocity perpendicular to the wall that is opposite to that of the real vortex, and thus ensures that the fluid does

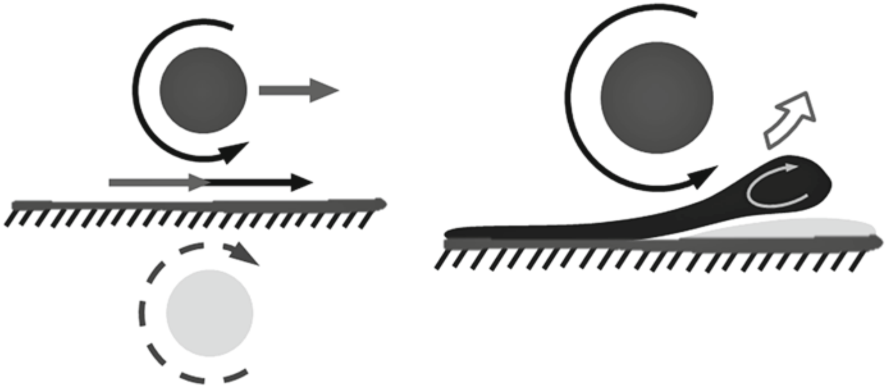

Fig 2.13 The interaction of a vortex with the wall produces two separate effects. First (*left panel*), the condition of impermeability is satisfied by a distortion to the vortex-induced flow that is equivalent to having an opposite vortex placed symmetrically below the wall. The presence of such a "image" vortex increases the tangential velocity next to the wall, and induces a translation velocity to the otherwise still vortex. The second effect (*right panel*) is due to viscous adherence, the development of a boundary layer and eventually a vortex-induced separation

not penetrate into it. On the contrary, the tangential velocity has the same sign of that due to the real vortex and therefore the velocity adjacent to the wall increases (*splash effect*). In addition, the image vortex also induces a velocity to the real vortex that accelerates or decelerates (depending on the direction of the circulation) with respect to the background flow because of this *image effect*. For example, a (clockwise) vortex that just formed from a wall underneath is decelerated by the image below the same wall, while it accelerates when it approaches a wall on the opposite side.

Second, in addition to the image effect, a vortex near a wall also influences the development of the boundary layer because of the viscous adherence condition at such a wall. A vortex creates a local velocity gradient along the wall, acceleration followed by deceleration (or vice versa depending on the direction of rotation). This perturbation, as previously discussed in Sect. 2.3, may give rise to a vortex-induced separation of the boundary layer and to the formation of secondary vortices as it is sketched in Fig. 2.13 (right panel).

When the vortex-boundary interaction described above applies to a tract of a three-dimensional vortex tube, it eventually affects the following three-dimensional dynamics. First, the image effect gives a local stretching and deformation of a vortex filament. Second, when the vortex gets closer, it eventually interacts directly with the vortex-induced vorticity distribution. This is an interaction between oppositely circulating vorticity. That gives rise to the local wind-up of the wall vorticity around the approaching vortex and to reconnection with its vortex lines. Eventually, the vortex crops by dissipation in the regions closer to the wall; this unbalances the three-dimensional vortex structures that tends to rapidly further deform and develop small structures that are eventually dissipated.

2.8 AMention to Turbulence

Let us enter smoothly into the realm of fully developed turbulence by deepening a little further the concept of interaction between three-dimensional vortices introduced in the previous chapter. We have said there that the interaction between two vortices first deforms the overall, *large scale* geometry of the vortex loops then, after sequences of reconnections, breaking of vortices and further deformations, it eventually transforms the original vorticity into several irregular small structures. Such *small scale* elements present sharp velocity gradient, viscous friction, and are rapidly dissipated.

Now consider that large vortices are continuously generated, formed from the surrounding boundary. The resulting flow witnesses the simultaneous presence of these large structures with others of all intermediate sizes from these down to the smallest vortices dominated by viscosity. A measure of the complexity of such a flow can be provided by from the amount of such contemporary vortices, measured by the ratio between the largest scale, say L, and the smallest one, that is indicated with η. When L is comparable to η, the flow is a regular one. For example, if the system creates continuously vortex loops of size L, the resulting flow is a sequence of individual rings that decay as time proceeds because there are no vortices of a smaller size. On the opposite end, when L is much larger than η, the flow presents the just generated vortex loop of size L, the previously generated vortices that broke down into smaller structure of size, say, $L/2$, and a large number of interacting vortices of progressively small size up to the smallest ones. The order of magnitude of this complexity can be estimated from the phenomenological theory of turbulence (due to Kolmogorov in 1941 and reported, for example, in Frisch 1995, Sect. 7.4)

$$\frac{L}{\eta} \approx Re^{3/4} \tag{2.15}$$

where *Re* is the *Reynolds number*

$$Re = \frac{UL}{\nu} \tag{2.16}$$

that was previously introduced in (1.21). When the Reynolds number is large enough, the flow presents fluctuations on velocity and vorticity over a wide range of scales, and can be classified as a *turbulent flow*.

An increased friction between fluid elements and enhanced energy dissipation with respect to regular fluid motion characterizes turbulence. In fact, the development of turbulence is the strategy used by fluids to dissipate the excess energy. When a fluid motion presents a large density of energy (high velocity), the fluid may be unable, in a regular motion, to maintain equilibrium between viscous dissipation and the external energy source; in that case, it increases the particle paths by developing swirling motions and small scales with higher shear rate to increases viscous dissipation up to equilibrium. In fact, the Reynolds number also represents the ratio between the kinetic energy introduced in the large scales, proportional to ρU^2, and their ability to dissipate with shear stress, grossly estimable as proportional to $\rho \nu U/L$. When the Reynolds number increases above a certain threshold, smaller scales develop to enhance dissipation. In other words, regular flow becomes unstable and turbulence appears. Every realization of flow motion presents a *critical value of the Reynolds number* above which the motion develops turbulence. The value of the critical Reynolds number was mentioned in Sect. 1.6 to be about 2,300 in the case of steady flow in a circular vessel.

Turbulence enhances energy dissipation and therefore it is normally a threat of excessive energy consumption in the vascular circulation. Another property is the unpredictability of its chaotic fluctuations that makes turbulent flows difficult to control, model, and manage. On the other side, turbulence has several positive implications; first of all, it makes life possible by enhancing mixing and diffusion. While viscous diffusion is an extremely efficient mechanism to distribute substances at very small scales, turbulent dispersion dominates mixing at larger scales. For example, viscous diffusion length, which grows proportionally to $\sqrt{t}$ (see Eq. 1.16), in water takes a few hundredth of a second to reach 1 mm, a few second for 1 cm, and over 1 h for 1 m. On the contrary, the *accelerated turbulent dispersion* dominates the mixing and heat propagation at scales sufficiently larger, typically above a few millimeters (in general as soon as the Reynolds number is above a few thousands). It is evident how turbulence is ubiquitous in nature and how it ensures the mixing that is experienced in everyday life.

In general, we may think of turbulence as a system of entangles and interacting vortex elements of disparate sizes. Ranging from the large size generated by the boundaries, to the smaller size where the flow is smoothed out by viscous effects. These vortices are not clear individual structures like those discussed in previous chapters. These are the *turbulent eddies*, loosely defined blob of vorticity of some size and arbitrary shape coming from the breakdown of the large unstable vortices generated from the surrounding boundaries. Turbulence is thus a sea of eddies that is stretched and twisted by the velocity field that is induced by vorticity field itself. Following Davidson (2004, Sect. 2.4), turbulence is a spatially complex distribution of vorticity that exhibits a wide and continuous distribution of scales and advects itself in a chaotic manner.

The overall dynamics of turbulence is normally described in terms of *energy cascade* (Davidson 2004, Sects. 1.6 and 3.2). An external energy input (slope of a channel, a pumping pressure) pushes a fluid within its boundaries, across an orifice, around an obstacle, along an irregular vessel bend. The flow thus generates energetic vortices whose size is comparable with that of the container, these vortices interact and produce smaller eddies, which further interact producing turbulent eddies of progressively smaller size. While eddies become smaller, velocity gradients are larger, and viscous shear stresses become increasingly capable to dissipate kinetic energy into heat. At the lower end of this energy cascade, very small eddies are entirely dissipated and do not generate anything smaller. Thus energy is injected in the turbulent flow at large scales, it cascades toward smaller scales, and it is dissipated in the smallest scales of the flow by viscous friction. The example in Fig. 2.14 shows the co-presence of vortices of different scale in a two-dimensional turbulent flow.

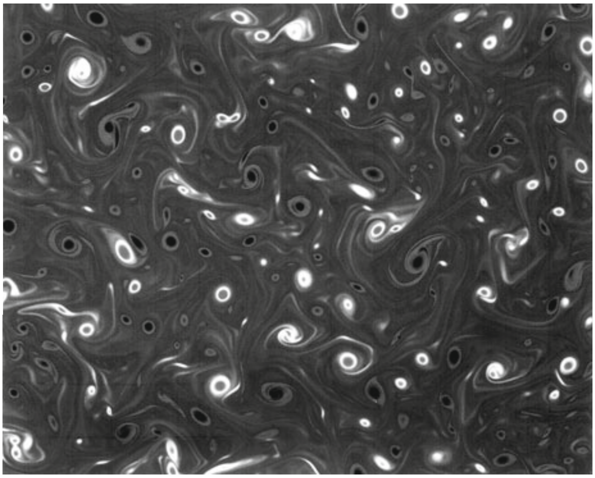

Fig 2.14 Vorticity field in a two-dimensional decaying turbulent flow. The vorticity distribution is shown in *gray scale*, clockwise vorticity is *white*, counter-clockwise is *black*. The flow field presents interacting turbulent eddies, from vortices to shear layers, with a few order of magnitude differences between their sizes

[AU1]

The most common strategy to tackle the problem of turbulence relies of statistical methods, searching for a description of the average motion (responsible for transport) and of its fluctuations (responsible for dispersion). This is such a common practice that the study of turbulence is often considered that of *statistical fluid mechanics*. However, turbulence is not a random process, it is actually a deterministic phenomenon; a turbulent flow is a solution of the mathematical equations governing fluid motion (the Navier-Stokes Eq. 1.14). The underlying deterministic nature of turbulence often emerges in the form of *coherent structures*, a somehow collectively organized dynamics, developing within a random background. These coherent structures are often the large scale vortices generated during vortex formation processes, possibly modified for the presence of turbulence. Or they are vortices that develop from the instability of a parallel flow, typically vortex layer, like in boundary layer flows (Davidson 2004, Sect. 4.2.6), in some other cases they emerge from coalescence of incoherent background vorticity, like the vortices in two-dimensional flows of Fig. 2.14 or three-dimensional filaments. Coherent vortex structures typically contain most part of the energy and therefore they represent the fundamental objects in the analysis of each specific turbulent flow. In other words, the concepts of vortex formation discussed above find application to turbulent flows as well, keeping in mind the presence of the turbulent background, and extending the concept of vortex formation loosely to vortex structures developing for instability of smooth flows.

In the cardiovascular system, turbulent flows are rarely encountered. The largest scales of motion achievable in the arterial network cannot exceed the vessel size, of a few centimeters at most. The Reynolds number is normally well below 1,000, with the exception of the very largest vessels. The flow in the ascending aortic and, sometime, in the left ventricular cavity can reach values of the Reynolds number up to some thousands, just above the critical threshold during a short interval near the systolic or diastolic peaks. In any case, when any turbulence develops, it is *weak turbulence* with an energetic level that does not influence appreciably the main dynamics and vortex formation processes. Such a weak turbulence simply increases the dissipation level through more intense interaction between larger vortices. It should be remarked that the highest levels of turbulence, if any, in an unsteady pulsatile flow are recorded during the deceleration after the peak of the flow. In fact, although the instantaneous Reynolds number has decreased, the flow has been filled with the energy during the maximum velocity and has to dissipate such energy during deceleration. Deceleration enhances shear layer instability phenomena, and boundary layer separation, which support turbulence.

Weak turbulence may develop in the diastolic filling of the left ventricle when the mitral jet impacts onto the walls, as it may occur in a dilated heart with a large cardiac output. The most frequent appearance of turbulence occurs in the aortic artery, particularly in the ascending part. Here the tri-leaflet geometry of the aortic valve provokes a rather complex three-dimensional vortex formation that, associated with the large Reynolds number (roughly from 3,000–8,000 at peak systole), gives rise to interactions that produce small scales vorticity and weak turbulence. This is even enhanced in presence of mechanical valves because of the more irregular interactions between the unnatural geometry of vortices and shear layer behind the artificial orifice. Specific examples will be treated in the next parts of the book.

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